IC-32 **Optimal Control** **Winter 2006**

Benoît Chachuat
ME C2 401, Ph: 33844, `benoit.chachuat@epfl.ch`

Problem Set #4

1. Consider the problem of finding the smooth curve $y(x), x_A \leq x \leq x_B$, in the vertical plane (x, y), joining given points $A = (x_A, y_A)$ and $B = (x_B, y_B)$, $x_A < x_B$, and such that a material point sliding along $y(x)$ without friction from A to B, under gravity and with initial speed $v_A \geq 0$, reaches B in a minimal time (see Fig. 1).

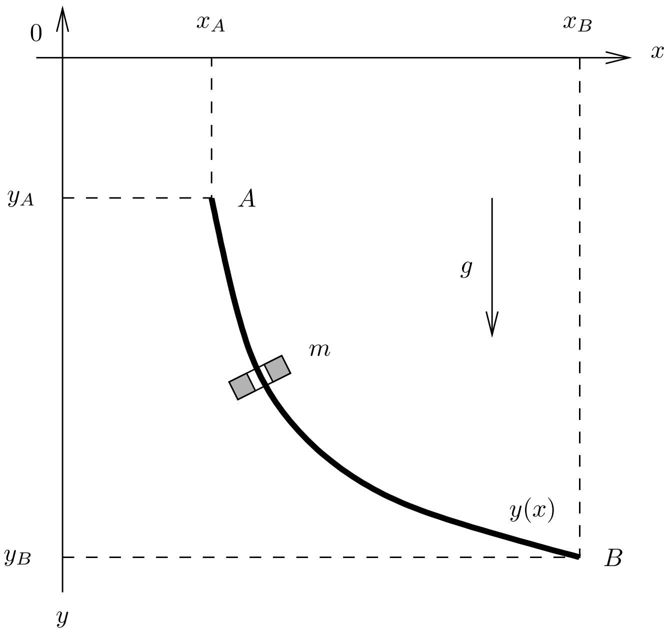

Figure 1: Brachistochrone problem.

(a) Consider the case where $x_A = y_A = 0$. Show that the problem can be expressed in mathematical terms as the following problem of the calculus of variations:

$$\begin{aligned} \text{minimize:} \quad & \mathcal{J}(y) = \int_0^{x_B} \sqrt{\frac{1 + \dot{y}(x)^2}{v_A^2 - 2gy(x)}} \mathrm{d}x \qquad (1) \\ \text{subject to:} \quad & y \in \mathcal{D} := \{y \in \mathcal{C}^1[0, x_B] : y(0) = 0, y(x_B) = y_B\}, \end{aligned}$$

where g denotes the gravity acceleration.

(b) We wish to calculate an approximate solution to this problem by parameterizing the curve $y(x)$

using Lagrange polynomials of order $N \geq 1$:

$$Y(\tilde{\mathbf{y}}; x) \;:=\; \sum_{k=0}^{N} \tilde{y}_k \prod_{\substack{j=0 \\ j \neq k}}^{N} \frac{x - X_j}{X_k - X_j},$$

where $\tilde{\mathbf{y}} = (\tilde{y}_0, \ldots, \tilde{y}_N)^{\mathsf{T}}$; and $X_0, \ldots, X_N$ are $N+1$ points in $[0, x_B]$. Here, we shall consider equally spaced points:

$$X_0 := 0, \quad \text{and} \quad X_k := X_{k-1} + \frac{x_B}{N}, \quad k = 1, \ldots, N.$$

Using this parameterization, (1) is transformed into the following (finite-dimensional) constrained NLP problem:

$$\begin{aligned}
\text{mininimize:} \quad & f(\tilde{\mathbf{y}}) := \int_0^{x_B} \sqrt{\frac{1 + Z(\tilde{\mathbf{y}}; x)^2}{v_A^2 - 2gY(\tilde{\mathbf{y}}; x)}} \mathrm{d}x \\
\text{subject to:} \quad & Y(\tilde{\mathbf{y}}; 0) \;=\; 0 \\
& Y(\tilde{\mathbf{y}}; x_B) \;=\; y_B \\
& \tilde{y}_k \leq \frac{v_A^2}{2g}, \quad k = 0, \ldots, N,
\end{aligned}$$

with

$$\begin{aligned}
X_k \;&:=\; \frac{k}{N} x_B, \quad k = 0, \ldots, N \\
Y(\tilde{\mathbf{y}}; x) \;&:=\; \sum_{k=0}^{N} \tilde{y}_k \prod_{\substack{j=0 \\ j \neq k}}^{N} \frac{x - X_j}{X_k - X_j} \\
Z(\tilde{\mathbf{y}}; x) \;&:=\; \sum_{k=0}^{N} \tilde{y}_k \sum_{\substack{i=0 \\ i \neq k}}^{N} \frac{1}{X_k - X_i} \prod_{\substack{j=0 \\ j \neq i,k}}^{N} \frac{x - X_j}{X_k - X_j}.
\end{aligned}$$

Compute an optimal solution $\tilde{\mathbf{y}}^\star$ to this problem using the function `fmincon` in MATLAB®'s Optimization Toolbox:

- Consider $N = 10$ stages, and take the values $x_B = 1$, $y_B = -0.5$ and $v_A = 1$ for the parameters;
- Specify the **initial guess** as $\tilde{y}_k^0 = \frac{k}{N} y_B$, for $k = 0, \ldots, N$;
- Make sure that the medium-scale **SQP algorithm**, with Quasi-Newton update and line-search, is the selected solver in `fmincon`;
- Set the solution point tolerance, function tolerance and constraint tolerance to 10^{-7} in `fmincon`;
- Use the function `quad` or `quadl` to compute the value of the integral cost, with tolerance set to 10^{-7};
- For simplicity, let `fmincon` calculate a **finite-difference approximation** for the gradients of the objective function and constraints. In particular, set the minimum change in variables for finite differencing to 10^{-6};
- M-files calculating Lagrange polynomials (`lagrange.m`) and their derivative (`dlagrange.m`) can be retreived from the class website.