Optimal Control
Lectures 22-24: Variational Methods

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Motivation

Simplest Problem of CV:

$$\min \ \mathcal{J}(\mathbf{x}) \triangleq \int_{t_1}^{t_2} \ell(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) \, dt$$
$$\text{s.t.} \ \mathbf{x}(t_1) = \mathbf{x}_1, \quad \mathbf{x}(t_2) = \mathbf{x}_2$$
$$\mathbf{x} \in \hat{\mathcal{C}}^1[t_1, t_2]^{n_x}$$

$\Longrightarrow$

NCOs: Euler & Legendre

$$\frac{d}{dt}\ell_{\dot{\mathbf{x}}}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)) = \ell_{\mathbf{x}}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t))$$
$$\ell_{\dot{\mathbf{x}}\dot{\mathbf{x}}}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)) \succeq \mathbf{0}$$

for each $t \in [t_1, t_2]$

Simplest Problem of OC:

$$\min \ \mathcal{J}(\mathbf{u}) \triangleq \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t)) \, dt$$
$$\text{s.t.} \ \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0$$
$$\mathbf{u} \in \hat{\mathcal{C}}[t_0, t_f]^{n_u}$$

$\Longrightarrow$

NCOs:

?

- Helpful to single out candidate optimal controls
- Beware: May provide a nonempty set of candidates, even though the problem may not have any solution!

Recalls: Necessary Cond. for Optimality in Linear Spaces

- Let $\mathcal{D}$ be a subset of a normed linear space $(\mathcal{U}, \|\cdot\|)$
- Consider a functional $\mathcal{J} : \mathcal{D} \to \mathbb{R}$

$\mathcal{D}$-Admissible Directions

A direction $\boldsymbol{\omega}$ is admissible for $\mathcal{J}$ at $\mathbf{u}^*$ if:

1. $\delta\mathcal{J}(\mathbf{u}^*; \boldsymbol{\omega})$ exists; and
2. $\exists \delta > 0$ such that: $\mathbf{u}^* + \eta\boldsymbol{\omega} \in \mathcal{D}, \forall \eta \in \mathcal{B}_\delta(0)$

Necessary Conditions

If $\mathbf{u}^*$ is a (local) minimizer of $\mathcal{J}$ on $\mathcal{D}$, then

$$\delta\mathcal{J}(\mathbf{u}^*; \boldsymbol{\omega}) = \left.\frac{\partial}{\partial\eta}\mathcal{J}(\mathbf{u}^* + \eta\boldsymbol{\omega})\right|_{\eta=0} = 0,$$

for every $\mathcal{D}$-admissible directions $\boldsymbol{\omega}$ at $\mathbf{u}^*$.

Derivation of the Euler-Lagrange Conditions

Idea: Define the functional $\mathcal{J}$ as follows,

$\mathbf{u} \in \mathcal{C}[t_0, t_f]^{n_u}$ $\longrightarrow$ $\boxed{\int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t)) \, dt \quad \text{with: } \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0}$ $\longrightarrow$ $\mathcal{J}(\mathbf{u}) \in \mathbb{R}$

Procedure:

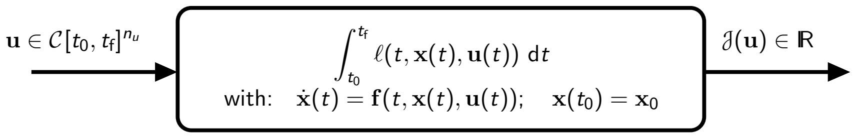
- **Control Variations Around $\mathbf{u}^*$:**
  - $\mathbf{v}(t; \eta) \triangleq \mathbf{u}^*(t) + \eta\boldsymbol{\omega}(t)$, with $\boldsymbol{\omega} \in \mathcal{C}[t_0, t_f]^{n_u}$
  - Corresponding response $\mathbf{y}(t; \eta) \in \mathcal{C}^1[t_0, t_f]$; guaranteed to exist, be unique and continuously-differentiable for $\eta \in \mathcal{B}_\delta(0)$
- **First Variation:**

