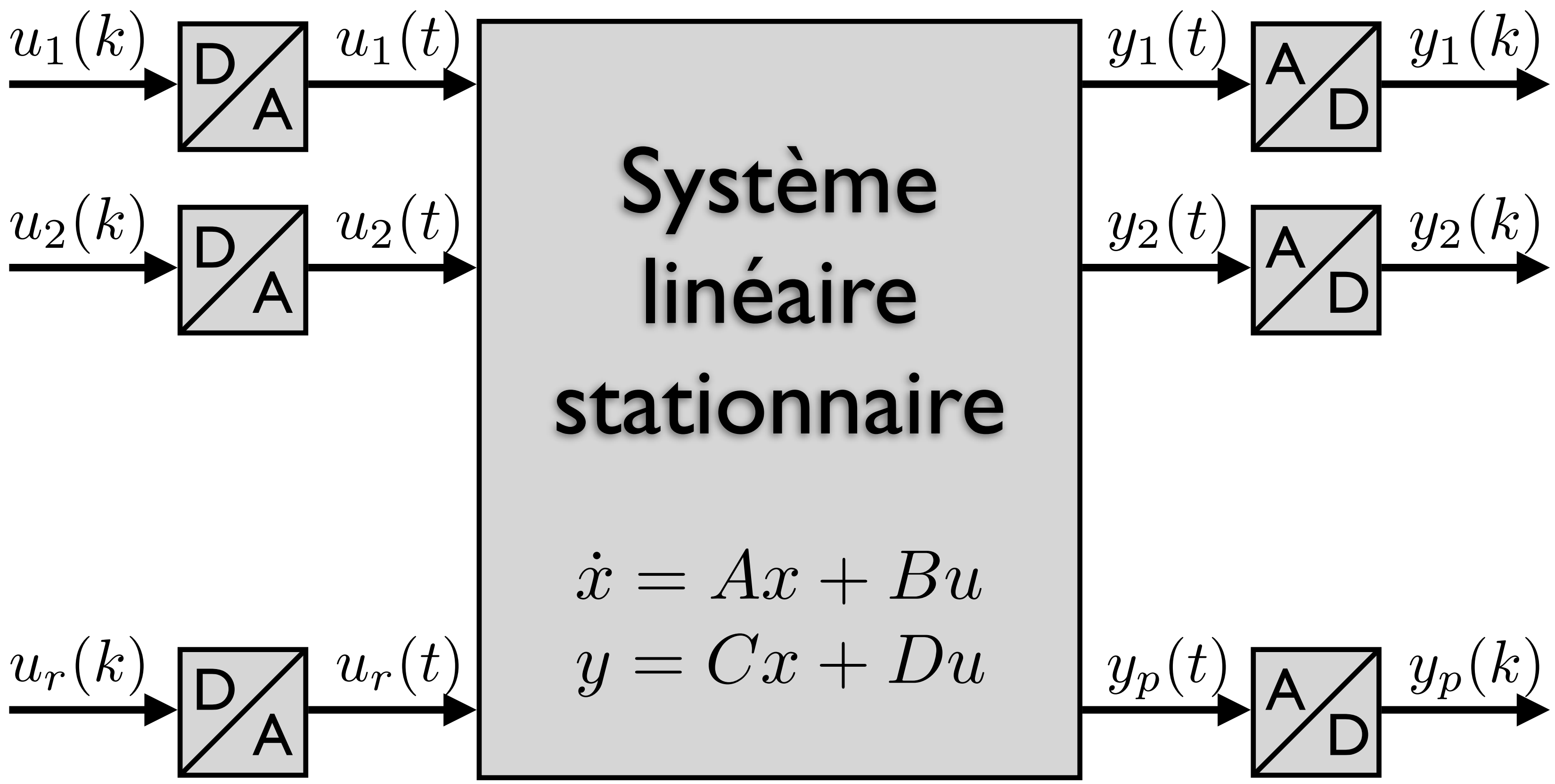
Discrétisation



Comment prendre explicitement en compte les convertisseurs ?

