IC-32 Optimal Control Spring 2011

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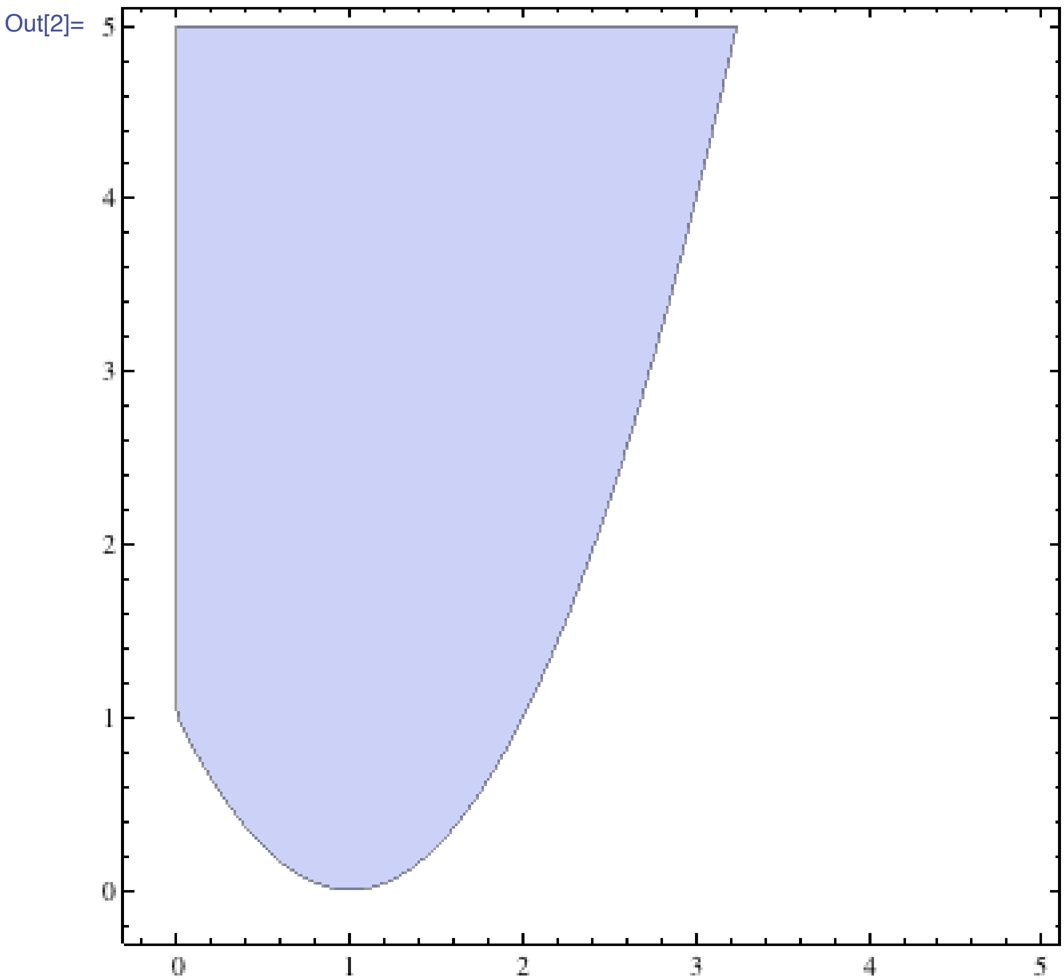
1. Consider the constraints

$$\begin{cases} \mathbf{g}_1(\mathbf{x}) := -\mathbf{x}_1 \le 0 \\ \mathbf{g}_2(\mathbf{x}) := -\mathbf{x}_2 \le 0 \\ \mathbf{g}_3(\mathbf{x}) := (\mathbf{x}_1 - 1)^2 - \mathbf{x}_2 \le 0 \end{cases}$$

a) Sketch the feasible region:



b) Show that the point $\bar{\mathbf{x}} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is feasible but not regular

Feasibility is obvious.

Regularity can be checked by investigating the rank of matrices whose rows are composed of the gradients of the active constraints.