$$\delta\mathcal{J}(\mathbf{u}^*; \boldsymbol{\omega}) = \int_{t_0}^{t_f} \left(\ell_{\mathbf{u}}^*[t] + \boldsymbol{\lambda}(t)^\mathsf{T}\mathbf{f}_{\mathbf{u}}^*[t]\right) \boldsymbol{\omega}(t) \, dt - \boldsymbol{\lambda}(t_f)^\mathsf{T}\mathbf{y}_\eta(t_f; 0)$$
$$\int_{t_0}^{t_f} \left(\ell_{\mathbf{x}}^*[t] + \boldsymbol{\lambda}(t)^\mathsf{T}\mathbf{f}_{\mathbf{x}}^*[t] + \dot{\boldsymbol{\lambda}}(t)^\mathsf{T}\right) \mathbf{y}_\eta(t; 0) \, dt$$

  - Guaranteed to exist for every $\omega \in \mathcal{C}[t_0, t_f]$ and $\boldsymbol{\lambda} \in \mathcal{C}^1[t_0, t_f]$

Derivation of the Euler-Lagrange Conditions

Idea: Define the functional $\mathcal{J}$ as follows,



Procedure:

- **Specialization:**
  - Adjoint variables to satisfy the ODEs:

$$\dot{\boldsymbol{\lambda}}(t)^{\mathsf{T}} = -\ell_{\mathbf{x}}^*[t] - \boldsymbol{\lambda}(t)^{\mathsf{T}} \mathbf{f}_{\mathbf{x}}^*[t]$$

    with the terminal conditions $\boldsymbol{\lambda}(t_f) = \mathbf{0}$
  - Optimal control to satisfy the algebraic conditions:

$$0 = \ell_{\mathbf{u}}^*[t] + \boldsymbol{\lambda}(t)^{\mathsf{T}} \mathbf{f}_{\mathbf{u}}^*[t]$$

The Euler-Lagrange Conditions

Theorem. Consider the problem

$$\min_{\mathbf{u}} \quad \mathcal{J}(\mathbf{u}) \triangleq \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

for $\mathbf{u} \in \mathcal{C}[t_0, t_f]^{n_u}$, with fixed endpoints $t_0 < t_f$, where ℓ and $\mathbf{f}$ are continuous in $(t, \mathbf{x}, \mathbf{u})$ and have continuous first partial derivatives with respect to $\mathbf{x}$ and $\mathbf{u}$ for all $(t, \mathbf{x}, \mathbf{u}) \in [t_0, t_f] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$.
Suppose that $\mathbf{u}^* \in \mathcal{C}[t_0, t_f]^{n_u}$ is a (local) minimizer for the problem, with corresponding response $\mathbf{x}^* \in \mathcal{C}^1[t_0, t_f]^{n_x}$. Then, there is a vector function $\boldsymbol{\lambda}^* \in \mathcal{C}^1[t_0, t_f]^{n_x}$ such that the triple $(\mathbf{u}^*, \mathbf{x}^*, \boldsymbol{\lambda}^*)$ satisfies the so called Euler-Lagrange equations for each $t_0 \leq t \leq t_f$:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); & \mathbf{x}(t_0) = \mathbf{x}_0 \\
\dot{\boldsymbol{\lambda}}(t) &= -\ell_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}(t)) - \boldsymbol{\lambda}(t)^{\mathsf{T}} \mathbf{f}_{\mathbf{x}}(t, \mathbf{x}(t), \mathbf{u}(t)); & \boldsymbol{\lambda}(t_f) = \mathbf{0} \\
\mathbf{0} &= \ell_{\mathbf{u}}(t, \mathbf{x}(t), \mathbf{u}(t)) + \boldsymbol{\lambda}(t)^{\mathsf{T}} \mathbf{f}_{\mathbf{u}}(t, \mathbf{x}(t), \mathbf{u}(t))
\end{aligned}$$

The Euler-Lagrange Conditions (cont'd)

Remarks:

- Complete set of necessary conditions:
  - $2 \times n_x$ ODEs in the variables $\mathbf{x}(t)$ and $\boldsymbol{\lambda}(t)$, with initial/terminal conditions
  - n_u AEs in the variables $\mathbf{u}(t)$