Résoudre analytiquement

$$\dot{x} = Ax + Bu$$

Avec un maintien de *u* entre *k* et *k*+1

Solution de l'équation d'état

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Solution générale = **solution homogène** + solution particulière

Cas scalaire

$$x(t_0) = x_0$$

Cas vectoriel

$$\dot{x}(t) = ax(t)$$

$$\dot{x}(t) = Ax(t)$$

$$x = e^{at}c_2 = e^{a(t-t_0)}x_0$$

$$e^{\tau} = 1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3 + \dots$$

$$x = \left[1 + a(t-t_0) + \frac{1}{2!}a^2(t-t_0)^2 + \frac{1}{3!}a^3(t-t_0)^3 + \dots\right]x_0$$

$$x = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + a_3(t-t_0)^3 + \dots$$

Par analogie

$$x = A_0 + A_1(t-t_0) + A_2(t-t_0)^2 + A_3(t-t_0)^3 + \dots$$

Solution de l'équation d'état homogène

$$\dot{x}(t) = Ax(t)$$

$$x = A_0 + A_1 (t - t_0) + A_2 (t - t_0)^2 + A_3 (t - t_0)^3 + \ldots$$

$$x(t_0) = A_0 \qquad x(t_0) = x_0$$

$$\dot{x} = A_1 + 2A_2 (t - t_0) + 3A_3 (t - t_0)^2 + \ldots$$

$$\dot{x}(t_0) = A_1$$

$$\dot{x}(t) = Ax(t)$$
$$\dot{x}(t_0) = Ax(t_0) = Ax_0$$

$$\ddot{x} = 2A_2 + 6A_3 (t - t_0) + \ldots$$

$$\ddot{x}(t_0) = 2A_2$$

$$\ddot{x}(t) = A\dot{x}(t)$$
$$\ddot{x}(t_0) = A\dot{x}(t_0) = A^2 x_0$$

$$x(t) = \underbrace{\left[I + A(t - t_0) + \frac{A^2}{2!}(t - t_0)^2 + \frac{A^3}{3!}(t - t_0)^3 + \ldots \right]}_{\equiv e^{A(t-t_0)}} x_0$$

Exponentielle de matrice

$$e^{A(t-t_0)} = I + A\left(t-t_0\right) + \frac{A^2}{2!}\left(t-t_0\right)^2 + \frac{A^3}{3!}\left(t-t_0\right)^3 + \ldots = \sum_{k=0}^{\infty} \frac{A^k}{k!}\left(t-t_0\right)^k$$

$$x\left(t\right) = e^{A(t-t_0)}x_0$$

$$\begin{aligned}
&x\left(t_1\right) = e^{A(t_1-t_0)}x\left(t_0\right) \\
&x\left(t_2\right) = e^{A(t_2-t_1)}x\left(t_1\right) = e^{A(t_2-t_1)}e^{A(t_1-t_0)}x\left(t_0\right) \\
&t_2 = t_0 \\
&x\left(t_0\right) = e^{A(t_0-t_1)}e^{A(t_1-t_0)}x\left(t_0\right)
\end{aligned}$$

$$I = e^{A(t_0-t_1)}e^{A(t_1-t_0)} = M^{-1}M$$

$$\left[e^{A(t_1-t_0)}\right]^{-1} = e^{A(t_0-t_1)} = e^{-A(t_1-t_0)}$$

Solution particulière de l'équation d'état

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$x(t) = e^{A(t-t_0)}v(t)$$

$$\underbrace{Ae^{A(t-t_0)}v(t) + e^{A(t-t_0)}\dot{v}(t)}_{\dot{x}(t)} = A\underbrace{e^{A(t-t_0)}v(t)}_{x(t)} + Bu(t)$$

$$\dot{v}(t) = \left[e^{A(t-t_0)}\right]^{-1} Bu(t) = e^{-A(t-t_0)}Bu(t)$$

$$v(t) = \int_{t_0}^{t} e^{-A(\tau-t_0)}Bu(\tau)\,d\tau$$

$$x(t) = e^{A(t-t_0)} \int_{t_0}^{t} e^{-A(\tau-t_0)}Bu(\tau)\,d\tau = \int_{t_0}^{t} e^{A(t-t_0)}e^{A(t_0-\tau)}Bu(\tau)\,d\tau$$

$$x(t) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Solution complète de l'équation d'état

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

réponse libre + réponse forcée (produit de convolution)

