Systèmes discrets linéaires et stationnaires

Solution

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

$$x(k) = \underbrace{\Phi^{k-k_0} x(k_0)}_{\text{Réponse libre}} + \underbrace{\sum_{l=k_0}^{k-1} \Phi^{k-l-1} \Gamma u(l)}_{\text{Réponse forcée}}$$

Matrice de transfert et stabilité

$$Y(z) = \left[C(zI-\Phi)^{-1}\Gamma + D\right] U(z) = H(z) U(z)$$

$$H(z) = \begin{bmatrix} H_{11}(z) & \dots & H_{1r}(z) \\ \vdots & & \vdots \\ H_{p1}(z) & \dots & H_{pr}(z) \end{bmatrix} = [H_{ij}(z)] = \left[\frac{H_{ij}^*(z)}{\det(zI-\Phi)}\right]$$

$$\text{Pôles } z_i \text{ des } H_{ij} \text{ solution de: } \det(zI-\Phi) = 0$$

$$\text{Valeurs propres } v_i \text{ de } \Phi \text{ solution de: } \det(\lambda I-\Phi) = 0$$

$$z_i = v_i$$

$$\text{Asymptotiquement stable si: } |v_i| < 1 \text{ pour } i = 1, \dots, n$$

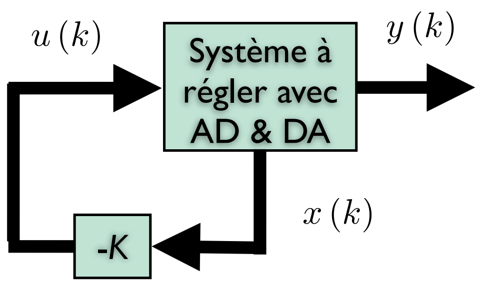
Commande d'état

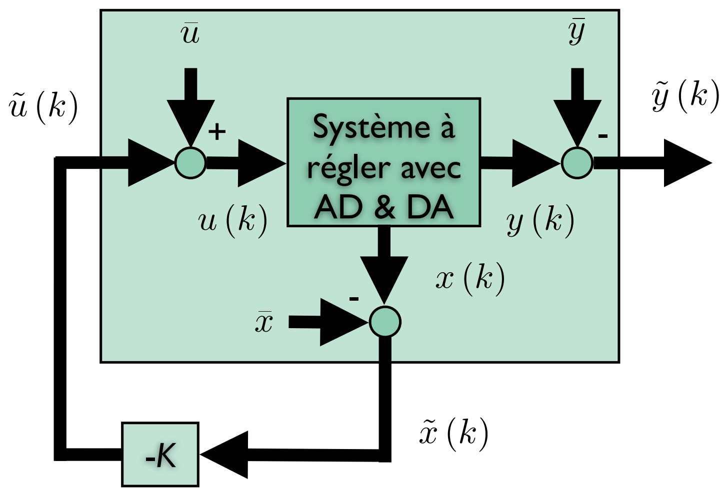
Système à régler

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \qquad \begin{aligned} \tilde{x}(k+1) &= \Phi \tilde{x}(k) + \Gamma \tilde{u}(k) \\ \tilde{y}(k) &= C\tilde{x}(k) + D\tilde{u}(k) \end{aligned}$$

Régulateur

$$u(k) = -Kx(k) \qquad \tilde{u}(k) = -K\tilde{x}(k)$$





Commande d'état

Système à régler

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Régulateur $u(k) = -Kx(k)$

Système en boucle fermée (BF)

$$\begin{aligned} x(k+1) &= \overbrace{(\Phi - \Gamma K)}^{\Phi_{BF}} x(k) \\ y(k) &= \underbrace{(C - DK)}_{C_{BF}} x(k) \end{aligned}$$

$$X(z) = (zI - \Phi_{BF})^{-1} zx(0) = W(z)x(0)$$

$$X_i(z) = W_{i1}(z)x_1(0) + \ldots + W_{ij}(z)x_j(0) + \ldots + W_{in}(z)x_n(0)$$

$$Z^{-1}\left[W_{ij}(z)\right] = \sum_i c_i z_i^k + 2\sum_i \left|c_i^*\right| r_i^k \cos(k\omega_i + \varphi_i) + \sum_i \left[c'_{1i} z_i^k + c'_{2i} k z_i^{k-1} + \ldots\right]$$

La commande d'état ramène l'état à zéro $x(k) \to 0$ pour $k \to \infty$

En variables écart, la commande d'état ramène l'état à l'état nominal $\tilde{x}(k) \to 0$ ou $x(k) \to \bar{x}$ pour $k \to \infty$

