Optimal Control

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Correction Pb Set 7 – 1st part

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1. **[Free Problem of the calculus of variations]**

Find candidate extremals for the functional $\mathbf{J}(\mathbf{x}) = \int_0^{\mathbf{T}} \left[\mathbf{c}_1(\dot{\mathbf{x}}(\mathbf{t}))^2 + \mathbf{c}_2\mathbf{x}(\mathbf{t})\right]\mathbf{dt}$,

with x(0) = 0 and x(T) = b.

(a) T = 1. Here both final time and final states are fixed.

Euler Equation :

$$\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{l}_{\dot{\mathbf{x}}} = \mathbf{l}_{\mathbf{x}}$$

We get :

$$\frac{\mathbf{d}}{\mathbf{dt}}\left(2\mathbf{c}_1\dot{\bar{\mathbf{x}}}(\mathbf{t})\right) = \mathbf{c}_2$$

$$\ddot{\bar{\mathbf{x}}}(\mathbf{t}) = \frac{\mathbf{c}_2}{2\mathbf{c}_1} \Rightarrow \dot{\bar{\mathbf{x}}}(\mathbf{t}) = \frac{\mathbf{c}_2}{2\mathbf{c}_1}\mathbf{t} + \alpha$$

$$\bar{\mathbf{x}}(\mathbf{t}) = \frac{\mathbf{c}_2}{4\mathbf{c}_1}\mathbf{t}^2 + \alpha\mathbf{t} + \beta$$

As x(0) = 0 we have $\beta = 0$.

Also x(1) = b implies $\mathbf{b} = \frac{\mathbf{c}_2}{4\mathbf{c}_1} + \alpha$.

Thus : $\bar{\mathbf{x}}(\mathbf{t}) = \frac{\mathbf{c}_2}{4\mathbf{c}_1}\mathbf{t}^2 + \left(\mathbf{b} - \frac{\mathbf{c}_2}{4\mathbf{c}_1}\right)\mathbf{t}$

(b) T is now let free, we can use the results obtained previously by applying the Euler equation, but we need also to use the transveral conditions to determine T and α, using that x(T) equals b.

Again, the Euler equation together with x(0) = 0 leads to : $\bar{\mathbf{x}}(\mathbf{t}) = \frac{\mathbf{c}_2}{4\mathbf{c}_1}\mathbf{t}^2 + \alpha\mathbf{t}$

The transversal condition at T reads : $\left[\mathbf{l} - \dot{\mathbf{x}}^{\mathbf{T}}\mathbf{l}_{\dot{\mathbf{x}}} + \phi_{\mathbf{t}}\right]_{\mathbf{T}} = 0 \Rightarrow \left[\mathbf{l} - \dot{\mathbf{x}}^{\mathbf{T}}\mathbf{l}_{\dot{\mathbf{x}}}\right]_{\mathbf{T}} = 0$,

as there is no terminal cost.

Thus, $\left[c_1\left(\dot{x}(T)\right)^2 + c_2\bar{x}(T)\right] - \dot{x}(T).2c_1.\dot{x}(T) = 0 \Rightarrow c_2\bar{x}(T) - c_1\left(\dot{x}(T)\right)^2 = 0$,

Reinjecting $\dot{x}(T) = \frac{c_2}{4c_1}T + \alpha$ and $\bar{x}(T) = \frac{c_2}{4c_1}T^2 + \alpha T$ leads to $\alpha = 0$. Thus

$\bar{x}(t) = \frac{c_2}{4c_1}t^2$, as x(T) = b, we get $T = \sqrt{\frac{4bc_1}{c_2}}$.

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2. **[Problems with free End-Points]**
Find candidate extremals for the

functional : $J(x) = \int_0^1 \left[\frac{1}{2}(\dot{x}(t))^2 + x(t)(\dot{x}(t)+1)\right]dt$

Where x(0) and x(1) can be chosen freely.

Applying the Euler equation $\frac{d}{dt}l_{\dot{x}} = l_x$ leads to : $\frac{d}{dt}\left(\dot{x}+\bar{x}\right) = \dot{x}+1 \Rightarrow \ddot{\bar{x}}(t) = 1$

Thus :

$\dot{x}(t) = t + \alpha$

$\bar{x}(t) = \frac{t^2}{2} + \alpha t + \beta$

We need to use the transversal conditons at t=0 and 1, for determining α and β.
As there is no terminal cost in J, the transversal conditions read :

$\left[l_{\dot{x}}\right]_{t=0} = 0 \Rightarrow \dot{x}(0) + \bar{x}(0) = 0 \Rightarrow \alpha + \beta = 0$

$\left[l_{\dot{x}}\right]_{t=1} = 0 \Rightarrow \dot{x}(1) + \bar{x}(1) = 0 \Rightarrow 1 + \alpha + \frac{1}{2} + \alpha + \beta = 0$

Solving this 2 by 2 system of linear equations gives :

$\alpha = -\frac{3}{2}$

$\beta = \frac{3}{2}$

$\bar{x}(t) = \frac{t^2}{2} - \frac{3}{2}t + \frac{3}{2}$

from where we can deduce the values of x(0) and x(1).

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3. **[Problems with Piecewise C1 extremals]**
Find piecewise C1 candidate extremals for the functional :

$J(x) = \int_0^4 \left[\left(\dot{x}(t)-1\right)^2\left(\dot{x}(t)+1\right)^4\right]dt$, such that there is only one corner point and x(0) = 0 and x(4) = 2.