At $\bar{\mathbf{x}}$ the two last constraints are active. Thus we should consider the following matrix:

$$\mathbf{M}(\bar{\mathbf{x}}) = \begin{bmatrix} \nabla \mathbf{g}_2(\bar{\mathbf{x}}) \\ \nabla \mathbf{g}_3(\bar{\mathbf{x}}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2(\bar{\mathbf{x}}_1 - 1) & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}, \text{ which is clearly of rank 1.}$$

Thus, $\bar{\mathbf{x}}$ is not a regular point.

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2. We consider the problem [P], such that:

$$\min_{\mathbf{x}_1,\mathbf{x}_2} \left[\mathbf{f}(\mathbf{x})\right] := 2\mathbf{x}_1^2 + 2\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_2^2 - 10\mathbf{x}_1 - 10\mathbf{x}_2$$

$$\mathbf{s.t.}: \begin{array}{l} \mathbf{g}_1 := \mathbf{x}_1^2 + \mathbf{x}_2^2 \le 5 \\ \mathbf{g}_2 := 3\mathbf{x}_1 + \mathbf{x}_2 \le 6 \end{array}$$

a) Assume both constraints are inactive. The problem is to find a stationary point of f that satisfies the constraints.

$$\nabla\left(\mathbf{f}\left(\mathbf{x}^*\right)\right) = \begin{bmatrix} 4\mathbf{x}_1^* + 2\mathbf{x}_2^* - 10 & 2\mathbf{x}_1^* + 2\mathbf{x}_2^* - 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

A stationary point for a) is $\mathbf{x}^* = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$, for which the first constraint is violated.

b) Assume both constraints are active. Thus we have $\mathbf{v}_1^* \ge 0$ and $\mathbf{v}_2^* \ge 0$. We look for a point such that: $\nabla\left(\mathbf{f}\left(\mathbf{x}^*\right)\right) + \begin{bmatrix} \mathbf{v}_1^* & \mathbf{v}_2^* \end{bmatrix} \cdot \begin{bmatrix} \nabla \mathbf{g}_1\left(\mathbf{x}^*\right) \\ \nabla \mathbf{g}_2\left(\mathbf{x}^*\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

That is, we have to solve the following set of equations:

$$\begin{cases} \left[4 + 2\mathbf{v}_1^*\right]\mathbf{x}_1^* + 2\mathbf{x}_2^* - 10 + 3\mathbf{v}_2^* = 0 \\ \left[2 + 2\mathbf{v}_1^*\right]\mathbf{x}_2^* + 2\mathbf{x}_1^* - 10 + \mathbf{v}_2^* = 0 \\ \left(\mathbf{x}_1^*\right)^2 + \left(\mathbf{x}_2^*\right)^2 = 5 \\ 3\mathbf{x}_1^* + \mathbf{x}_2^* = 6 \end{cases}$$

Which has two solutions:

$$\begin{bmatrix} \mathbf{x}_1^* & \mathbf{x}_2^* & \mathbf{v}_1^* & \mathbf{v}_2^* \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} \frac{1}{10}\left(18 - \sqrt{14}\right) \\ \frac{3}{10}\left(2 + \sqrt{14}\right) \\ \frac{1}{10}\left(16 - 7\sqrt{14}\right) \\ \frac{1}{2}\left(-1 + \sqrt{14}\right) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x}_1^* & \mathbf{x}_2^* & \mathbf{v}_1^* & \mathbf{v}_2^* \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} \frac{1}{10}\left(18 + \sqrt{14}\right) \\ \frac{3}{10}\left(2 - \sqrt{14}\right) \\ \frac{1}{10}\left(16 + 7\sqrt{14}\right) \\ \frac{1}{2}\left(-1 - \sqrt{14}\right) \end{bmatrix}$$

Checking the signs of the Lagrange multipliers leads to the rejection of both points, as $\mathbf{v}_1^* < 0$ for the 1st point, while $\mathbf{v}_2^* < 0$ for the second.