Principe de synthèse de la commande d'état

Système à régler

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Régulateur $u(k) = -Kx(k)$

Système en boucle fermée (BF)

$$\begin{aligned} x(k+1) &= \overbrace{(\Phi - \Gamma K)}^{\Phi_{BF}} x(k) \\ y(k) &= \underbrace{(C - DK)}_{C_{BF}} x(k) \end{aligned}$$

$$\begin{aligned} \det(\lambda I - \Phi_{BF}) &= \det(\lambda I - \Phi + \Gamma K) &= \\ \alpha_c(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) &= \\ &= \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n &= 0 \end{aligned}$$

Identification terme à terme dans le cas SIMO (une entrée)

Double intégrateur (5.1.2)

Système en BF

$$\begin{aligned} x(k+1) &= \overbrace{(\Phi - \Gamma K)}^{\Phi_{BF}} x(k) \\ y(k) &= \underbrace{(C - DK)}_{C_{BF}} x(k) \end{aligned} \qquad \Phi = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \quad \Gamma = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}$$

$$\det(\lambda I - \Phi + \Gamma K) = (\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

Choix des valeurs propres $\lambda_{1,2} = 0.8 \pm 0.25j$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} h^2/2 \\ h \end{bmatrix}\begin{bmatrix} K_1 & K_2 \end{bmatrix}\right) = (\lambda - 0.8 - 0.25j)(\lambda - 0.8 + 0.25j)$$

$$\det\left(\begin{bmatrix} \lambda - 1 + \frac{h^2}{2}K_1 & -h + \frac{h^2}{2}K_2 \\ hK_1 & \lambda - 1 + hK_2 \end{bmatrix}\right) = \lambda^2 - 1.6\lambda + 0.7$$

$$\lambda^2 + \underbrace{\left(hK_2 + \frac{h^2}{2}K_1 - 2\right)}_{-1.6}\lambda + \underbrace{\left(\frac{h^2}{2}K_1 - hK_2 + 1\right)}_{0.7} = \lambda^2 - 1.6\lambda + 0.7$$

$$K_1 = 0.1/h^2 = 10 \ \text{ et } \ K_2 = 0.35/h = 3.5 \ \text{ pour } \ h = 0.1\text{s}$$

FT → Modèle d'état (2.5)

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_n z^{-n}}$$

$$w(k+1) = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{bmatrix}}_{\Phi_w} w(k) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{g_w} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \ldots & b_n - a_n b_0 \end{bmatrix}}_{c_w^T} w(k) + b_0 u(k)$$

$$\det(\lambda I - \Phi_w) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n \lambda + a_n = 0$$

Modèle d'état physique → artificiel (A.1.2)

$$\left| \begin{array}{rcl} x(k+1) & = & \Phi x(k) + gu(k) \\ y(k) & = & c^T x(k) + du(k) \end{array} \right.$$

Transformation

$$\left| \begin{array}{rcl} w & = & Px \\ x & = & P^{-1}w \end{array} \right.$$

$$\left| \begin{array}{rcl} P^{-1}w(k+1) & = & \Phi P^{-1}w(k) + gu(k) \\ y(k) & = & c^T P^{-1}w(k) + du(k) \end{array} \right.$$

$$\left| \begin{array}{rcl} w(k+1) & = & P\Phi P^{-1}w(k) + Pgu(k) \\ y(k) & = & c^T P^{-1}w(k) + du(k) \end{array} \right.$$

Quelle matrice de transformation P choisir ?

$$\left| \begin{array}{rcl} w(k+1) & = & \underbrace{P\Phi P^{-1}}_{?} w(k) + \underbrace{Pg}_{?} u(k) \\ y(k) & = & \underbrace{c^T P^{-1}}_{?} w(k) + du(k) \end{array} \right.$$

Modèle d'état physique → artificiel

$$G = \left[\begin{array}{cccc} Ig & \Phi g & \ldots & \Phi^{n-1}g \end{array} \right] \qquad G^{-1} = \begin{bmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_n^T \end{bmatrix} \qquad P = \begin{bmatrix} e_n^T \Phi^{n-1} \\ e_n^T \Phi^{n-2} \\ \vdots \\ e_n^T I \end{bmatrix}$$

Construction de Pg

$$I = G^{-1}G \qquad \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1^T Ig & e_1^T \Phi g & \ldots & e_1^T \Phi^{n-1} g \\ e_2^T Ig & e_2^T \Phi g & \ldots & e_2^T \Phi^{n-1} g \\ \vdots & \vdots & & \vdots \\ e_n^T Ig & e_n^T \Phi g & & e_n^T \Phi^{n-1} g \end{bmatrix}$$

$$Pg = \begin{bmatrix} e_n^T \Phi^{n-1} g \\ e_n^T \Phi^{n-2} g \\ \vdots \\ e_n^T Ig \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Modèle d'état physique → artificiel

$$I = PP^{-1} = \begin{bmatrix} e_n^T\Phi^{n-1} \\ e_n^T\Phi^{n-2} \\ \vdots \\ e_n^T I \end{bmatrix} P^{-1} \qquad \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_n^T\Phi^{n-1}P^{-1} \\ e_n^T\Phi^{n-2}P^{-1} \\ \vdots \\ e_n^T P^{-1} \end{bmatrix}$$

