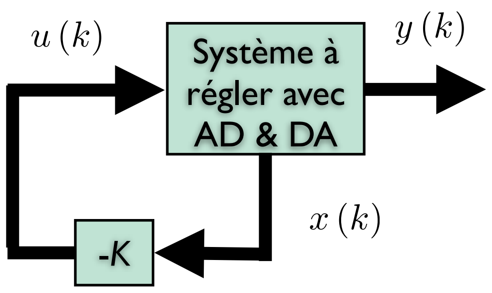
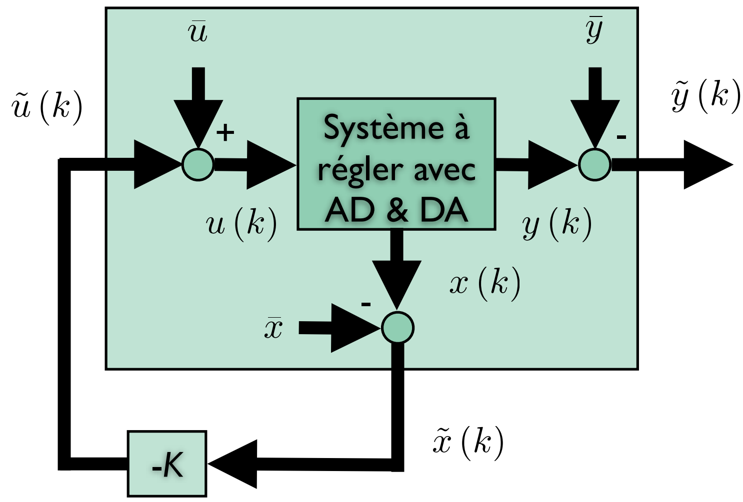
Commande d'état

Système à régler

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \qquad \begin{aligned} \tilde{x}(k+1) &= \Phi \tilde{x}(k) + \Gamma \tilde{u}(k) \\ \tilde{y}(k) &= C\tilde{x}(k) + D\tilde{u}(k) \end{aligned}$$

Régulateur $u(k) = -Kx(k)$ $\qquad \tilde{u}(k) = -K\tilde{x}(k)$





Résumé commande d'état SISO

Système à régler

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Commande $u(k) = -Kx(k)$

Système en boucle fermée (BF)

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma \mathrm{K})\, x(k) &= \Phi_{bf} x(k) \\ y(k) &= (C - DK)\, x(k) &= C_{bf} x(k) \end{aligned}$$

$$\begin{aligned} \alpha_c(\lambda) = \det(\lambda I - \Phi_{bf}) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) \\ &= \lambda^n + \alpha_1 \lambda^{n-1} + \ldots \lambda a_{n-1} + \alpha_n = 0 \end{aligned}$$

Formule d'Ackermann

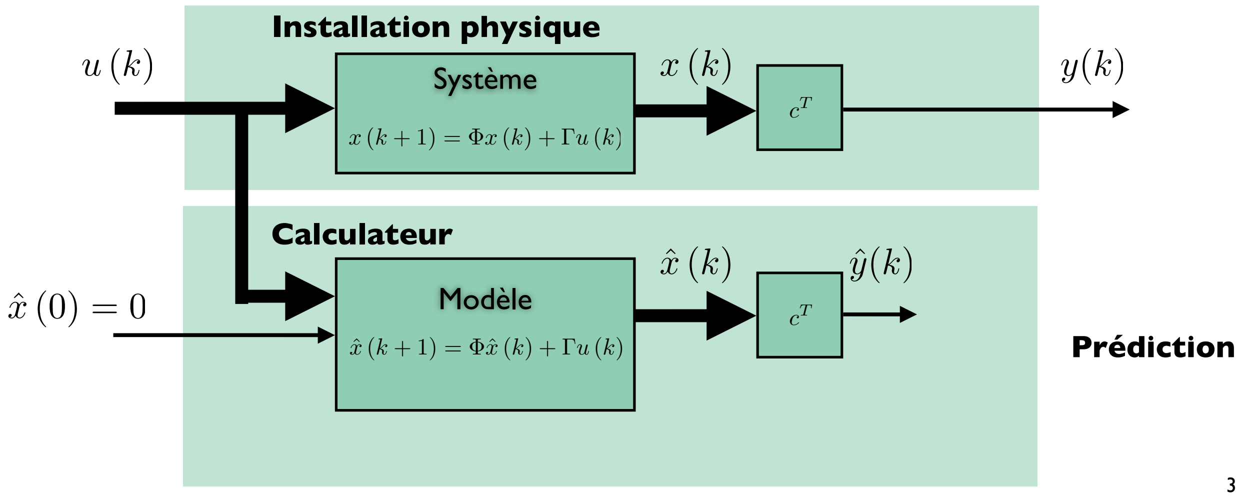
$$\begin{aligned} K &= \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} G^{-1} \alpha_c(\Phi) \\ &= \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} \Gamma & \Phi\Gamma & \ldots & \Phi^{n-1}\Gamma \end{bmatrix}^{-1} \alpha_c(\Phi) \end{aligned}$$