Discrétisation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Convertisseurs AD

$$t_0 = kh$$
$$t = kh + h$$

$$x(kh+h) = e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{A(kh+h-\tau)}Bu(\tau)\,d\tau$$

$$t = kh$$

$$y(kh) = Cx(kh) + Du(kh)$$

Convertisseurs DA

$$u(\tau) = u(kh)$$
$$kh \leq \tau < kh + h$$

$$x(kh+h) = e^{Ah}x(kh) + \int_{kh}^{kh+h} e^{A(kh+h-\tau)}B\underbrace{u(\tau)}_{u(kh)}\,d\tau$$

Discrétisation

$$x\left(kh+h\right)=e^{Ah}x\left(kh\right)+\int\limits_{kh}^{kh+h}e^{A(kh+h-\tau)}B\underbrace{u\left(\tau\right)}_{u(kh)}d\tau$$

$$\eta=kh+h-\tau \qquad\qquad d\eta=-d\tau$$

$$x\left(kh+h\right)=e^{Ah}x\left(kh\right)+\left[-\int\limits_{h}^{0}e^{A\eta}Bd\eta\right]u\left(kh\right)$$

$$y\left(kh\right)=Cx\left(kh\right)+Du\left(kh\right)$$

$$x\left(k+1\right)=\underbrace{\left[e^{Ah}\right]}_{\Phi}x\left(k\right)+\underbrace{\left[\int\limits_{0}^{h}e^{A\eta}d\eta\right]B}_{\Gamma}u\left(k\right)$$

$$y\left(k\right)=Cx\left(k\right)+Du\left(k\right)$$

Solution de l'équation d'état analogique, linéaire et stationnaire + Discrétisation

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$x(k+1) = \underbrace{\Phi}_{e^{Ah}} x(k) + \underbrace{\Gamma}_{\left[\int_0^h e^{A\eta}d\eta\right]B} u(k)$$

$$y(k) = Cx(k) + Du(k)$$

Exponentielle de matrice: Série

$$\Phi = e^{Ah} = I + Ah + \frac{A^2}{2!}h^2 + \ldots + \frac{A^i}{i!}h^i + \ldots$$

$$\Gamma = \int_0^h e^{A\eta} d\eta B = \left[Ih + \frac{A}{2!}h^2 + \frac{A^2}{3!}h^3 + \ldots + \frac{A^i}{(i+1)!}h^{i+1} + \ldots \right] B$$

$$\Psi = I + \frac{A}{2!}h + \frac{A^2}{3!}h^2 + \ldots + \frac{A^i}{(i+1)!}h^i + \ldots$$

$$\Phi = I + Ah\Psi$$

$$\Gamma = \Psi h B$$

$$\Psi \cong I + \frac{Ah}{2}\left(I + \frac{Ah}{3}\left(\ldots \frac{Ah}{N-1}\left(I + \frac{Ah}{N} \right)\right)\right)$$

Discrétisation de systèmes stationnaires

<table>
<tr><th>Modèle linéaire</th><th colspan="2">Modèle non linéaire</th></tr>
<tr><td rowspan="3"></td><td colspan="2">$$\begin{aligned}\dot{x}(t) &= f\left[x(t), u(t)\right] \\ y(t) &= g\left[x(t), u(t)\right]\end{aligned}$$</td></tr>
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<tr><td colspan="3">Discrétisation exacte $\Phi = e^{Ah}$ $\Gamma = \int_0^h e^{A\eta} d\eta \; B$</td></tr>
<tr><td colspan="2">$$\begin{aligned}x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k)\end{aligned}$$</td><td>$$\begin{aligned}\tilde{x}(k+1) &= \Phi \tilde{x}(k) + \Gamma \tilde{u}(k) \\ \tilde{y}(k) &= C\tilde{x}(k) + D\tilde{u}(k)\end{aligned}$$</td></tr>
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Double intégrateur: Série

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u \quad \text{et} \quad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

$$\Phi = e^{Ah} = I + Ah + \frac{A^2}{2!}h^2 + \ldots + \frac{A^i}{i!}h^i + \ldots$$

