Optimal Control
Lectures 17-18: Problem Formulation

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Spring 2009

Optimal Control Problem Formulation

Objective

"Determine the control signals that will cause a controlled system to satisfy the physical constraints and, at the same time, minimize (or maximize) some performance criterion"

Main Steps of Problem Formulation:

1. Specification/modeling of the controlled system
   - Admissible controls
   - Response and dynamical model
2. Specification of a performance criterion
   - Lagrange, Mayer and Bolza forms
3. Specification of physical constraints to be satisfied
   - Path and terminal constraints

Controlled System

A controlled system is characterized by:

- State variables $\mathbf{x} = [x_1 \ \cdots \ x_{n_x}]^\mathsf{T}$, $n_x \geq 1$, describing its internal behavior — phase space
  - e.g., coordinates, velocities, concentrations, flow rates, etc.
- Control variables $\mathbf{u} = [u_1 \ \cdots \ u_{n_u}]^\mathsf{T}$, $n_u \geq 1$, describing the controller positions — control space
  - e.g., force, voltage, temperature, etc.

Model Development Objective:

- Develop the simplest model that accurately predicts the system behavior to all foreseeable control decisions"
- Typically, in the form of Ordinary Differential Equations (ODEs),

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)),$$

with specified initial conditions,

$$\mathbf{x}(t_0) = \mathbf{x_0}$$

Controlled System

System Response:

A solution $\mathbf{x}(t; t_0, \mathbf{x}_0, \mathbf{u}(\cdot))$ to the differential equations is called a response to the control $\mathbf{u}(\cdot)$ for the initial conditions $\mathbf{x}_0$ at t_0

Important Questions:

- Does a response exist for a given control and initial conditions?
- Is this response unique?

Admissible Controls

- Restriction to a certain control region, $\mathbf{u}(t) \in U \subset \mathbb{R}^{n_u}$
  - E.g., $U \triangleq \{\mathbf{u} \in \mathbb{R}^{n_u} : |u_j| \leq 1, \forall j\}, \quad U \triangleq \{\mathbf{u} \in \mathbb{R}^{n_u} : \phi(\mathbf{u}) = 0\}$
- Which class for the time-varying controls?
  - Continuous controls: $\mathbf{u} \in \mathcal{C}[t_0, t_f]^{n_u}$ — may not be a large enough class
  - More generally, piecewise continuous control: $\mathbf{u} \in \hat{\mathcal{C}}[t_0, t_f]^{n_u}$

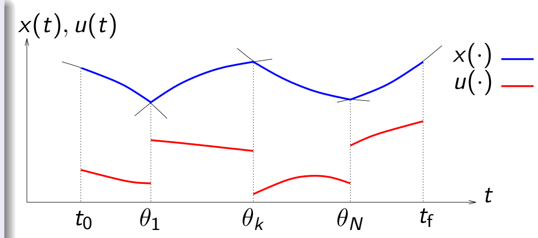
Finite partition:

$$t_0 = \theta_0 < \cdots < \theta_N < \theta_{N+1} = t_f$$

and

$$\mathbf{u}|_{[\theta_k, \theta_{k+1}]} \in \mathcal{C}[\theta_k, \theta_{k+1}]^{n_u}$$

for each $k = 0, \ldots, N$



- Allow instantaneous control jumps
  - **Underlying assumption:** inertia-less controllers

Admissible Controls

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Class of Admissible Controls $\mathcal{U}[t_0, t_f]$

A piecewise continuous control $\mathbf{u}(\cdot)$, defined on some time interval $t_0 \leq t \leq t_f$, with range in the control region U,

$$\mathbf{u}(t) \in U, \quad \forall t \in [t_0, t_f],$$

is said to be an admissible control

- Every admissible control is bounded
  - **Do you see why?**

Performance Criterion

- A functional used for quantitative evaluation of a system's performance
  - Can depend on both the control and state variables
  - Can depend on the initial and/or terminal times too (if not fixed)

Functional Forms

- **Lagrange form:** $\mathcal{J}(\mathbf{u}) \triangleq \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$
- **Mayer form:** $\mathcal{J}(\mathbf{u}) \triangleq \varphi(t_0, \mathbf{x}(t_0), t_f, \mathbf{x}(t_f))$
- **Bolza form:** $\mathcal{J}(\mathbf{u}) \triangleq \varphi(t_0, \mathbf{x}(t_0), t_f, \mathbf{x}(t_f)) + \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$

- Lagrange, Mayer and Bolza functional forms are equivalent!