6.1.1 Estimation d'état (MISO)

Système à observer
$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= c^T x(k) \end{aligned}$$

Observateur
$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k)$$

Erreur d'estimation
$$\delta(k) = \hat{x}(k) - x(k)$$
$$\delta(k+1) = \hat{x}(k+1) - x(k+1) = \Phi \delta(k)$$



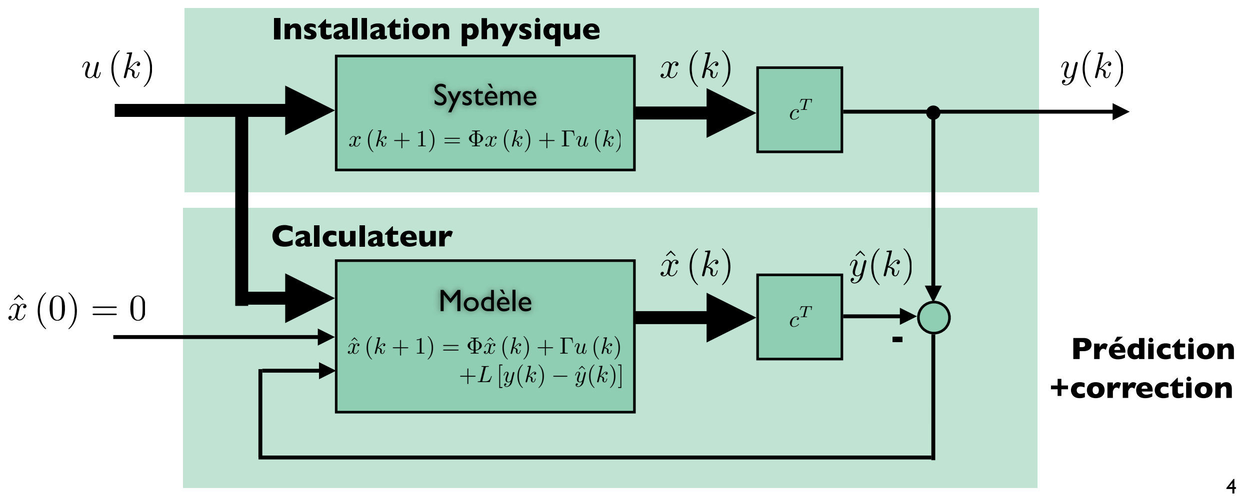
6.1.2 Estimation d'état

Système à observer
$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= c^T x(k) \end{aligned}$$

Observateur
$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + L\left[y(k) - \hat{y}(k)\right]$$

Erreur d'estimation
$$\delta(k+1) = \left[\Phi - Lc^T\right]\delta(k) = \Phi_e \delta(k)$$

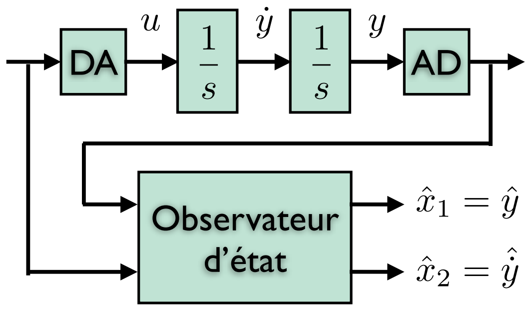
$$\alpha_e(\lambda) = \det(\lambda I - \Phi_e) = (\lambda - \beta_1)(\lambda - \beta_2)\ldots(\lambda - \beta_n) = \det(\lambda I - \Phi + Lc^T) = 0$$



