IC-32 | # Optimal Control | **Winter 2006**

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Problem Set #5 (With Corrections)

1. Consider the functional
$$\mathcal{J}(x) := \int_0^1 \left([\dot{x}(t)]^2 + 12\ t\ x(t)\right)\ \mathrm{d}t,$$
for $x \in \mathcal{D} := \{x \in \mathcal{C}^1[0,1] : x(0) = 1, x(1) = 2\}$.

   (a) Find stationary functions for this problem, which satisfy the specified end-point conditions.

   Solution. Since the Lagrangian function is dependent of t, x and $\dot{x}$, no first-integral can be considered for the problem. The Euler's equation reads
$$\ddot{x}(t) = 6t.$$
Integrating this equation twice yields,
$$x(t) = t^3 + c_1 t + c_2,$$
where c_1 and c_2 are constants of integration. These two constants can be determined from the end-point conditions, as $c_1 = 0$ and $c_2 = 1$. The resulting stationary function is
$$\bar{x}(t) := t^3 + 1.$$

   (b) What can be said about the resulting stationary functions: Do they give a local minimum? A local maximum? A global minimum? A global maximum? Or, neither of these?

   Solution. Since the Lagrangian function $\ell(t, y, z) := z^2 + 12\ t\ y$ is differentiable and jointly convex in (y, z), the first-order sufficiency theorem ensures that every stationary function for ℓ gives a global minimum for $\mathcal{J}$ on $\mathcal{D}$. But since $\bar{x}(t) := t^3 + 1$ is the unique stationary function for ℓ, $\bar{x}(t)$ is a global minimizer for $\mathcal{J}$ on $\mathcal{D}$.
Observe that the possibility of $\bar{x}$ giving a local or global maximum for $\mathcal{J}$ on $\mathcal{D}$ is also precluded by Legendre second-order necessary condition, since $\ell_{zz} = 2 \geq 0$ along $\bar{x}(t)$.

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2. Consider the functional
$$\mathcal{J}(x) := \int_{t_1}^{t_2} \left([x(t)]^2 + a x(t)\dot{x}(t) + b[\dot{x}(t)]^2\right)\ \mathrm{d}t,$$
for $x \in \mathcal{D} := \{x \in \mathcal{C}^1[t_1, t_2] : x(t_1) = x_1, x(t_2) = x_2\}$.

(a) Find stationary functions for this problem, by distinguishing between the cases where $b = 0$, $b > 0$ and $b < 0$. Which stationary functions are candidate (local or global) minimizers for $\mathcal{J}$ on $\mathcal{D}$? Candidate (local or global) maximizers for $\mathcal{J}$ on $\mathcal{D}$?

Solution. The Lagrangian function for this problem is

$$\ell(t, y, z) = y^2 + ayz + bz^2,$$

and we have

$$\begin{aligned} \ell_y(t, x(t), \dot{x}(t)) &= 2x(t) + a\dot{x}(t) \\ \ell_z(t, x(t), \dot{x}(t)) &= ax(t) + 2b\dot{x}(t). \end{aligned}$$

That is, the Euler's equation reads

$$b\ddot{x}(t) = x(t), \tag{1}$$

and we have the following cases:

case $b = 0$ Trivially, the unique stationary function is

$$\bar{x}(t) = 0, \quad t_1 \leq t \leq t_2.$$

Observe, in particular, that the only stationary function satisfying the end-point requirement on $\mathcal{C}^1[t_1, t_2]$ is with $x_1 = x_2 = 0$.

case $b > 0$ The stationary functions are given by

$$\bar{x}(t) = c_1 \exp\left(\frac{t}{\sqrt{b}}\right) + c_2 \exp\left(-\frac{t}{\sqrt{b}}\right), \quad t_1 \leq t \leq t_2,$$

where c_1 and c_2 are constants of integration to be determined from the end-point conditions $x(t_1) = x_1$ and $x(t_2) = x_2$. Note that b being positive, the Lagrangian $\ell(t, y, z)$ is jointly convex in (y, z). Hence, the first-order sufficiency theorem ensures that $\bar{x}(t)$ gives a global minimum for $\mathcal{J}$ on $\mathcal{D}$.