$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \frac{h^2}{2} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} = e^{Ah}$$

$$\Gamma = \int_0^h e^{A\eta} d\eta B = \int_0^h \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix} d\eta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \eta|_0^h & \frac{\eta^2}{2}\Big|_0^h \\ 0 & \eta|_0^h \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} h & \frac{h^2}{2} \\ 0 & h \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix}$$

$$x(k+1) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u(k) \quad \text{et} \quad y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

Algorithme de Leverrier (analogique)

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] \quad \text{et} \quad \Phi = e^{Ah}$$

$$(sI - A)^{-1} = \frac{adj\,(sI - A)}{det\,(sI - A)} = \frac{H_0 s^{n-1} + H_1 s^{n-2} + H_2 s^{n-3} + \ldots + H_{n-1}}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}$$

$$\begin{aligned} H_0 &= I \\ H_1 &= AH_0 + a_1 I \\ H_2 &= AH_1 + a_2 I \\ &\vdots \\ H_{n-1} &= AH_{n-2} + a_{n-1} I \\ H_n &= AH_{n-1} + a_n I = 0 \end{aligned}$$

$$\begin{aligned} a_1 &= -tr\,(AH_0) \\ a_2 &= -\tfrac{1}{2} tr\,(AH_1) \\ &\vdots \\ a_{n-1} &= -\tfrac{1}{n-1} tr\,(AH_{n-2}) \\ a_n &= -\tfrac{1}{n} tr\,(AH_{n-1}) \end{aligned}$$

Double intégrateur: Leverrier

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u \quad \text{et} \quad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] \quad \text{et} \quad \Phi = e^{Ah}$$

$$H_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a_1 = -tr\,(AH_0) = -tr\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0, \quad H_1 = AH_0 + a_1 I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$a_2 = -\frac{1}{2}tr\,(AH_1) = -\frac{1}{2}tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0, H_2 = AH_1 + a_2 I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{(contrôle)}$$

$$(sI - A)^{-1} = \frac{H_0 s + H_1}{s^2 + a_1 s + a_2} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}{s^2} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{donc} \quad e^{Ah} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

Théorème de Cayley-Hamilton

Soit *A* une matrice *n* x *n*:

$$f(A) = p(A) = \alpha_0 A^{n-1} + \alpha_1 A^{n-2} + \ldots + \alpha_{n-1} I$$

Les coefficients α_i sont solution de:

$$f(\lambda_i) = p(\lambda_i) \qquad i = 1, \ldots, n$$

Les λ_i sont les valeurs propres de *A*, soit les solutions de:

$$\det(\lambda I - A) = |\lambda I - A| = 0$$

Pour des valeurs propres de multiplicité m_i

$$\left| \begin{array}{rcl} f^{(1)}(\lambda_i) & = & p^{(1)}(\lambda_i) \\ & \vdots & \\ f^{(m_i-1)}(\lambda_i) & = & p^{(m_i-1)}(\lambda_i) \end{array} \right.$$

Double intégrateur: Cayley-Hamilton (p.62)

$$\dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{B} u \quad \text{et} \quad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

$$f(A) = e^{Ah} = \alpha_0 A + \alpha_1 I \qquad \left| \begin{array}{l} e^{\lambda_1 h} = \alpha_0 \lambda_1 + \alpha_1 \\ e^{\lambda_2 h} = \alpha_0 \lambda_2 + \alpha_1 \end{array} \right.$$

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix} \right| = \lambda^2 = 0, \quad \lambda_1 = \lambda_2 = 0$$

$$e^{\lambda_1 h} = \alpha_0 \lambda_1 + \alpha_1 \quad \rightarrow \quad 1 = \alpha_1$$

$$\frac{d}{d\lambda_2} e^{\lambda_2 h} = \frac{d}{d\lambda_2} \left(\alpha_0 \lambda_2 + \alpha_1 \right) \quad \rightarrow \quad h e^{\lambda_2 h} = \alpha_0 \quad \rightarrow \quad h = \alpha_0$$

$$e^{Ah} = h \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

Solution complète de l'équation d'état

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

réponse libre + réponse forcée (produit de convolution)