6.1.3 Double intégrateur

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}}_{\Phi} x(k) + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c^T} x(k)$$



$$\alpha_e(\lambda) = \det\left[\lambda I - \Phi_e\right] = (\lambda - \beta_1)(\lambda - \beta_2) = \det\left[\lambda I - \Phi + Lc^T\right] = 0$$

$$\left| \begin{bmatrix} \lambda - 1 & -h \\ 0 & \lambda - 1 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right| = \lambda^2 - 0.8\lambda + 0.32 = 0$$

$$\left| \begin{bmatrix} \lambda - 1 + L_1 & -h \\ L_2 & \lambda - 1 \end{bmatrix} \right| = \lambda^2 + \underbrace{(L_1 - 2)}_{-0.8} \lambda + \underbrace{(L_2 h - L_1 + 1)}_{0.32} = 0$$

$$L_1 = 1.2 \qquad\qquad L_2 = \frac{0.52}{h}$$

A.2.1 FT → Modèle d'état

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}}$$

$$v(k+1) = \underbrace{\begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & 0 & \ldots & 0 & 1 \\ -a_n & 0 & \ldots & 0 & 0 \end{bmatrix}}_{\Phi_v} v(k) + \underbrace{\begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix}}_{g_v} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}}_{c_v^T} v(k) + \underbrace{b_0}_{d} u(k)$$

$$\det(\lambda I - \Phi_v) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n \lambda + a_n = 0$$

A2.2 Modèle d'état physique → artificiel

$$\left|\begin{array}{rcl} x(k+1) & = & \Phi x(k) + g u(k) \\ y(k) & = & c^T x(k) + d u(k) \end{array}\right.$$

Transformation

$$\left|\begin{array}{rcl} x & = & Rv \\ v & = & R^{-1}x \end{array}\right.$$

$$\left|\begin{array}{rcl} Rv(k+1) & = & \Phi R v(k) + g u(k) \\ y(k) & = & c^T R v(k) + d u(k) \end{array}\right.$$

$$\left|\begin{array}{rcl} v(k+1) & = & R^{-1}\Phi R v(k) + R^{-1} g u(k) \\ y(k) & = & c^T R v(k) + d u(k) \end{array}\right.$$

Quelle matrice de transformation R choisir ?

$$\left|\begin{array}{rcl} v(k+1) & = & \underbrace{R^{-1}\Phi R}_{?}\, v(k) + \underbrace{R^{-1} g}_{?}\, u(k) \\ y(k) & = & \underbrace{c^T R}_{?}\, v(k) + d u(k) \end{array}\right.$$

Modèle d'état physique → artificiel

$$Q = \begin{bmatrix} c^T I \\ c^T \Phi \\ \vdots \\ c^T \Phi^{n-1} \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix}$$

$$R = \begin{bmatrix} \Phi^{n-1} e_n & \dots & \Phi e_n & I e_n \end{bmatrix}$$

Construction de $c^T R$ $\quad I = Q Q^{-1}$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} = \begin{bmatrix} c^T I e_1 & c^T I e_2 & \dots & c^T I e_n \\ c^T \Phi e_1 & c^T \Phi e_2 & \dots & c^T \Phi e_n \\ \vdots & \vdots & & \vdots \\ c^T \Phi^{n-1} e_1 & c^T \Phi^{n-1} e_2 & \dots & c^T \Phi^{n-1} e_n \end{bmatrix}$$

$$c^T R = \begin{bmatrix} c^T \Phi^{n-1} e_n & \dots & c^T \Phi e_n & c^T I e_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} = c_v^T$$

Modèle d'état physique → artificiel

$$R^{-1}\Phi R = \left(R^{-1}\Phi\right) R = \left[\begin{array}{cccc} R^{-1}\Phi^n e_n & \ldots & R^{-1}\Phi^2 e_n & R^{-1}\Phi e_n \end{array}\right]$$

$$I = R^{-1}R = \left[\begin{array}{cccc} R^{-1}\Phi^{n-1} e_n & \ldots & R^{-1}\Phi e_n & R^{-1} I e_n \end{array}\right] = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix}$$

$$R^{-1}\Phi^n e_n = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_{n-1} \\ -a_n \end{bmatrix}$$

$$R^{-1}\Phi R = \begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & 0 & & & 1 \\ -a_n & 0 & \ldots & 0 & 0 \end{bmatrix} = \Phi_v$$