  But, two-point boundary value problem (TPBVP)!
- In the case that $\mathbf{f}(t, \mathbf{x}, \mathbf{u}) \triangleq \mathbf{u}$, the Euler-Lagrange equations reduce to the Euler equation,

$$\frac{d}{dt} \ell_{\mathbf{u}}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)) = \ell_{\mathbf{x}}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)),$$

  with the natural boundary conditions $\ell_{\mathbf{u}}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t))|_{t_f} = 0$
- In the case that the objective function contains the terminal cost $\phi(t_f, \mathbf{x}(t_f))$, the adjoint terminal conditions become

$$\boldsymbol{\lambda}(t_f) = \phi_{\mathbf{x}}(t_f, \mathbf{x}(t_f))$$

The Euler-Lagrange Conditions (cont'd)

Remarks:

- Advantageous to define the Hamiltonian function,

$$\mathcal{H}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \triangleq \ell(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{f}(t, \mathbf{x}, \mathbf{u})$$

  - Euler-Lagrange Equations: $\dot{\mathbf{x}} = \mathcal{H}_{\boldsymbol{\lambda}}$, $\dot{\boldsymbol{\lambda}} = -\mathcal{H}_{\mathbf{x}}$, $\mathbf{0} = \mathcal{H}_{\mathbf{u}}$
  - Autonomous case: $\frac{d}{dt}\mathcal{H} = \mathcal{H}_t = 0$
- Beware: Same set of conditions for minimize and maximize problems!
  - Second-order conditions: $\mathcal{H}_{\mathbf{uu}} \succeq \mathbf{0}$ (minimize), $\mathcal{H}_{\mathbf{uu}} \preceq \mathbf{0}$ (maximize)

Applying the Euler-Lagrange Conditions

Class Exercise: Single out candidate optimal controls to the problem:

$$\min_u \quad \mathcal{J}(u) \triangleq \int_0^1 \left[\tfrac{1}{2}u(t)^2 - x(t)\right] \, \mathrm{d}t$$
$$\text{s.t.} \quad \dot{x}(t) = 2\,[1 - u(t)]\,; \quad x(0) = 1.$$

Piecewise Continuous Extremals

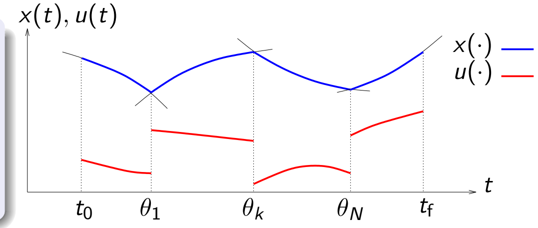
In either one of two situations:

- A continuous control $\mathbf{u} \in \mathcal{C}[t_0, t_f]^{n_u}$ satisfying the Euler-Lagrange equations does not exist
- Improved results are sought in the more general class of piecewise continuous controls, $\mathbf{u} \in \hat{\mathcal{C}}[t_0, t_f]^{n_u}$

Finite partition:

$t_0 = \theta_0 < \cdots < \theta_N < \theta_{N+1} = t_f$

with $\mathbf{u}|_{[\theta_k, \theta_{k+1}]} \in \mathcal{C}[\theta_k, \theta_{k+1}]^{n_u}$, for each $k = 0, \ldots, N$



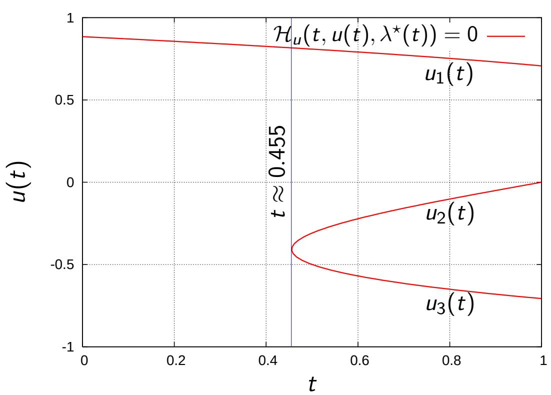
What conditions must hold at the corner points $\theta_1, \ldots, \theta_N$ of a piecewise continuous extremal?