c) Assume now that the 1st constraint is inactive, while the 2nd is active. Hence, $\mathbf{v}_1^* = 0$ and $\mathbf{v}_2^* \geq 0$. We look for a point such that:

$$\nabla\left(\mathbf{f}\left(\mathbf{x}^*\right)\right) + \mathbf{v}_2^*.\left[\nabla \mathbf{g}_2\left(\mathbf{x}^*\right)\right] = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

We have the following set of equations to solve:

$$\begin{cases} 4\mathbf{x}_1^* + 2\mathbf{x}_2^* - 10 + 3\mathbf{v}_2^* = 0 \\ 2\mathbf{x}_1^* + 2\mathbf{x}_2^* - 10 + \mathbf{v}_2^* = 0 \\ 3\mathbf{x}_1^* + \mathbf{x}_2^* = 6, \quad \text{as } g_2 \text{ is act.} \end{cases}$$

The candidate solution point is: $\begin{bmatrix} \mathbf{x}_1^* & \mathbf{x}_2^* & \mathbf{v}_2^* \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{24}{5} & -\frac{2}{5} \end{bmatrix}$. This point is clearly not acceptable as $\mathbf{v}_2^* < 0$.

d) Finally, assume that the 1st constraint is active, while the 2nd is inactive. Hence, $\mathbf{v}_1^* \geq 0$ and $\mathbf{v}_2^* = 0$. We look for a point such that:

$$\nabla\left(\mathbf{f}\left(\mathbf{x}^*\right)\right) + \mathbf{v}_1^*.\left[\nabla \mathbf{g}_1\left(\mathbf{x}^*\right)\right] = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

We have the following set of equations to solve:

$$\begin{cases} 4\mathbf{x}_1^* + 2\mathbf{x}_2^* - 10 + 2\mathbf{v}_1^*\mathbf{x}_1^* = 0 \\ 2\mathbf{x}_1^* + 2\mathbf{x}_2^* - 10 + 2\mathbf{v}_1^*\mathbf{x}_2^* = 0 \\ (\mathbf{x}_1^*)^2 + (\mathbf{x}_2^*)^2 = 5, \quad \text{as } g_1 \text{ is act.} \end{cases}$$

The candidate solution point is: $\begin{bmatrix} \mathbf{x}_1^* & \mathbf{x}_2^* & \mathbf{v}_1^* \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. This point is clearly acceptable as $\mathbf{v}_1^* > 0$. In addition, the 2nd constraint is satisfied (and inactive) as 3.1+2 = 5 < 6.

We thus have found the solution.

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3. This problem shows a good example of a badly formulated problem.

Sketching the feasible region shows that the only feasible point for this problem is [0 0]. At this point both constraints are active. However, this point is not a KKT point, as this point is not regular for both equality and inequality constraint.

Thus the best is clearly to reformulate the problem while getting rid of x_2, whose value at optimum is known.

$$\min_{\mathbf{x}_1} \mathbf{x}_1$$

$$\textbf{s.t.} \quad \mathbf{x}_1^2 \leq 0$$

Thus we can clearly show that [0] is the only feasible point and thus the solution of the reformulated problem, from which we can state that the solution of the original problem is [0 0]

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4. This problem is not difficult by itself. However, it shows that a good graphical illustration helps simplifying the problem, as it allows to reduce the number of scenarios to investigate.