Probably the best way to solve the problem is to construct the candidate solution.

First, the fact that J(x) is bounded from below by 0 is obvious as J(x) is the integral between 0 and 4 of positive terms.

Also $J(x) = 0$, if $\dot{x}(t) = 1$ or $\dot{x}(t) = -1$. As we are only looking for solutions for which there is only one corner point we only have two solutions that satisfy the endpoint conditions:

$$(1):\begin{cases}\bar{\mathbf{x}}(\mathbf{t}) = -\mathbf{t} & \forall \mathbf{t} \in [0;\ 1] \\ \bar{\mathbf{x}}(\mathbf{t}) = \mathbf{t} & \forall \mathbf{t} \in [1;\ 4]\end{cases}$$

and

$$(2):\begin{cases}\bar{\mathbf{x}}(\mathbf{t}) = \mathbf{t} & \forall \mathbf{t} \in [0;\ 3] \\ \bar{\mathbf{x}}(\mathbf{t}) = -\mathbf{t} & \forall \mathbf{t} \in [3;\ 4]\end{cases}$$

with such functions, it is easy to show that endpoint conditions are satisfied.
To check if these solutions are acceptable we need to verify the Weiestrass Erdman corner conditions, i.e. the fact that :

$$\mathbf{l}_{\dot{\mathbf{x}}}\left(\mathbf{c},\bar{\mathbf{x}}(\mathbf{c}),\dot{\bar{\mathbf{x}}}\left(\mathbf{c}^-\right)\right) = \mathbf{l}_{\dot{\mathbf{x}}}\left(\mathbf{c},\bar{\mathbf{x}}(\mathbf{c}),\dot{\bar{\mathbf{x}}}\left(\mathbf{c}^+\right)\right).$$

As $\mathbf{l} = (\dot{\mathbf{x}}(\mathbf{t})-1)^2(\dot{\mathbf{x}}(\mathbf{t})+1)^4 = (\dot{\mathbf{x}}(\mathbf{t})-1)^2(\dot{\mathbf{x}}(\mathbf{t})+1)^2(\dot{\mathbf{x}}(\mathbf{t})+1)^2$, we have :

$$\begin{aligned}\mathbf{l} &= (\dot{\mathbf{x}}(\mathbf{t})-1)^2(\dot{\mathbf{x}}(\mathbf{t})+1)^2(\dot{\mathbf{x}}(\mathbf{t})+1)^2 \\ &= \left(\dot{\mathbf{x}}(\mathbf{t})^2-1\right)^2(\dot{\mathbf{x}}(\mathbf{t})+1)^2 \\ &= \left(\dot{\mathbf{x}}(\mathbf{t})^4-2\dot{\mathbf{x}}(\mathbf{t})^2+1\right)\left(\dot{\mathbf{x}}(\mathbf{t})^2+2\dot{\mathbf{x}}(\mathbf{t})^2+1\right) \\ &= \dot{\mathbf{x}}(\mathbf{t})^6+2\dot{\mathbf{x}}(\mathbf{t})^5-\dot{\mathbf{x}}(\mathbf{t})^4-4\dot{\mathbf{x}}(\mathbf{t})^3-\dot{\mathbf{x}}(\mathbf{t})^2+2\dot{\mathbf{x}}(\mathbf{t})+1\end{aligned}$$

And :

$$\mathbf{l}_{\dot{\mathbf{x}}}\big|_{\bar{\mathbf{x}}} = 6\dot{\bar{\mathbf{x}}}(\mathbf{t})^5+10\dot{\bar{\mathbf{x}}}(\mathbf{t})^4-4\dot{\bar{\mathbf{x}}}(\mathbf{t})^3-12\dot{\bar{\mathbf{x}}}(\mathbf{t})^2-2\dot{\bar{\mathbf{x}}}(\mathbf{t})+2$$

For solution (1) we have : $\dot{\bar{\mathbf{x}}}\left(\mathbf{c}^-\right) = -1$, and $\dot{\bar{\mathbf{x}}}\left(\mathbf{c}^+\right) = 1$ while we have : $\dot{\bar{\mathbf{x}}}\left(\mathbf{c}^-\right) = +1$, and $\dot{\bar{\mathbf{x}}}\left(\mathbf{c}^+\right) = -1$ for solution (2), the position of the cormer point being at t = 1 and t = 3, respectively.
Replacing $\dot{\bar{\mathbf{x}}}(\mathbf{t})$ by alternatively -1 and 1 leads to :

$\mathbf{l}_{\dot{\mathbf{x}}}\big|_{\dot{\bar{\mathbf{x}}}=-1} = -6+10-4.(-1)-12-2.(-1)+2=0$, and to

$\mathbf{l}_{\dot{\mathbf{x}}}\big|_{\dot{\bar{\mathbf{x}}}=1} = 6+10-4.(1)-12-2.(1)+2=0.$

Thus, both solutions (1) and (2) are such that J = 0 (i.e. extremals), such that endpoint conditions are satisfied, such that they are piecewise C1 with one corner point and such that the continuity of $\mathbf{l}_{\dot{\mathbf{x}}}$ is preserved at the corner point ; i.e. (1) and (2) are solutions to Problem 3.