Physical Constraints

- Functional equalities/inequalities restricting the range of values that can be assumed by control and/or state variables

Types of Constraints

- **Point constraints:** $\psi^{\mathrm{i}}(\bar{t}, \mathbf{x}(\bar{t})) \leq 0, \quad \psi^{\mathrm{e}}(\bar{t}, \mathbf{x}(\bar{t})) = 0, \quad \bar{t} \in [t_0, t_f]$
  - E.g., terminal state constraint $x_k(t_f) \leq x_k^{U,f}$
- **Isoperimetric (integral) constraints:**

$$\int_{t_0}^{t_f} \kappa^{\mathrm{i}}(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \leq 0, \quad \int_{t_0}^{t_f} \kappa^{\mathrm{e}}(t, \mathbf{x}(t), \mathbf{u}(t))\, dt = 0$$

  - Can always be reformulated as a point constraints! (easier to handle)
- **Path constraints:**

$$\kappa^{\mathrm{i}}(t, \mathbf{x}(t), \mathbf{u}(t)) \leq 0, \quad \kappa^{\mathrm{e}}(t, \mathbf{x}(t), \mathbf{u}(t)) = 0, \quad \forall t \in [t_0, t_f]$$

  - E.g., input path constraints $\mathbf{u}^L \leq \mathbf{u}(t) \leq \mathbf{u}^U$, $\forall t \in [t_0, t_f]$; (pure) state path constraints $x_k(t) \leq x_k^U$, $\forall t \in [t_0, t_f]$

A Quite General Optimal Control Formulation

Optimal Control Problem

Determine $\mathbf{u} \in \hat{\mathcal{C}}^1[t_0, t_f]^{n_u}$ that

$$\begin{aligned}
\text{minimize:} \quad & \mathcal{J}(\mathbf{u}) \triangleq \phi(\mathbf{x}(t_f)) + \int_{t_0}^{t_f} \ell(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \\
\text{subject to:} \quad & \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{x}(t_0) = \mathbf{x_0} \\
& \psi_j^{\mathrm{i}}(\mathbf{x}(t_f)) \leq 0, \quad j = 1, \ldots, n_\psi^{\mathrm{i}} \\
& \psi_j^{\mathrm{e}}(\mathbf{x}(t_f)) = 0, \quad j = 1, \ldots, n_\psi^{\mathrm{e}} \\
& \kappa_j^{\mathrm{i}}(t, \mathbf{x}(t), \mathbf{u}(t)) \leq 0, \quad j = 1, \ldots, n_\kappa^{\mathrm{i}} \\
& \kappa_j^{\mathrm{e}}(t, \mathbf{x}(t), \mathbf{u}(t)) = 0, \quad j = 1, \ldots, n_\kappa^{\mathrm{e}} \\
& \mathbf{u}(t) \in [\mathbf{u}^L, \mathbf{u}^U]
\end{aligned}$$

Open-Loop vs. Closed Loop Optimal Control

Open-Loop Optimal Control

An optimal control law of the form

$$\mathbf{u}^*(t) = \boldsymbol{\omega}(t; t_0, \mathbf{x_0}), \quad \mathbf{t} \in [\mathbf{t_0}, \mathbf{t_f}],$$

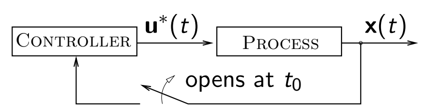
determined for a particular initial state value $\mathbf{x}(t_0) = \mathbf{x_0}$