case $b < 0$ The stationary functions are given by

$$\bar{x}(t) = c_1 \sin\left(\frac{t}{\sqrt{-b}}\right) + c_2 \cos\left(\frac{t}{\sqrt{-b}}\right), \quad t_1 \leq t \leq t_2,$$

where c_1 and c_2 are constants of integration to be determined from the end-point conditions $x(t_1) = x_1$ and $x(t_2) = x_2$. Note that b being negative, the Lagrangian $\ell(t, y, z)$ is no longer jointly convex in (y, z), and the first-order sufficiency theorem does not apply. However, $\ell(t, y, z)$ being concave in z, Legendre second-order necessary condition precludes the possibility of $\bar{x}$ being a local minimizer for $\mathcal{J}$ on $\mathcal{D}$. On the other hand, $\bar{x}$ could be a local maximizer for $\mathcal{J}$ on $\mathcal{D}$.

(b) How does the parameter a affect the solution? Why?

Solution. Observe first that Euler's equation (1) does *not* depend on a, nor do the end-point conditions. That is, the stationary solutions are unaffected by the value of a.

The reason for this is that

$$\int_{t_1}^{t_2} ax(t)\dot{x}(t)\, \mathrm{d}t = a \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}[x(t)^2]\, \mathrm{d}t = \frac{a}{2}\left(x_2^2 - x_1^2\right),$$

which is fixed since both end-points are specified. Therefore, this part of the cost remains identical for any admissible trajectory.

3. Consider the problem to

$$\begin{aligned}\text{minimize:}\quad & \mathcal{J}(x) = \int_0^T \exp(-rt)\ x(t)\ \mathrm{d}t\\ \text{subject to:}\quad & x \in \mathcal{D} := \left\{ x \in \mathcal{C}[0,T] : \int_0^T \sqrt{x(t)}\ \mathrm{d}t = A \right\},\end{aligned}$$

given $A \geq 0$ and $r > 0$.

(a) Reformulate this constrained problem into a free problem.
[Hint: consider the new phase variable $y \in \mathcal{C}^1[0,T]$ defined by $y(t) := \int_0^t \sqrt{x(s)}\ \mathrm{d}s$, $0 \leq t \leq T$.]

Solution. Following the suggested reformulation, we have

$$\dot{y}(t) \;=\; \sqrt{x(t)}, \tag{2}$$

for each $0 \leq t \leq T$. That is, the problem of minimizing $\mathcal{J}$ on $\mathcal{D}$ is equivalent to:

$$\begin{aligned}\text{minimize:}\quad & \tilde{\mathcal{J}}(x) = \int_0^T \exp(-rt)\ [\dot{y}(t)]^2\ \mathrm{d}t\\ \text{subject to:}\quad & x \in \tilde{\mathcal{D}} := \left\{ y \in \mathcal{C}^1[0,T] : y(0) = 0, y(T) = A \right\}.\end{aligned}$$

(b) Identify candidate solutions to this problem based on Euler's equation and Legendre condition.

Solution. The Lagrangian function for the reformulated problem is

$$\tilde{\ell}(t,z) = \exp(-rt)\ z^2,$$

and a first integral to the Euler equation is obtained as

$$\exp(-rt)\dot{y}(t)\ = c_1,$$

with c_1 a real constant. Integrating this equation yields the following stationary function

$$\bar{y}(t) \;:=\; \frac{c_1}{r}\exp(rt) + c_2,$$

with c_2 another constant of integration, and $r > 0$. Then, from the specified end-point conditions, we get $c_1 = Ar\exp(-rT)$ and $c_2 = 0$, so that

$$\bar{y}(t) \;:=\; A\exp(-r(T-t)).$$

Note that, $\tilde{\ell}$ being differentiable and convex in z, it satisfies the first-order sufficiency conditions, hence $\bar{y}$ is a global minimum for the problem to minimize $\tilde{\mathcal{J}}$ on $\tilde{\mathcal{D}}$. Finally, from (2) and since $A \geq 0$, the (continuous) function $\bar{x}$ defined by

$$\bar{x}(t) \;:=\; A^2r^2\exp(-2r(T-t)),$$

is a global minimizer for $\mathcal{J}$ on $\mathcal{D}$.