Discrétisation de systèmes stationnaires

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Solution de l'équation d'état discrète linéaire

$$\begin{aligned} x\left(k+1\right) &= \Phi x\left(k\right)+\Gamma u\left(k\right) \\ y\left(k\right) &= Cx\left(k\right)+Du\left(k\right) \end{aligned}$$

$$x(k_0+1)=\Phi x(k_0)+\Gamma u(k_0)$$

$$x(k_0+2)=\Phi x(k_0+1)+\Gamma u(k_0+1)$$
$$x(k_0+2)=\Phi\left[\Phi x(k_0)+\Gamma u(k_0)\right]+\Gamma u(k_0+1)$$
$$x(k_0+2)=\Phi^2 x(k_0)+\Phi\Gamma u(k_0)+\Gamma u(k_0+1)$$

$$x(k_0+3)=\Phi x(k_0+2)+\Gamma u(k_0+2)$$
$$x(k_0+3)=\Phi\left[\Phi^2 x(k_0)+\Phi\Gamma u(k_0)+\Gamma u(k_0+1)\right]+\Gamma u(k_0+2)$$
$$x(k_0+3)=\Phi^3 x(k_0)+\Phi^2\Gamma u(k_0)+\Phi\Gamma u(k_0+1)+\Gamma u(k_0+2)$$

$$x(k)=\Phi^{k-k_0}x(k_0)+\sum_{i=k_0}^{k-1}\Phi^{k-i-1}\Gamma u(i)$$

réponse libre + réponse forcée (produit de convolution)

Matrice de transfert discrète

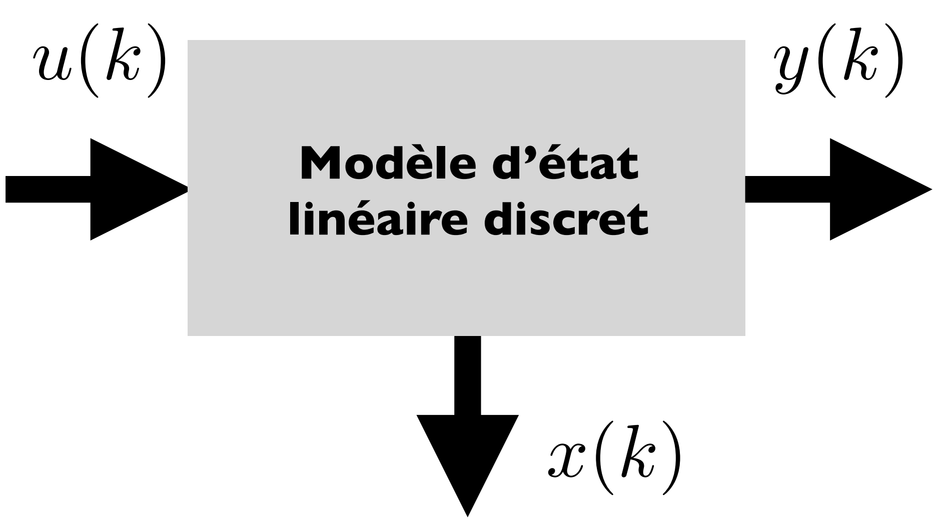
$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

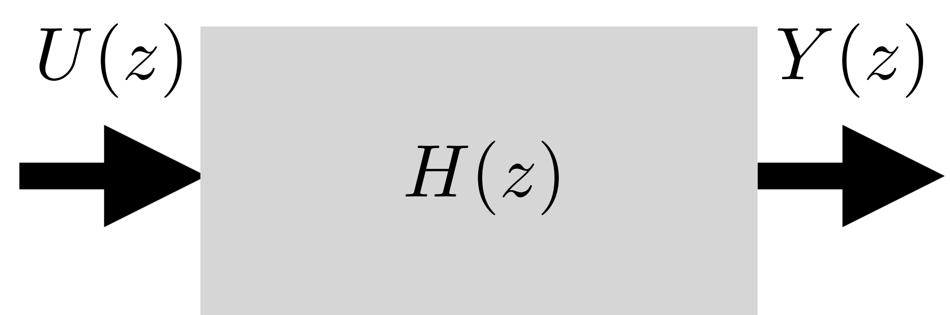
CI nulles

$$zX(z) - \Phi X(z) = \Gamma U(z)$$
$$Y(z) = CX(z) + DU(z)$$

$$(zI - \Phi)X(z) = \Gamma U(z)$$
$$Y(z) = CX(z) + DU(z)$$

$$X(z) = (zI - \Phi)^{-1}\Gamma U(z)$$
$$Y(z) = C\left[(zI - \Phi)^{-1}\Gamma U(z)\right] + DU(z)$$
$$Y(z) = \left[C(zI - \Phi)^{-1}\Gamma + D\right]U(z) \equiv H(z)U(z)$$