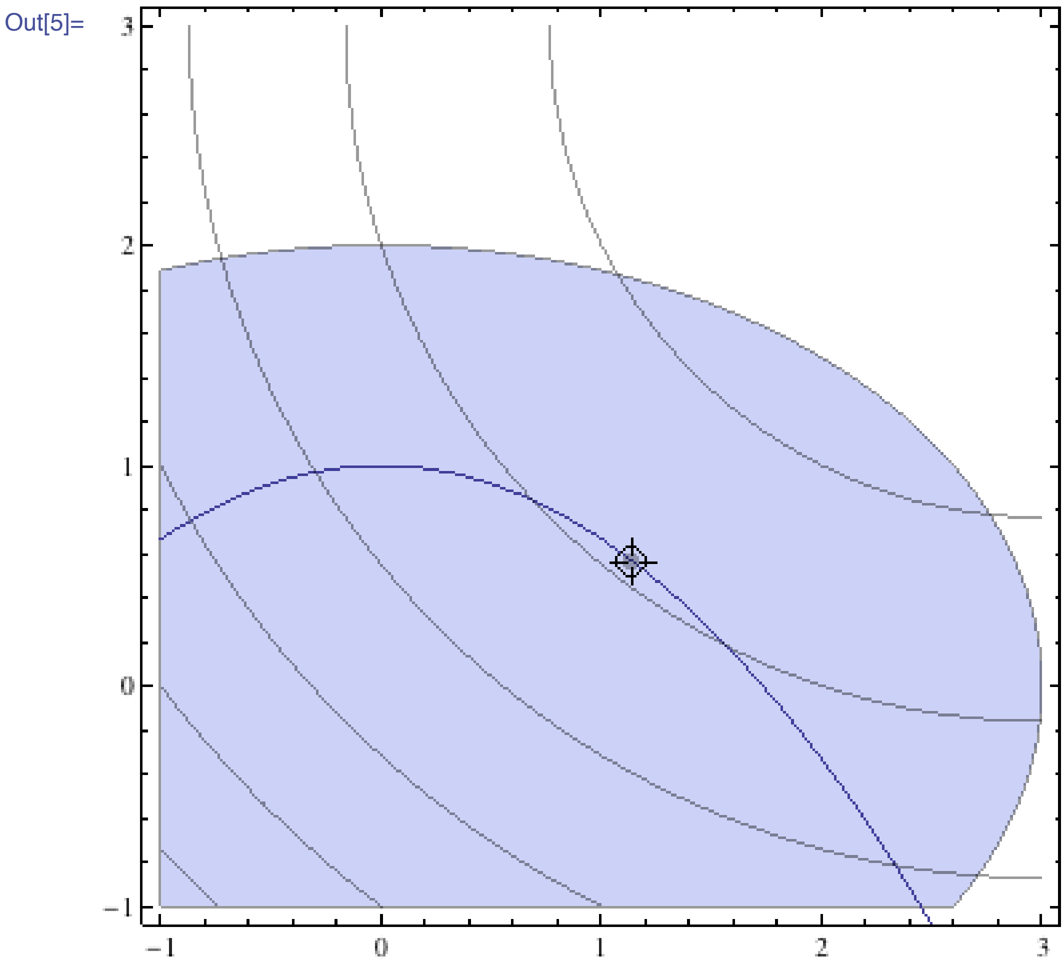
The problem [P] can be formulated as follows:

$$\min_{\mathbf{x}}\left[\mathbf{f}(\mathbf{x}) := (\mathbf{x}_1 - 3)^2 + (\mathbf{x}_2 - 3)^2\right]$$

$$\text{s.t.}\quad \begin{aligned} &\mathbf{g}_1 := 4\mathbf{x}_1^2 + 9\mathbf{x}_2^2 \le 36 \\ &\mathbf{g}_2 := \mathbf{x}_1^2 + 3\mathbf{x}_2 - 3 = 0 \\ &\mathbf{g}_3 := -\mathbf{x}_1 - 1 \le 0 \end{aligned}$$

The contours of the objective function will be circles centered at [3, 3]. In the problem of minimizing f, the objective will be to be as close as possible from [3, 3], while in the problem of maximizing f, the goal will be to be as far as possible. Also note that the 1st constraint defines an area in which the solution should lie. This area is delimited by an ellipse.

a)



From the Figure above, we clearly see that the only active constraint is the equality constraint, and the candidate solution point will be inside the area determined by the ellipse.

b)