Modèle d'état physique → artificiel

$$Q = \begin{bmatrix} c^T I \\ c^T \Phi \\ \vdots \\ c^T \Phi^{n-1} \end{bmatrix} \qquad Q^{-1} = \left[\begin{array}{cccc} e_1 & e_2 & \ldots & e_n \end{array}\right]$$

$$R = \left[\begin{array}{cccc} \Phi^{n-1} e_n & \ldots & \Phi e_n & I e_n \end{array}\right]$$

$x = Rv$

$$v(k+1) = \underbrace{\begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & 0 & \ldots & 0 & 1 \\ -a_n & 0 & \ldots & 0 & 0 \end{bmatrix}}_{\Phi_v \,=\, R^{-1}\Phi R} v(k) + \underbrace{\begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix}}_{g_v \,=\, R^{-1} g} u(k)$$

$$y(k) = \underbrace{\left[\begin{array}{ccccc} 1 & 0 & \ldots & 0 & 0 \end{array}\right]}_{c_v^T \,=\, c^T R} v(k) + \underbrace{b_0}_{d} u(k)$$

$$\det\left(\lambda I - \Phi_v\right) = \det\left(\lambda I - \Phi\right) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n\lambda + a_n = 0$$

6.2 Estimation d'état: Méthode constructive

Système à observer

$$\begin{aligned} v(k+1) &= \Phi_v v(k) + \Gamma_v u(k) = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{n-1} & 0 & \dots & 0 & 1 \\ -a_n & 0 & \dots & 0 & 0 \end{bmatrix} v(k) + \Gamma_v u(k) \\ y(k) &= c_v^T v(k) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} v(k) \end{aligned}$$

Erreur d'estimation $\delta_v(k+1) = \left[\Phi_v - L' c_v^T\right] \delta_v(k) = \Phi_{ve} \delta_v(k)$

$$\Phi_{ve} = \begin{bmatrix} -a_1 - L'_1 & 1 & 0 & \dots & 0 \\ -a_2 - L'_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{n-1} - L'_{n-1} & 0 & \dots & 0 & 1 \\ -a_n - L'_n & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\alpha_e(\lambda) = \alpha'_e(\lambda) = \det(\lambda I - \Phi_{ve}) = \lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_{n-1}\lambda + \alpha_n = 0$$

$$\det(\lambda I - \Phi_v + L' c_v^T) = \lambda^n + (a_1 + L'_1)\lambda^{n-1} + \dots + \left(a_{n-1} + L'_{n-1}\right)\lambda + (a_n + L'_n) = 0$$

$L'_i = \alpha_i - a_i$ $\qquad \hat{v}(k+1) = \Phi_v \hat{v}(k) + \Gamma_v u(k) + L'\left[y(k) - \hat{y}(k)\right]$

$v(k) = R^{-1} x(k)$ $\qquad \hat{x}(k+1) = R\Phi_v R^{-1}\hat{x}(k) + R\Gamma_v u(k) + \underbrace{RL'}_{L}\left[y(k) - \hat{y}(k)\right]$

6.2.1 Formule d'Ackermann

$$L = RL' = R \begin{bmatrix} \alpha_1 - a_1 \\ \vdots \\ \alpha_n - a_n \end{bmatrix} = R \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \underbrace{R \begin{bmatrix} -a_1 \\ \vdots \\ -a_n \end{bmatrix}}_{R^{-1}\Phi^n e_n}$$

$$L = \underbrace{\begin{bmatrix} \Phi^{n-1} e_n & \dots & \Phi e_n & e_n \end{bmatrix}}_{R} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \Phi^n e_n$$

$$L = \Phi^n e_n + \alpha_1 \Phi^{n-1} e_n + \alpha_2 \Phi^{n-2} e_n + \dots + \alpha_n I e_n$$