Algorithme de Leverrier (discret)

$$H(z) = C\left(zI - \Phi\right)^{-1}\Gamma + D$$

$$\left(zI - \Phi\right)^{-1} = \frac{adj\left(zI - \Phi\right)}{det\left(zI - \Phi\right)} = \frac{H_0 z^{n-1} + H_1 z^{n-2} + H_2 z^{n-3} + \ldots + H_{n-1}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n}$$

$$\begin{aligned} a_1 &= -tr\left(\Phi H_0\right) \\ a_2 &= -\tfrac{1}{2} tr\left(\Phi H_1\right) \\ &\vdots \\ a_{n-1} &= -\tfrac{1}{n-1} tr\left(\Phi H_{n-2}\right) \\ a_n &= -\tfrac{1}{n} tr\left(\Phi H_{n-1}\right) \end{aligned}$$

$$\begin{aligned} H_0 &= I \\ H_1 &= \Phi H_0 + a_1 I \\ H_2 &= \Phi H_1 + a_2 I \\ &\vdots \\ H_{n-1} &= \Phi H_{n-2} + a_{n-1} I \\ H_n &= \Phi H_{n-1} + a_n I = 0 \end{aligned}$$

Double intégrateur: Matrice de transfert

$$\Phi = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \qquad H(z) = C(zI-\Phi)^{-1}\Gamma + D$$

$$(zI-\Phi)^{-1} = \frac{adj(zI-\Phi)}{det(zI-\Phi)} = \frac{Iz+H_1}{z^2+a_1z+a_2} = \frac{1}{z^2-2z+1}\begin{bmatrix} z-1 & h \\ 0 & z-1 \end{bmatrix}$$

$$\left| \begin{array}{lcl} a_1 & = & -tr(\Phi) = -2 \\ H_1 & = & \Phi + a_1 I = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & h \\ 0 & -1 \end{bmatrix} \\ a_2 & = & -\frac{1}{2}tr(\Phi H_1) = -\frac{1}{2}tr\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1 \\ H_2 & = & \Phi H_1 + a_2 I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{cqfd} \end{array} \right.$$

$$H(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-1 & h \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \frac{1}{(z-1)^2}$$

$$H(z) = \begin{bmatrix} z-1 & h \end{bmatrix} \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \frac{1}{(z-1)^2} = \frac{(z-1)h^2/2 + h^2}{(z-1)^2} = \frac{h^2}{2}\frac{z+1}{(z-1)^2}$$

Stabilité

$$H\left(z\right) = C\left(zI - \Phi\right)^{-1}\Gamma + D$$

$$H(z) = \frac{C\mathrm{adj}(zI - \Phi)\Gamma}{\det(zI - \Phi)} + D = \frac{H^{*}(z)}{\det(zI - \Phi)}$$

$$H\left(z\right) = \begin{bmatrix} H_{11}\left(z\right) & \ldots & H_{1r}\left(z\right) \\ \vdots & & \vdots \\ H_{p1}\left(z\right) & \cdots & H_{pr}\left(z\right) \end{bmatrix} = \left[H_{ij}\left(z\right)\right] = \left[\frac{H_{ij}^{*}\left(z\right)}{\det\left(zI - \Phi\right)}\right]$$