Construction de

$$P\Phi P^{-1} = \begin{bmatrix} e_n^T\Phi^{n-1} \\ e_n^T\Phi^{n-2} \\ \vdots \\ e_n^T I \end{bmatrix} \Phi P^{-1} = \begin{bmatrix} e_n^T\Phi^{n}P^{-1} \\ e_n^T\Phi^{n-1}P^{-1} \\ \vdots \\ e_n^T \Phi P^{-1} \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

$$e_n^T\Phi^n P^{-1} = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \end{bmatrix}$$

Modèle d'état physique → artificiel

$$G = \begin{bmatrix} Ig & \Phi g & \dots & \Phi^{n-1}g \end{bmatrix} \qquad G^{-1} = \begin{bmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_n^T \end{bmatrix} \qquad P = \begin{bmatrix} e_n^T\Phi^{n-1} \\ e_n^T\Phi^{n-2} \\ \vdots \\ e_n^T I \end{bmatrix}$$

$w = Px$

$$w(k+1) = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}}_{\Phi_w = P\Phi P^{-1}} w(k) + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{g_w = Pg} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \dots & b_n - a_n b_0 \end{bmatrix}}_{c_w^T = c^T P^{-1}} w(k) + b_0 u(k)$$

$$\det(\lambda I - \Phi_w) = \det(\lambda I - \Phi) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n\lambda + a_n = 0$$

Commande d'état

Système à régler

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k) + Du(k)$$

SIMO
$\Gamma = g$

Commande $u(k) = -Kx(k)$

$$x(k+1) = (\Phi - \Gamma K)\, x(k) = \Phi_{BF} x(k)$$

Stabilité en BO $\det(\lambda I - \Phi) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$

Stabilité en BF
v.p. imposées

$$\det(\lambda I - \Phi_{BF}) = \lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n =$$
$$\det(\lambda I - \Phi + \Gamma K) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = \alpha_c(\lambda) = 0$$

Méthode constructive $w = Px \qquad w(k+1) = (\Phi_w - \Gamma_w K')\, w(k) = \Phi_{wBF} w(k)$

$$\alpha'_c(\lambda) = \alpha_c(\lambda) = 0 =$$
$$\det(\lambda I - \Phi_{wBF}) = \lambda^n + \alpha_1\lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n =$$
$$\det(\lambda I - \Phi_w + \Gamma_w K') = \lambda^n + (a_1 + K'_1)\lambda^{n-1} + \ldots + \left(a_{n-1} + K'_{n-1}\right)\lambda + (a_n + K'_n)$$

$$K'_i = \alpha_i - a_i \qquad u(k) = -K'w(k) = -\underbrace{K'P}_{K} x(k)$$

Formule d'Ackermann

$$K = K'P = \left[\begin{array}{cccc} (\alpha_1 - a_1) & (\alpha_2 - a_2) & \ldots & (\alpha_n - a_n) \end{array}\right] P$$

$$K = \left[\begin{array}{cccc} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{array}\right] P + \underbrace{\left[\begin{array}{cccc} -a_1 & -a_2 & \ldots & -a_n \end{array}\right] P}_{e_n^T \Phi^n P^{-1}}$$

$$K = \left[\begin{array}{cccc} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{array}\right] \left[\begin{array}{c} e_n^T\Phi^{n-1} \\ e_n^T\Phi^{n-2} \\ \vdots \\ e_n^T I \end{array}\right] + e_n^T\Phi^n$$

$$K = e_n^T\Phi^n + \alpha_1 e_n^T\Phi^{n-1} + \alpha_2 e_n^T\Phi^{n-2} + \ldots + \alpha_n e_n^T I$$

$$K = \left[e_n^T\right] \underbrace{\left[\Phi^n + \alpha_1\Phi^{n-1} + \alpha_2\Phi^{n-2} + \ldots + \alpha_n I\right]}_{\alpha_c(\Phi)}$$

$$K = \left[\begin{array}{ccccc} 0 & 0 & \ldots & 0 & 1 \end{array}\right] G^{-1}\alpha_c(\Phi)$$

$$K = \left[\begin{array}{ccccc} 0 & 0 & \ldots & 0 & 1 \end{array}\right] \left[\begin{array}{c|c|c|c} Ig & \Phi g & \ldots & \Phi^{n-1}g \end{array}\right]^{-1} \alpha_c(\Phi)$$

