IC-32	**Optimal Control**	**Winter 2006**

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1. In $\mathbb{R}^2$, consider the constraints

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_2 - (x_1 - 1)^2 &\geq 0. \end{aligned}$$

   (a) Sketch the feasible region.

   (b) Show that the point $x_1 = 1$, $x_2 = 0$ is feasible but not a regular point.

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2. Find a solution to the problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^2} \; & 2x_1^2 + 2x_1x_2 + x_2^2 - 10x_1 - 10x_2 \\ \text{s.t.} \; & x_1^2 + x_2^2 \leq 5 \\ & 3x_1 + x_2 \leq 6. \end{aligned}$$

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3. Find a solution to the problem

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^2} \; & x_1 \\ \text{s.t.} \; & h(\mathbf{x}) = x_2 = 0 \\ & \mathbf{x} \in X := \{\mathbf{x} : x_1^2 \leq x_2\}. \end{aligned}$$

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4. Consider the following NLP problem:

$$\begin{aligned} \min_{\mathbf{x}} \; & (x_1 - 3)^2 + (x_2 - 3)^2 \\ \text{s.t.} \; & 4x_1^2 + 9x_2^2 \leq 36 \\ & x_1^2 + 3x_2 = 3 \\ & \mathbf{x} \in X := \{\mathbf{x} : x_1 \geq -1\} \; . \end{aligned}$$

   (a) Sketch the feasible region and the contours of the objective function, then identify the optimum graphically.

   (b) Using this graphical information, determine the minimum point *precisely*, based on the first-order necessary conditions of optimality. Are the second-order sufficient conditions of optimality satisfied at this point?

(c) Repeat by replacing min by max in the problem statement.

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5. What is the rectangle of given perimeter p that has greatest area?

   (a) Formulate this problem has an NLP with equality constraint. Which are the regular and non-regular points for the constraint?

   (b) Identify candidate minimum point(s) by considering the stationary points of the Lagrangian function.

   (c) Are the second-order sufficient conditions satisfied at that (those) point(s)?

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6. Given a cardboard of area A to make a rectangular box, what is the maximum volume that can be attained?

   (a) Reformulate the problem as an unconstrained problem, then find candidate minimum points for the problem. [Hint: eliminate variable x_3 and then show that $x_1 = x_2$ must hold at an optimal solution]

   (b) Solve that problem directly, by using the method of Lagrange multipliers. [Hint: show that $x_1 = x_2 = x_3$ must hold at an optimal solution.]

   (c) Verify the second-order conditions.

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7. (*) *Optimal Control.* A one-dimensional dynamic process is governed by a difference equation

$$x(k+1) \; = f_k(x(k), u(k)), \tag{1}$$

with initial conditions $x(0) = x_0$. In this equation, the value of $x(k)$ is called the *state* at step k. Associated with this system, there is an *objective function* of the form

$$\mathcal{J} \; = \; \sum_{k=0}^{N} \ell_k(x(k), u(k)). \tag{2}$$

In addition, there is a *terminal constraint* of the form

$$g(x(N+1)) \; = \; 0. \tag{3}$$

The problem is to find the sequence of controls $u(0), u(1), \ldots, u(N)$ and corresponding state values that minimizes the objective function while satisfying the terminal constraint. It shall be assumed throughout this exercise that all functions have continuous first partial derivatives and that the regularity condition is satisfied.

   (a) Consider the case $N = 1$. Show that necessary conditions of optimality for the controls $u(0)$ and $u(1)$ are:

$$\begin{aligned} \frac{\partial \ell_0}{\partial u(0)} + \lambda(1)\frac{\partial f_0}{\partial u(0)} &= 0 \\ \frac{\partial \ell_1}{\partial u(1)} + \lambda(2)\frac{\partial f_1}{\partial u(1)} &= 0, \end{aligned}$$

where

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\begin{aligned}
\lambda(1) &= \lambda(2)\frac{\partial f_1}{\partial x(1)} + \frac{\partial \ell_1}{\partial x(1)} \\
\lambda(2) &= \mu\frac{\partial g}{\partial x(2)},
\end{aligned}
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for some scalar μ. How can μ be determined?

(b) Generalize the foregoing result to $N \geq 1$ by showing that necessary conditions of optimality for the controls $u(k)$, $k = 0, \ldots, N$ are:

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\frac{\partial \ell_k}{\partial u(k)} + \lambda(k+1)\frac{\partial f_k}{\partial u(k)} = 0, \quad k = 0, \ldots, N,
$$

where the $\lambda(k)$'s are determined from the backward difference equation

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\begin{aligned}
\lambda(k) &= \lambda(k+1)\frac{\partial f_k}{\partial x(k)} + \frac{\partial \ell_k}{\partial x(k)}, \quad k = 1, \ldots, N, \\
\lambda(N+1) &= \mu\frac{\partial g}{\partial x(N+1)},
\end{aligned}
$$

for some scalar μ.

(c) Repeat the exercise in the case where the state $\mathbf{x}(k)$ is an n_x-dimensional vector and the control $\mathbf{u}(k)$ is an n_u-dimensional vector at stage k.