The problem can be easily solved by the method of the Lagrange Multipliers for which only the equality constraint is considered.
We have to solve the following set of nonlinear equations:

$$\nabla \mathbf{f}(\mathbf{x}^*) + \boldsymbol{\lambda}\nabla \mathbf{g}_2(\mathbf{x}^*) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2(\mathbf{x}_1^* - 3) & 2(\mathbf{x}_2^* - 3) \end{bmatrix} + \boldsymbol{\lambda}\begin{bmatrix} 2\mathbf{x}_1^* & 3 \end{bmatrix} = [\ 0 \quad 0\ ]$$

$$(1+\boldsymbol{\lambda})\mathbf{x}_1^* - 3 = 0$$

i.e. $\quad 2\mathbf{x}_2^* - 6 + 3\boldsymbol{\lambda} = 0$

$$(\mathbf{x}_1^*)^2 + 3\mathbf{x}_2^* = 3, \quad \text{as } g_2 \text{ is active}$$

Using Mathematica, e.g., we can find the only (real) solution:
$\begin{bmatrix} \mathbf{x}_1^* & \mathbf{x}_2^* & \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} 1.14336 & 0.5642 & 1.6238 \end{bmatrix}$. Graphically, it is easily seen that the two other constraints are satisfied at this point.
b)
We have $\nabla \mathbf{g}_2(\mathbf{x}) = [\ 2\mathbf{x}_1 \quad 3\ ]$. Hence $\mathbf{E} = \begin{bmatrix} -3 \\ 2\mathbf{x}_1 \end{bmatrix}$ is a basis of the tangent subspace to the 2nd constraint at x.
We should check the sign of $\mathbf{E}^T\nabla_{xx}\mathbf{L}(\mathbf{x},\boldsymbol{\lambda})\mathbf{E}$, at the candidate minimum point .

$$\mathbf{L}(\mathbf{x},\boldsymbol{\lambda}) = \mathbf{f}(\mathbf{x}) + \boldsymbol{\lambda}\mathbf{g}_2(\mathbf{x}) \Rightarrow \nabla_{xx}\mathbf{L}(\mathbf{x},\boldsymbol{\lambda}) = \nabla_{xx}\mathbf{f} + \boldsymbol{\lambda}\nabla_{xx}\mathbf{g}_2$$

$$\Rightarrow \nabla_{xx}\mathbf{L}(\mathbf{x},\boldsymbol{\lambda}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \boldsymbol{\lambda}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2+2\boldsymbol{\lambda} & 0 \\ 0 & 2 \end{bmatrix}$$

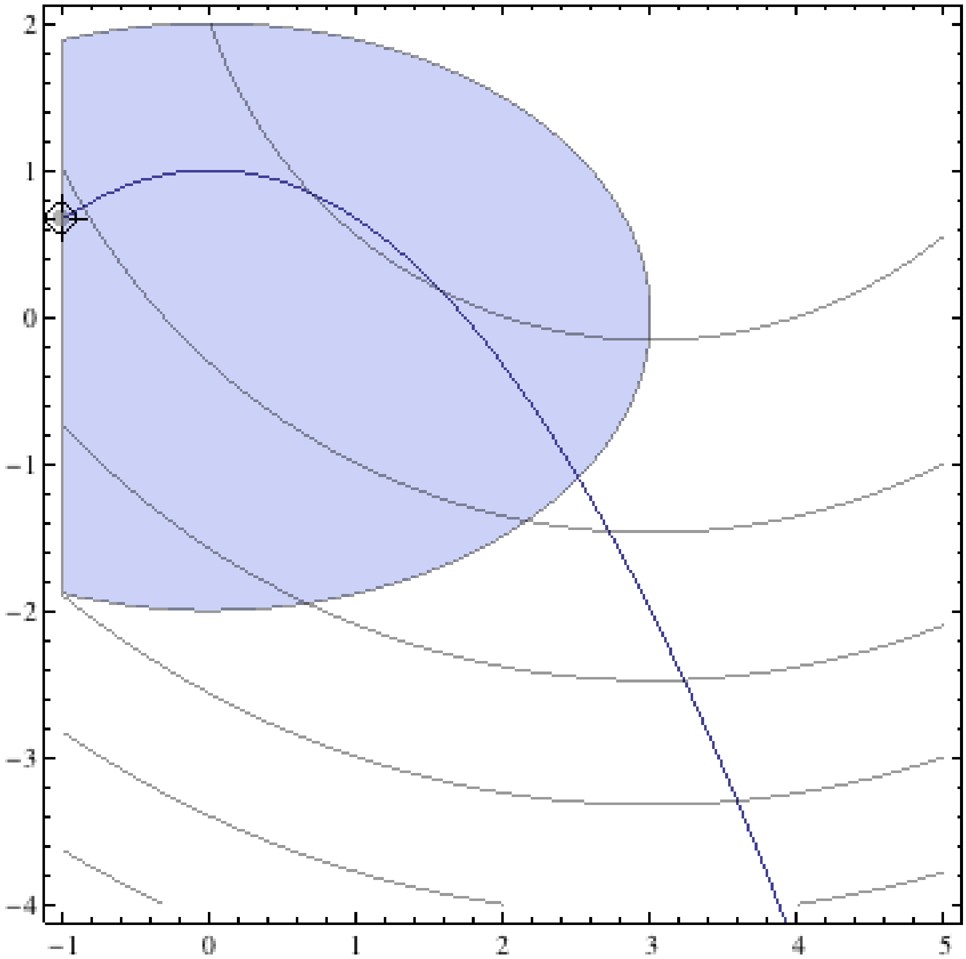
It follows that:

$$\mathbf{E}^T\nabla_{xx}\mathbf{L}(\mathbf{x}^*,\boldsymbol{\lambda})\mathbf{E} = \begin{bmatrix} -3 & 2\mathbf{x}_1^* \end{bmatrix} . \begin{bmatrix} 2+2\boldsymbol{\lambda}^* & 0 \\ 0 & 2 \end{bmatrix} . \begin{bmatrix} -3 \\ 2\mathbf{x}_1^* \end{bmatrix} = 57.68 > 0$$

Thus the 2nd order conditions hold and the candidate solution point is a minimum.

c) If we replace “min” by “max” in the problem formulation we see graphically that the solution should lie at the intersection of the equality constraint and of the 3rd constraint. Also note that it is easier to keep the optimization problem as a minimization problem and to change the sign of f(x).

Out[7]=



We now have the following set of nonlinear equations:

$$\nabla \mathbf{f}\left(\mathbf{x}^*\right)+\boldsymbol{\lambda}\nabla \mathbf{g}_2\left(\mathbf{x}^*\right)+\boldsymbol{\nu}\nabla \mathbf{g}_3\left(\mathbf{x}^*\right)=\begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2\left(\mathbf{x}_1^*-3\right) & -2\left(\mathbf{x}_2^*-3\right) \end{bmatrix}+\boldsymbol{\lambda}\begin{bmatrix} 2\mathbf{x}_1^* & 3 \end{bmatrix}+\boldsymbol{\nu}\begin{bmatrix} -1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\textbf{i.e.} \quad \begin{gathered} -2\left(\mathbf{x}_1^*-3\right)+2\boldsymbol{\lambda}\mathbf{x}_1^*-\boldsymbol{\nu}=0 \\ -2\left(\mathbf{x}_2^*-3\right)+3\boldsymbol{\lambda}=0 \\ \left(\mathbf{x}_1^*\right)^2+3\mathbf{x}_2^*=3 \\ \mathbf{x}_1^*=-1 \end{gathered}$$

From which we obtain the solution:

$$\begin{bmatrix} \mathbf{x}_1^* & \mathbf{x}_2^* & \boldsymbol{\lambda} & \boldsymbol{\nu} \end{bmatrix}=\begin{bmatrix} -1 & \frac{2}{3} & -1.5556 & 11.1111 \end{bmatrix}$$

5.
6.

These two exercises are straightforward and left to the students.

