IC-32 **Optimal Control** **Winter 2006**

Benoît Chachuat
ME C2 401, Ph: 33844, benoit.chachuat@epfl.ch

Problem Set #6

1. Consider the problem of finding the smooth curve $y(x), x_A \leq x \leq x_B$, in the vertical plane (x, y), joining given points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, $x_A < x_B$, and such that a material point sliding along $y(x)$ without friction from A to B, under gravity and with initial speed $v_A \geq 0$, reaches B in a minimal time (see Fig. 1).

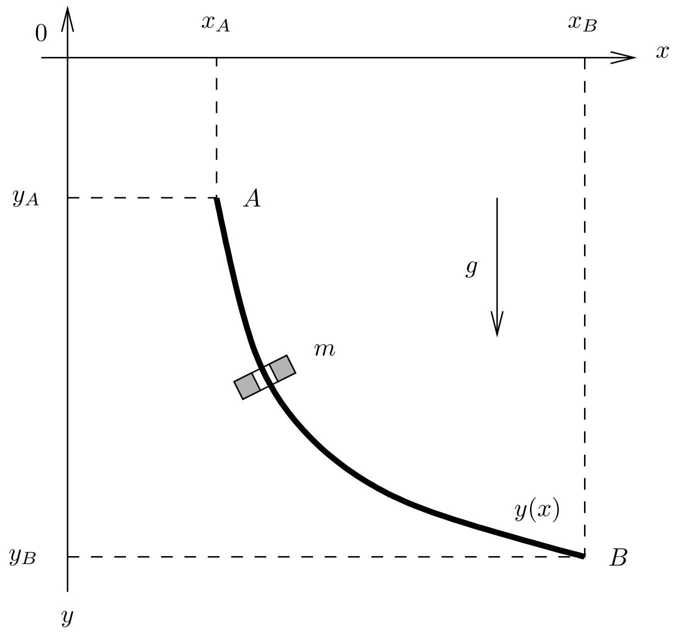

Figure 1: Brachistochrone problem.

We saw in a previous problem (Problem 4) that this formulation yields the following problem of the calculus of variations:

$$\begin{aligned} \text{minimize:} \quad & \mathcal{J}(y) = \int_{x_A}^{x_B} \sqrt{\frac{1 + \dot{y}(x)^2}{\frac{v_A^2}{2g} - y(x)}}\, \mathrm{d}x \qquad (1) \\ \text{subject to:} \quad & y \in \mathcal{D} := \{y \in \mathcal{C}^1[x_A, x_B] : y(x_A) = y_A, y(x_B) = y_B\}, \end{aligned}$$

where g denotes the gravity acceleration.

(a) Show that the stationary functions $\bar{y}$ of the Brachistochrone problem are those satisfying the system of differential equations

$$\dot{y}(x) = z(x) \tag{2}$$

$$\dot{z}(x) = \frac{1 + z(x)^2}{2\left(\frac{v_A^2}{2g} - y(x)\right)}, \tag{3}$$

subject to the split boundary conditions

$$y(x_A) = y_A, \quad \text{and} \quad y(x_B) = y_B. \tag{4}$$

(b) Stationary functions to the Brachistochrone problem can be obtained by computing the solutions to the differential equations (2,3) which satisfy the boundary conditions (4). However, the boundary conditions being split (some are specified at $x = x_A$ and others at $x = x_B$), a solution cannot be obtained based on classical solvers for initial value problems. Instead, the so-called *shooting* approach is considered here, which consists of guessing the missing initial condition $z(x_A) = z_A$ so that the corresponding curve $y(x)$ verifies the end-point condition $y(x_B) = y_B$.

i. In MATLAB®, write a program plotting $y(x_B)$ vs. z_A, for z_A in the range $[-10 : 0]$; in particular:
   - Use the function `ode15s` to integrate the differential equations (2,3), from the initial conditions (y_A, z_A);
   - Set both relative and absolute tolerances to 10^{-6} in `ode15s`;
   - Take the values $x_A = y_A = 0$, $x_B = 1$, $y_B = -0.5$ and $v_A = 1$ for the parameters (g can be set to $10\,\text{m/s}^2$, for simplicity).

   Estimate *visually* the value of the missing initial condition z_A giving the desired end-point condition $y(x_B) \approx y_B$.

ii. In order to refine the estimation of $\hat{z}_A$, we now want to implement an automatic optimization procedure. In MATLAB®, write a program solving the following optimization problem with simple bound constraints:

$$\min_{z_A^L \le z_A \le z_A^U} \quad (y(x_B) - y_B)^2,$$

   where $y(x_B)$ is obtained from the solution to the differential equations (2,3), with initial conditions $y(x_A) = y_A$ and $z(x_A) = z_A$; in particular:
   - Use the function `fmincon` to solve the optimization problem;
   - Consider the range $[z_A^L, z_A^U] := [-4, -3]$ for the optimization parameter z_A;
   - Set the solution point tolerance, function tolerance and constraint tolerance to 10^{-6} in `fmincon`;
   - For simplicity, let `fmincon` calculate a **finite-difference approximation** of the gradients of the objective function; in particular, set the minimum change in variables for finite differencing to 10^{-5};
   - Take the same values as before for the parameters x_A, y_A, x_B, y_B and v_A.

   Plot the resulting stationary curve $\bar{y}(x)$, and compare the results with those obtained for Problem 4.