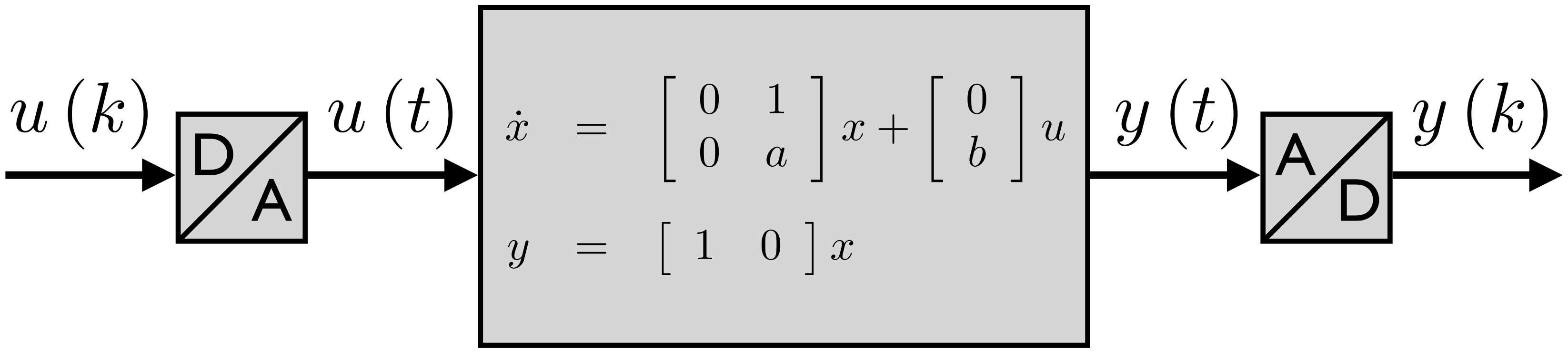
Pôles z_i des H_{ij} solution de: $\det\left(zI - \Phi\right) = 0$

Valeurs propres v_i de Φ solution de: $\det\left(\lambda I - \Phi\right) = 0$

$$z_i = v_i$$

Asymptotiquement stable si: $|v_i| < 1$ pour $i = 1, \ldots, n$

Entraînement discret



$$\left\{ \begin{array}{rcl} x(k+1) & = & \Phi x(k) + \Gamma u(k) \\ y(k) & = & C x(k) \end{array} \right.$$

$$\Phi = e^{Ah} = \begin{bmatrix} 1 & \frac{1}{a}\left(e^{ah}-1\right) \\ 0 & e^{ah} \end{bmatrix} \qquad \Gamma = \frac{b}{a} \begin{bmatrix} \frac{1}{a}\left(e^{ah}-1\right) - h \\ \left(e^{ah}-1\right) \end{bmatrix}$$

exemple: 4.1.12

$$(zI - \Phi)^{-1} = \frac{Iz + H_1}{z^2 + a_1 z + a_2} = \frac{1}{\left(z - e^{ah}\right)(z-1)} \begin{bmatrix} \left(z - e^{ah}\right) & \frac{1}{a}\left(e^{ah}-1\right) \\ 0 & z-1 \end{bmatrix}$$

$H(z) = C(zI - \Phi)^{-1}\Gamma + D$ Inutile ! Les valeurs propres donnent la même info

$$|\lambda I - \Phi| = \begin{bmatrix} \lambda - 1 & -\frac{1}{a}\left(e^{ah}-1\right) \\ 0 & \lambda - e^{ah} \end{bmatrix} = (\lambda - 1)\left(\lambda - e^{ah}\right) = 0 \rightarrow \left\{ \begin{array}{l} \lambda_1 = z_1 = 1 \\ \lambda_2 = z_2 = e^{ah} \end{array} \right.$$

Systèmes discrets linéaires et stationnaires

Solution

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma u(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

$$x(k) = \underbrace{\Phi^{k-k_0} x(k_0)}_{\text{Réponse libre}} + \underbrace{\sum_{l=k_0}^{k-1} \Phi^{k-l-1} \Gamma u(l)}_{\text{Réponse forcée}}$$

Matrice de transfert et stabilité

$$Y(z) = \left[C (zI - \Phi)^{-1} \Gamma + D \right] U(z) = H(z) U(z)$$

$$H(z) = \begin{bmatrix} H_{11}(z) & \dots & H_{1r}(z) \\ \vdots & & \vdots \\ H_{p1}(z) & \cdots & H_{pr}(z) \end{bmatrix} = \left[H_{ij}(z) \right] = \left[\frac{H^*_{ij}(z)}{\det(zI - \Phi)} \right]$$

Pôles z_i des H_{ij} solution de: $\det(zI - \Phi) = 0$

Valeurs propres v_i de Φ solution de: $\det(\lambda I - \Phi) = 0$

$$z_i = v_i$$

Asymptotiquement stable si: $|v_i| < 1$ pour $i = 1, \dots, n$