Résumé Commande d'état SIMO

Système à régler
$$\begin{aligned} x(k+1) &= \Phi x(k) + gu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Commande
$$u(k) = -Kx(k)$$

Système en boucle fermée (BF)
$$x(k+1) = (\Phi - gK)\, x(k) = \Phi_{BF} x(k)$$

$$\begin{aligned} \alpha_c(\lambda) = \det(\lambda I - \Phi_{bf}) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n) \\ &= \lambda^n + \alpha_1 \lambda^{n-1} + \dots \lambda a_{n-1} + \alpha_n = 0 \end{aligned}$$

Formule d'Ackermann

$$\begin{aligned} K &= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} G^{-1} \alpha_c(\Phi) \\ &= \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} Ig & | & \Phi g & | & \dots & | & \Phi^{n-1} g \end{bmatrix}^{-1} \alpha_c(\Phi) \end{aligned}$$

Double intégrateur (5.3.1)

Système en BF
$$\begin{aligned} x(k+1) &= \overbrace{(\Phi - \Gamma K)}^{\Phi_{BF}} x(k) \\ y(k) &= \underbrace{(C - DK)}_{C_{BF}} x(k) \end{aligned} \qquad \Phi = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \quad \Gamma = \begin{bmatrix} h^2/2 \\ h \end{bmatrix}$$

Choix des valeurs propres $\lambda_{1,2} = 0.8 \pm 0.25j$

$$\det(\lambda I - \Phi + \Gamma K) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - 1.6\lambda + 0.7$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} Ig & | & \Phi g \end{bmatrix}^{-1} \alpha_c(\Phi)$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} h^2/2 & 3h^2/2 \\ h & h \end{array} \right]^{-1} \left[\Phi^2 - 1.6\Phi + 0.7I\right]$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{h^3} \left[\begin{array}{c|c} -h & 3h^2/2 \\ h & -h^2/2 \end{array} \right] \left\{ \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}^2 - 1.6 \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} + 0.7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$K = \frac{\begin{bmatrix} h & -h^2/2 \end{bmatrix}}{h^3} \begin{bmatrix} 0.1 & 0.4h \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} \frac{0.1}{h^2} & \frac{0.35}{h} \end{bmatrix} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

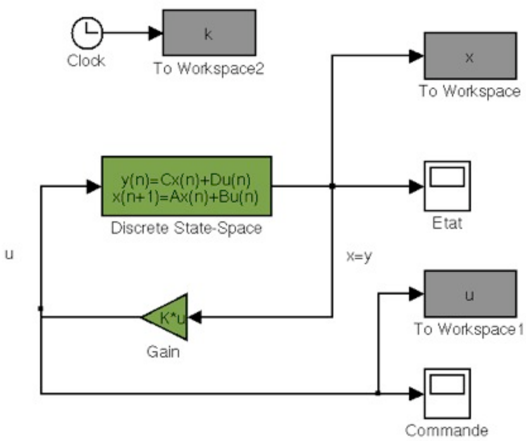
Double intégrateur (5.3.1)

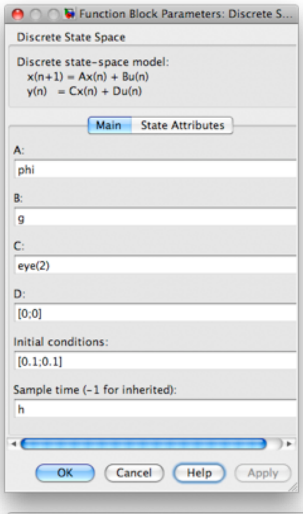
```
% Modèle discret

h = 0.1;

phi = [ 1  h   ;
        0  1  ];

g   = [ h^2/2  ;
        h     ];

% Matrice de gouvernabilité

G   = [ g phi*g];

% Coefficient du polynôme caractéristique en BF

alpha1 = -1.6;

alpha2 =  0.7;

% Gain de contre-réaction

K = [0 1] * inv(G) * (phi^2 + alpha1 * phi + alpha2 * eye(2));

% Calcul direct avec Ackermann

L = [0.8+0.25*1i  0.8-0.25*1i];

Kbis = acker(phi,g,L);
```

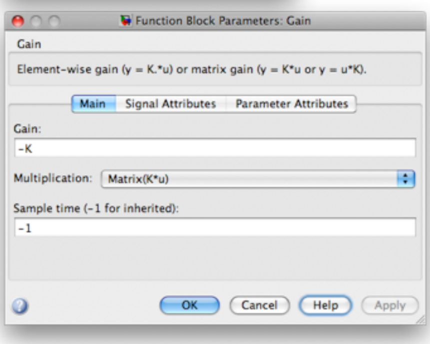
Code Matlab

Double intégrateur (5.3.1)

Simulation avec Simulink







Double intégrateur (5.3.1)