7. In this Simple Optimal Control Problem, the main objective is to show that NLP can be used for discrete systems as a way to solve OCP.
The 1st thing to realize is the number of degrees of freedom. As x(0) is fixed, it will not be a degree of freedom, while for all the other state values as well as for the control values, it will be degrees of freedom.
a) N=2

$$\min \quad \mathbf{J} = \mathbf{l}_0\left(\mathbf{x}(0),\mathbf{u}(0)\right) + \mathbf{l}_1\left(\mathbf{x}(1),\mathbf{u}(1)\right)$$

$$\textbf{s.t.} \quad \begin{aligned} &\mathbf{x}(1) = \mathbf{f}_0\left(\mathbf{x}(0),\mathbf{u}(0)\right) \\ &\mathbf{x}(2) = \mathbf{f}_1\left(\mathbf{x}(1),\mathbf{u}(1)\right) \\ &\mathbf{g}(\mathbf{x}(2)) = 0 \end{aligned}$$

Note that we treat here the state equations as equality constraints (which makes sense).
Thus we can form the Lagrangian:

$$\mathbf{L}\left(\mathbf{u}(0),\mathbf{u}(1),\mathbf{x}(1),\mathbf{x}(2),\boldsymbol{\lambda}(1),\boldsymbol{\lambda}(2),\boldsymbol{\mu}\right) = \mathbf{J} + \boldsymbol{\lambda}(1).(\mathbf{f}_0 - \mathbf{x}(1)) + \boldsymbol{\lambda}(2).(\mathbf{f}_1 - \mathbf{x}(2)) + \boldsymbol{\mu}\mathbf{g}\left(\mathbf{x}(2)\right)$$

The stationarity of L implies:

$$(1): \quad \nabla_{\mathbf{u}(0)}\mathbf{L} = \frac{\partial \mathbf{l}_0}{\partial \mathbf{u}(0)} + \boldsymbol{\lambda}(1)\frac{\partial \mathbf{f}_0}{\partial \mathbf{u}(0)} = 0$$

$$(2): \quad \nabla_{\mathbf{u}(1)}\mathbf{L} = \frac{\partial \mathbf{l}_1}{\partial \mathbf{u}(1)} + \boldsymbol{\lambda}(2)\frac{\partial \mathbf{f}_1}{\partial \mathbf{u}(1)} = 0$$

$$(3): \quad \nabla_{\mathbf{x}(1)}\mathbf{L} = \frac{\partial \mathbf{l}_1}{\partial \mathbf{x}(1)} - \boldsymbol{\lambda}(1) + \boldsymbol{\lambda}(2)\frac{\partial \mathbf{f}_1}{\partial \mathbf{x}(1)} = 0$$

$$(4): \quad \nabla_{\mathbf{x}(2)}\mathbf{L} = -\boldsymbol{\lambda}(2) + \boldsymbol{\mu}\frac{\partial \mathbf{g}}{\partial \mathbf{x}(2)} = 0$$

$$(5): \quad \nabla_{\boldsymbol{\lambda}(1)}\mathbf{L} = \mathbf{f}_0\left(\mathbf{x}(0),\mathbf{u}(0)\right) - \mathbf{x}(1) = 0$$

$$(6): \quad \nabla_{\boldsymbol{\lambda}(2)}\mathbf{L} = \mathbf{f}_1\left(\mathbf{x}(1),\mathbf{u}(1)\right) - \mathbf{x}(2) = 0$$

$$(7): \quad \nabla_{\boldsymbol{\mu}}\mathbf{L} = \mathbf{g}(\mathbf{x}(2)) = 0$$

The four first equations correspond to the answers to the question a) of Problem 7. We thus have 7 equations for 7 unknowns ! Typically, the equations related to the states will be integrated forward in time (eq. 5 & 6), while Equations (3) and (4) are solved backward in time. The real difficulty lies in the fact that x(2), and thus the terminal constraint depends on the choice of the controls u(0) and u(1). We can for instance solve iteratively the complete set of equations for each pair of u's, and repeat until convergence, i.e. when g(x(2)) = 0.
b) and c) Given a), both b) and c) follow in a straightforward manner, provided attention is being paid (i) to the fact that x(0) will never be a degree of freedom and (ii) to the rigor in the numbering of the equations. It is thus left to the students.