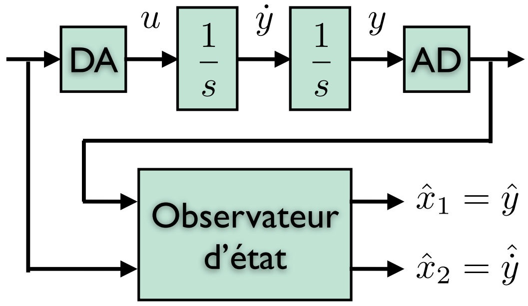
$$L = \underbrace{\left[\Phi^n + \alpha_1 \Phi^{n-1} + \alpha_2 \Phi^{n-2} + \dots + \alpha_n I\right]}_{\alpha_e(\Phi)} e_n$$

$$L = \alpha_e(\Phi) Q^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \alpha_e(\Phi) \begin{bmatrix} c^T I \\ c^T \Phi \\ \vdots \\ c^T \Phi^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Double intégrateur

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}}_{\Phi} x(k) + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c^T} x(k)$$



$$L = \alpha_e(\Phi) \begin{bmatrix} c^T \\ c^T\Phi \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L = \left\{ \underbrace{\begin{bmatrix} 1 & 2h \\ 0 & 1 \end{bmatrix}}_{\Phi^2} - 0.8 \underbrace{\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}}_{\Phi} + 0.32 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 \\ 1 & h \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Observateur à réponse pile

Système à observer
$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= c^T x(k) \end{aligned}$$

Observateur
$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + L\left[y(k) - \hat{y}(k)\right]$$

Erreur d'estimation
$$\delta(k+1) = \hat{x}(k) - x(k) = \left[\Phi - Lc^T\right] \delta(k)$$

Annuler l'erreur d'estimation après *n* périodes d'échantillonnage, avec

$$x(0) = 0, \qquad \delta(0) = -x(0) = -x_0$$

L'équation d'erreur étant une équation d'état homogène, sa solution est

$$\begin{aligned} \delta(n) &= (\Phi - Lc^T)^n \delta(0) = 0 \\ \delta(n) &= (R\Phi_v R^{-1} - RL'c_v^T R^{-1})^n \delta(0) \\ \delta(n) &= \left[R\left(\Phi_v - L'c_v^T\right) R^{-1}\right]^n \delta(0) \\ \delta(n) &= R^n \left(\Phi_v - L'c_v^T\right)^n \left(R^{-1}\right)^n \delta(0) \end{aligned}$$

Observateur à réponse pile

$$\delta(n) = R^n \left(\Phi_v - L' c_v^T\right)^n \left(R^{-1}\right)^n \delta(0)$$

Avec les valeurs propres régissants la dynamique d'estimation placées au centre du cercle unité

$$\alpha_e(\lambda) = \lambda^n$$

$$L'_i = \alpha_i - a_i = -a_i$$

$$\Phi_{ve} = \Phi_v - L' c_v^T = \begin{bmatrix} -a_1 - L'_1 & 1 & 0 & \dots & 0 \\ -a_2 - L'_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{n-1} - L'_{n-1} & 0 & \dots & 0 & 1 \\ -a_n - L'_n & 0 & \dots & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

C'est une matrice nilpotente

$$\left(\Phi_{ve}\right)^n = \left(\Phi_v - L' c_v^T\right)^n = 0 \qquad \text{donc} \qquad \delta(n) = 0$$

Résumé estimation d'état SISO

Système à observer
$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= c^T x(k) \end{aligned}$$

Observateur
$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + L\left[y(k) - \hat{y}(k)\right]$$

$$\begin{aligned} \delta(k) &= \hat{x}(k) - x(k) \qquad \text{Erreur d'estimation} \\ \delta(k+1) &= \hat{x}(k+1) - x(k+1) \\ &= \Phi \hat{x}(k) + \Gamma u(k) + L c^T \left[x(k) - \hat{x}(k)\right] - \Phi x(k) - \Gamma u(k) \\ &= \left[\Phi - L c^T\right] \delta(k) = \Phi_e \delta(k) \end{aligned}$$

$$\alpha_e(\lambda) = \det(\lambda I - \Phi_e) = (\lambda - \beta_1)(\lambda - \beta_2)\dots(\lambda - \beta_n) = 0$$

Formule d'Ackermann pour l'observateur

$$L = \alpha_e(\Phi) Q^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \alpha_e(\Phi) \begin{bmatrix} c^T \\ c^T \Phi \\ \vdots \\ c^T \Phi^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

6.5 Structure complète de la commande d'état