Piecewise Continuous Extremals (cont'd)

Class Exercise: Consider the optimal control problem:

$$\min_u \quad \mathcal{J}(u) \triangleq \int_0^1 \left[u(t)^2 - u(t)^4 - x(t)\right] \, \mathrm{d}t$$
$$\text{s.t.} \quad \dot{x}(t) = -u(t); \quad x(0) = 1.$$

Single out candidate optimal controls for this problem?



Piecewise Continuous Extremals (cont'd)

Regular vs. Non-Regular Problems

- A problem is said to be regular if the condition $\mathcal{H}_\mathbf{u}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) = \mathbf{0}$ has a unique solution, for each $t, \mathbf{x}, \boldsymbol{\lambda}$
- Only non-regular problems can exhibit corner points!

Corner Conditions

A corner point can only occur at a time instant θ such that:

$$\mathbf{x}^*(\theta^-) = \mathbf{x}^*(\theta^+)$$
$$\boldsymbol{\lambda}^*(\theta^-) = \boldsymbol{\lambda}^*(\theta^+)$$
$$\mathcal{H}(\theta^-, \mathbf{x}^*(\theta), \mathbf{u}^*(\theta^-), \boldsymbol{\lambda}^*(\theta)) = \mathcal{H}(\theta^+, \mathbf{x}^*(\theta), \mathbf{u}^*(\theta^+), \boldsymbol{\lambda}^*(\theta))$$

The Mangasarian Sufficient Conditions

Theorem. Consider the problem

$$\min_{\mathbf{u}} \quad \mathcal{J}(\mathbf{u}) \triangleq \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0,$$

for $\mathbf{u} \in \mathcal{C}[t_0, t_f]^{n_u}$, with fixed endpoints $t_0 < t_f$, where ℓ and $\mathbf{f}$ are continuous in $(t, \mathbf{x}, \mathbf{u})$, have continuous first partial derivatives with respect to $\mathbf{x}$ and $\mathbf{u}$.

Suppose that:

- ℓ and $\mathbf{f}$ are [strictly] jointly convex in $(\mathbf{x}, \mathbf{u})$, for all $(t, \mathbf{x}, \mathbf{u}) \in [t_0, t_f] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$
- The triple $\mathbf{u}^\star \in \mathcal{C}[t_0, t_f]^{n_u}$, $\mathbf{x}^\star \in \mathcal{C}^1[t_0, t_f]^{n_x}$ and $\boldsymbol{\lambda}^\star \in \mathcal{C}^1[t_0, t_f]^{n_x}$ satisfies the Euler-Lagrange conditions
- $\lambda^*(t) \geq \mathbf{0}$, for all $t \in [t_0, t_f]$

Then, $\mathbf{u}^\star$ is a [strict] global minimizer

Applying the Euler-Lagrange Conditions

Class Exercise: Find a global minimizer $\mathbf{u}^* \in \mathcal{C}^{[}t_0, t_f]$ for the problem:

$$\min_{u} \quad \mathcal{J}(u) \triangleq \int_0^1 \left[\tfrac{1}{2} u(t)^2 - x(t)\right] dt$$
$$\text{s.t.} \quad \dot{x}(t) = 2\left[1 - u(t)\right]; \quad x(0) = 1.$$

Recalls: Method of Lagrange Multipliers in Linear Spaces

- Let $\mathcal{J}$ and $\mathcal{G}_1, \ldots, \mathcal{G}_{n_g}$ be functionals defined in a normed linear space $(\mathcal{U}, \|\cdot\|)$ and have continuous Gâteaux derivatives

Suppose that:

- $\mathbf{u}^*$ is a (local) minimizer of $\mathcal{J}$ subject to $\mathcal{G}_1, \ldots, \mathcal{G}_{n_g} \leq 0$
- there exist n_g independent directions $\omega_1, \ldots, \omega_{n_g} \in \mathcal{U}$ such that:

$$\begin{vmatrix} \delta\mathcal{G}_1(\mathbf{u}^*; \boldsymbol{\omega}_1) & \cdots & \delta\mathcal{G}_1(\mathbf{u}^*; \boldsymbol{\omega}_{n_g}) \\ \vdots & \ddots & \vdots \\ \delta\mathcal{G}_{n_g}(\mathbf{u}^*; \boldsymbol{\omega}_1) & \cdots & \delta\mathcal{G}_{n_g}(\mathbf{u}^*; \boldsymbol{\omega}_{n_g}) \end{vmatrix} \neq 0$$

