IC-32 **Optimal Control** **Winter 2006**

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Problem Set #7

1. **[Free Problem of the Calculus of Variations]** Consider the functional

$$\mathcal{J}(x) := \int_0^T c_1[\dot{x}(t)]^2 + c_2\, x(t)\ \mathrm{d}t,$$

for $x \in \mathcal{D} := \{x \in \mathcal{C}^1[0,T] : x(0) = 0, x(T) = b\}$.

   (a) Find candidate extremals when the terminal time is fixed as $T = 1$,

   (b) Find candidate extremals when the terminal time T is let free.

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2. **[Problems with Free End-Points]** Find candidate extremals for the functional

$$\mathcal{J}(x) := \int_0^1 \frac{1}{2}[\dot{x}(t)]^2 + x(t)(\dot{x}(t) + 1)\ \mathrm{d}t,$$

for $x \in \mathcal{D} := \mathcal{C}^1[0,1]$ (i.e., $x(0)$ and $x(1)$ can be chosen freely).

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3. **[Problems with Piecewise $\mathcal{C}^1$ Extremals]** Consider the functional

$$\mathcal{J}(x) := \int_0^4 (\dot{x}(t) - 1)^2(\dot{x}(t) + 1)^4\ \mathrm{d}t,$$

for $x \in \mathcal{D} := \{x \in \hat{\mathcal{C}}^1[0,T] : x(0) = 0, x(4) = 2\}$.

   (a) Find candidate extremals that have just one corner point.

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4. **[Problems with Isoperimetric Constraints]**

   (a) Use the method of Lagrange multipliers to identify candidate solutions to the problem

$$\begin{aligned} \text{minimize:} \quad & \mathcal{J}(x) := \int_0^T \exp(-rt)\ x(t)\ \mathrm{d}t \\ \text{subject to:} \quad & x \in \mathcal{D} := \left\{ x \in \mathcal{C}^1[0,T] : \int_0^T \sqrt{x(t)}\ \mathrm{d}t = A \right\}, \end{aligned}$$

   given $A \geq 0$ and $r > 0$.

(b) Same question for the problem

$$\begin{aligned} \text{minimize:} \quad & \mathcal{J}(x) := \int_0^b \sqrt{1 + x(t)}\mathrm{d}t \\ \text{subject to:} \quad & x \in \mathcal{D} := \left\{ x \in \mathcal{C}^1[0, b] : \int_0^b x(t)\ \mathrm{d}t = c \right\}, \end{aligned}$$

given $b > 0$ and $c > 0$.

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5. **[Problems with End-Point Equality Constraints]** Find the curves in $\mathcal{C}^1[0, T]$ for which

$$\mathcal{J}(x) := \int_0^T \frac{\sqrt{1 + [\dot{x}(t)]^2}}{x(t)}\ \mathrm{d}t,$$

can have extrema subject to $x(0) = 0$, in the following cases:

(a) the point $(T, x(T))$ must be on the line $x = t - 5$;

(b) the point $(T, x(T))$ must be on the circle $(t - 9)^2 + x^2 = 9$.

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6. **[Problems with End-Point Inequality Constraints]** Consider the functional

$$\mathcal{J}(x) := \int_0^1 [\dot{x}(t)]^2 + 10\, t\, x(t)\ \mathrm{d}t,$$

for $\mathcal{D} := \{x \in \mathcal{C}^1[0, 1] : x(0) = 1\}$.

(a) Find candidate extremals when the right end-point condition is specified as $x(1) = 2$.

(b) Same question when the right end-point condition is specified as $\frac{3}{2} \leq x(1) \leq \frac{5}{2}$.

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7. **[A Simple Optimal Control Problem]** Consider the following optimal control problem:

$$\begin{aligned} \text{minimize:} \quad & \mathcal{J}(u) := [x(1)]^2 + \int_0^1 [u(t)]^2\ \mathrm{d}t, \\ \text{subject to:} \quad & \dot{x}(t) = x(t) + u(t); \quad x(0) = 1. \end{aligned}$$

(a) Identify candidate optimal controls for $u \in \mathcal{C}^[0, 1]$.
[Hint. Reformulate the problem as a classical problem of the calculus of variations.]