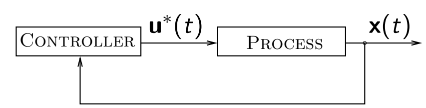
Closed-Loop Optimal Control

A feedback control law of the form

$$\mathbf{u}^*(t) = \boldsymbol{\omega}(t, \mathbf{x}(t)), \quad t \in [t_0, t_f],$$

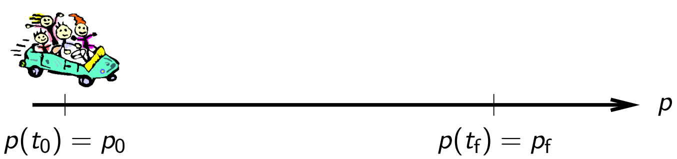
i.e., calculating the optimal control actions $\mathbf{u}^*(t)$ corresponding to the current state $\mathbf{x}(t)$





Pros and Cons?

Class Exercise: Car Control Problem

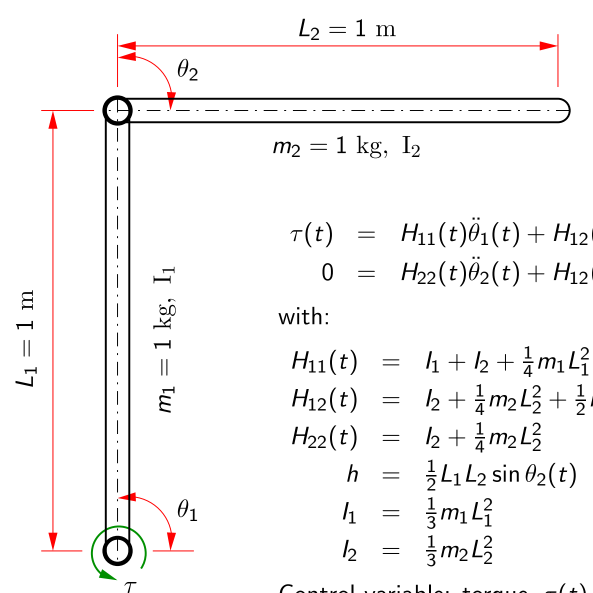


Optimal Control Formulation:

"Drive a car, initially parked at position p_0, to its final destination p_f (parking), in minimum time"

- State, $p(t), \dot{p}(t)$: position and speed
- Control, $u(t)$: force due to acceleration (≥ 0) or deceleration (≤ 0)

Class Exercise: Double Pendulum



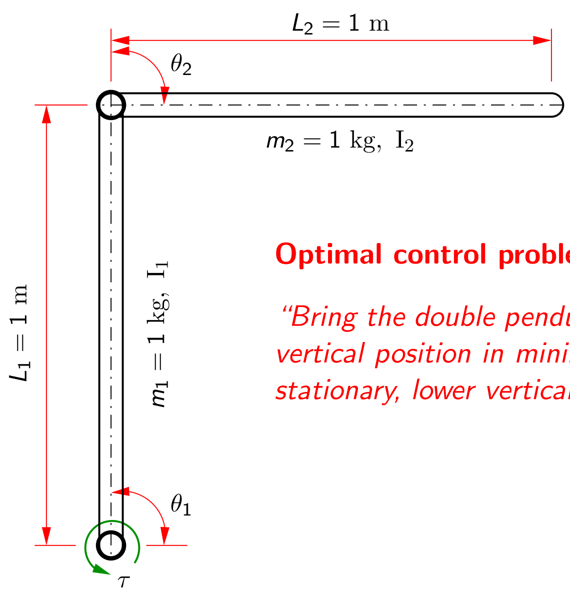
$$\begin{aligned}
\tau(t) &= H_{11}(t)\ddot{\theta}_1(t) + H_{12}(t)\ddot{\theta}_2(t) - 2h\dot{\theta}_1(t)\dot{\theta}_2(t) - h\dot{\theta}_2^2(t) \\
0 &= H_{22}(t)\ddot{\theta}_2(t) + H_{12}(t)\ddot{\theta}_1(t) + h\dot{\theta}_1^2(t)
\end{aligned}$$