Then, there exists $\boldsymbol{\nu}^* \in \mathbb{R}^{n_g}$ such that:

$$\delta\mathcal{J}(\mathbf{u}^*; \boldsymbol{\omega}) + \sum_{k=1}^{n_g} \nu_k^* \delta\mathcal{G}_k(\mathbf{u}^*; \boldsymbol{\omega}) = 0, \quad \forall \boldsymbol{\omega} \in \mathcal{U}$$
$$\nu_k \mathcal{G}_k(\mathbf{u}^*) = 0, \qquad \nu_k \geq 0$$

Problems with General Terminal Constraints

Consider the problem to find $(\mathbf{u}, t_f) \in \mathcal{C}[t_0, T]^{n_u} \times \mathbb{R}$ to:

$$\min_{\mathbf{u}, t_f} \quad \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt + \phi(t_f, \mathbf{x}(t_f))$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x}_0$$
$$\psi_k(t_f, \mathbf{x}(t_f)) \leq 0, \quad k = 1, \ldots, n_\psi$$

where ℓ and $\mathbf{f}$ are continuous in $(t, \mathbf{x}, \mathbf{u})$, have continuous first partial derivatives with respect to $\mathbf{x}$ and $\mathbf{u}$.

Application of the **method of Lagrange multipliers** with:

- $\mathcal{U} \triangleq \mathcal{C}[t_0, T]^{n_u} \times \mathbb{R}$
- $\mathcal{J}(\mathbf{u}, t_f) \triangleq \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt + \phi(t_f, \mathbf{x}(t_f))$
- $\mathcal{G}_k(\mathbf{u}, t_f) \triangleq \psi_k(t_f, \mathbf{x}(t_f)), \ k = 1, \ldots, n_\psi$

Problems with General Terminal Constraints (cont'd)

- Euler-Lagrange Equations ($\mathcal{H} \stackrel{\Delta}{=} \ell + \boldsymbol{\lambda}^\mathsf{T}\mathbf{f}$):

$$\left.\begin{aligned} \dot{\mathbf{x}} &= \mathcal{H}_{\boldsymbol{\lambda}} \\ \dot{\boldsymbol{\lambda}} &= -\mathcal{H}_{\mathbf{x}} \\ \mathbf{0} &= \mathcal{H}_{\mathbf{u}} \end{aligned}\right\}, \quad t_0 \leq t \leq t_f;$$

- Legendre-Clebsch Condition:

$$\mathcal{H}_{\mathbf{uu}} \text{ semi-definite positive}, \quad t_0 \leq t \leq t_f;$$

- Transversal Conditions:

$$\left[\mathcal{H} + \phi_t + \boldsymbol{\nu}^\mathsf{T}\boldsymbol{\psi}_t\right]_{t_f} = 0, \text{ if } t_f \text{ is free}$$

$$\left[\boldsymbol{\lambda} - \phi_{\mathbf{x}} + \boldsymbol{\nu}^\mathsf{T}\boldsymbol{\psi}_{\mathbf{x}}\right]_{t_f} = \mathbf{0}$$

$$\left[\boldsymbol{\nu}^\mathsf{T}\boldsymbol{\psi}\right]_{t_f} = \mathbf{0}, \quad \boldsymbol{\nu} \geq \mathbf{0}, \text{ if } \boldsymbol{\psi} \text{ is inequality}$$

$\boldsymbol{\psi}$ satisfy a regularity condition

Solving Problems with Terminal Constraints

Class Exercise: Find candidate solution to the following problem:

$$\begin{aligned} \min_u \quad & \mathcal{J}(u) \stackrel{\Delta}{=} \int_0^1 \tfrac{1}{2} u(t)^2 \, dt \\ \text{s.t.} \quad & \dot{x}(t) = u(t) - x(t); \quad x(0) = 1 \\ & x(1) = 0 \end{aligned}$$