with:

$$\begin{aligned}
H_{11}(t) &= I_1 + I_2 + \tfrac{1}{4}m_1L_1^2 + m_2\left[L_1^2 + \tfrac{1}{4}L_2^2 + 2L_1L_2\cos\theta_2(t)\right] \\
H_{12}(t) &= I_2 + \tfrac{1}{4}m_2L_2^2 + \tfrac{1}{2}m_2L_1L_2\cos\theta_2(t) \\
H_{22}(t) &= I_2 + \tfrac{1}{4}m_2L_2^2 \\
h &= \tfrac{1}{2}L_1L_2\sin\theta_2(t) \\
I_1 &= \tfrac{1}{3}m_1L_1^2 \\
I_2 &= \tfrac{1}{3}m_2L_2^2
\end{aligned}$$

Control variable: torque, $\tau(t)$

Class Exercise: Double Pendulum



Optimal control problem:

"Bring the double pendulum to stationary, upper vertical position in minimum time, starting from stationary, lower vertical position"

Feasibility and (Global) Optimality

Feasible Control

A control $\mathbf{u} \in \mathcal{U}[t_0, t_f]$ is said to be feasible if:

1. $\mathbf{x}(t; x_0, \mathbf{u}(\cdot))$ is defined for each $t \in [t_0, t_f]$
2. $\mathbf{u}$ and $\mathbf{x}$ satisfy all of the physical constraints

The set of feasible controls, $\Omega[t_0, t_f]$, is defined as:

$$\Omega[t_0, t_f] \triangleq \{\mathbf{u} \in \mathcal{U}[t_0, t_f] : \mathbf{u} \text{ is feasible}\}$$

(Globally) Optimal Control (Minimize Case)

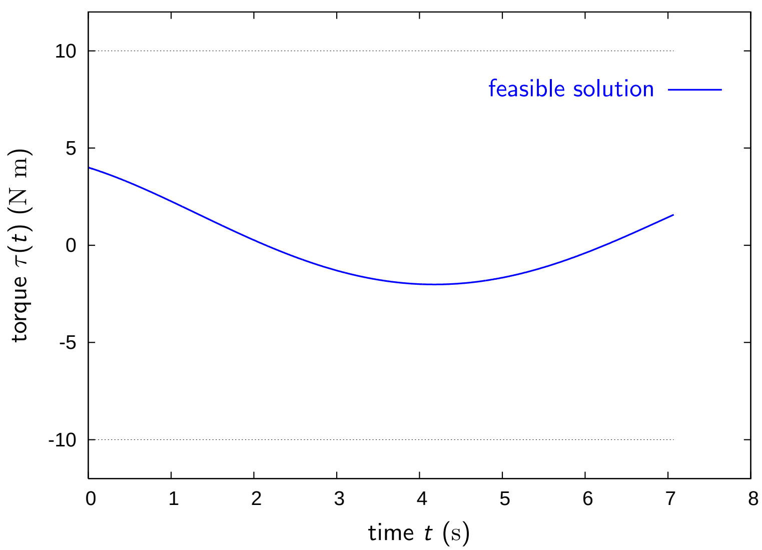
A control $\mathbf{u}^* \in \mathcal{U}[t_0, t_f]$ is said to be optimal if:

$$\mathcal{J}(\mathbf{u}^*) \leq \mathcal{J}(\mathbf{u}), \quad \forall \mathbf{u} \in \Omega[t_0, t_f]$$

- This optimality condition is global in nature, i.e. it does not require consideration of a norm

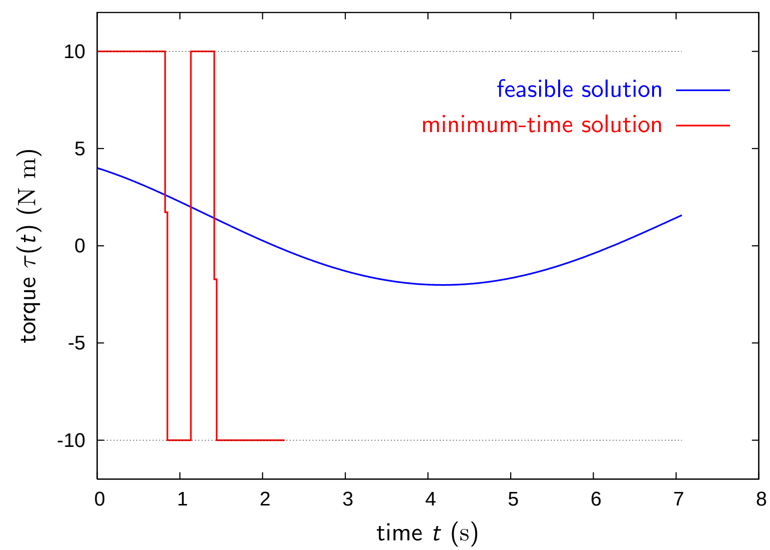
Class Exercise: Double Pendulum (cont'd)

A feasible solution:



Class Exercise: Double Pendulum (cont'd)

The optimal solution:



Local Optimality

Locally Optimal Control (Minimize Case)

A control $\mathbf{u}^* \in \mathcal{U}[t_0, t_f]$ is said to be a local optimum if:

$$\exists \delta > 0 \text{ such that: } \mathcal{J}(\mathbf{u}^*) \leq \mathcal{J}(\mathbf{u}), \quad \forall \mathbf{u} \in \mathcal{B}_\delta(\mathbf{u}^*) \cap \Omega[t_0, t_f]$$

Specification of a Norm/Distance on $\mathcal{U}[t_0, t_f]$:

- **Weak Minima:**

$$\mathcal{B}_\delta^{1,\infty}(\mathbf{u}^*) \triangleq \left\{ \mathbf{u} \in \mathcal{U}[t_0, t_f] : \sup_{t \in \bigcup_{k=0}^{N} (\theta_k, \theta_{k+1})} \|\mathbf{x}(t) - \mathbf{x}^*(t)\| + \|\mathbf{u}(t) - \mathbf{u}^*(t)\| < \delta \right\}$$

  - Similar considerations as with Euler's equation — leads to the Euler-Lagrange equations

- **Strong Minima:**

$$\mathcal{B}_\delta^{\infty}(\mathbf{u}^*) \triangleq \left\{ \mathbf{u} \in \mathcal{U}[t_0, t_f] : \sup_{t \in \bigcup_{k=0}^{N} (\theta_k, \theta_{k+1})} \|\mathbf{x}(t) - \mathbf{x}^*(t)\| < \delta \right\}$$

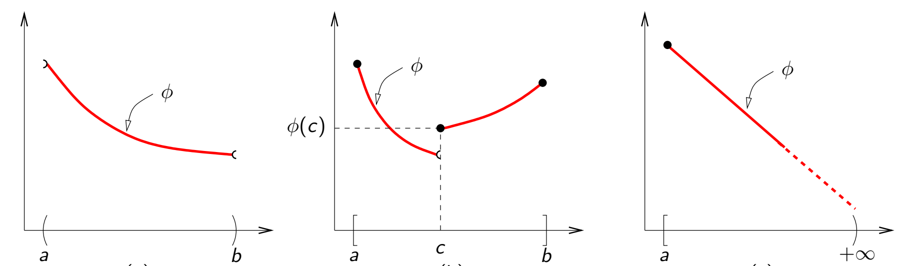
  - Similar considerations as with Weierstrass' condition — leads to the Pontryagin Maximum Principle

Existence of an Optimal Solution

Recalls: Weierstrass' Theorem

Let $P \subset \mathbb{R}^n$ be a nonempty, compact set, and let $\phi : P \to \mathbb{R}$ be continuous on P. Then, the problem $\min\{\phi(p) : p \in P\}$ attains its minimum, that is, there exists a minimizing solution to this problem.

Why closedness of P? Why continuity of ϕ? Why boundedness of P?



(a) (b) (c)