NON-EXISTENCE OF FIBERWISE LOCALIZATION FOR CROSSED MODULES

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Abstract. We prove that localization functors of crossed modules of groups do not always admit fiberwise (or relative) versions. To do so we characterize the existence of a fiberwise localization by a certain normality condition and compute explicit examples and counter-examples. In fact, some nullification functors do not behave well and we also prove that the fiber of certain nullification functors, known as acyclization functors in other settings such as groups or spaces, is not acyclic.

Introduction

As soon as localization functors have been introduced in homotopy theory and algebra, fiberwise techniques have been developed in order to reduce certain questions about extensions to easier ones. In homotopy theory for example May, [May80], highlighted the fundamental role of fiberwise localization in Sullivan's influential article [Sul74] on the Adams conjecture. Let us also mention the use of fiberwise plus constructions in algebraic K-theory, which explains how Quillen's plus construction is related to the lower K-theory groups, see Berrick's book [Ber82], or Arlettaz' survey [Arl00, Section 3]. Fiberwise localization also plays a prominent part in Farjoun's [Far96, Chapter I], and the conjunction between general fiberwise techniques and K-theoretical motivations led then Berrick and Farjoun to their work [BF03]. Finally, let us mention that in the modern approach to homotopy theory by ∞-categorical methods fiberwise localization appears in the recent work of Gepner and Kock [GK17], in the form of factorization *systems* in relation to the univalence axiom.

In group theory, Casacuberta and Descheemaeker noticed in [CD05] that localization functors admit a relative version, where group extensions replace fibration sequences. One way to obtain such a construction is to adapt Hilton's construction from [Hil83]. Original applications were related to algebraic K-theory again, via the plus-construction, while more recent computations include purely group theoretical work by Flores and the second author, [FS18], and the study of conditional flatness, see [FS15]. In the latter, fiberwise localization was a key tool to understand the difference between homotopical localization for spaces and group theoretical localization.

However, many arguments one can perform for groups make sense in any semi-abelian category in the sense of Janelidze, Márki, and Tholen, [JMT02]. They provided axioms that capture the properties not only of the categories of groups, but also non-unital rings, crossed modules, Lie algebras, cocommutative Hopf algebras over a field [GSV19], etc. Roughly speaking, semi-abelian categories are to groups what abelian categories are to abelian groups. In joint work with Gran, [GS22], the second author studied thus the behavior of localization functors in an arbitrary semi-abelian category with respect to extensions, with a specific

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focus on the preservation of certain properties under pullbacks. Most statements depend on the existence of fiberwise localization functors and since they always exist in the category of groups, this project grew out of the desire to understand what happens in the next most obvious category of interest to both algebraists and homotopy theorists, namely crossed modules of groups.

Crossed modules have been studied by Whitehead in [Whi49]. They serve as a combinatorial model for connected 2-types, i.e. spaces with vanishing homotopy groups in degrees ≥ 3 and have been used extensively in homotopy theory, see for example work of Brown and Higgins, [BH78]. Crossed modules also enjoy nice categorical properties and have been studied from the algebraic viewpoint. Our starting point is Norrie's [Nor90], where she establishes most of the constructions we need for crossed modules of groups. Janelidze extended this widely and defined the notion of internal crossed modules in any semi-abelian category [Jan03]. He generalized the result of Brown and Spencer [BS76] and proved that the category of internal crossed modules in a semi-abelian category $\mathcal{C}$ is equivalent to the category of internal groupoids in $\mathcal{C}$, which forms again a semi-abelian category [BG02].

When working with extensions of crossed modules and localization functors, one wishes to have a fiberwise or a relative version at hand. It means that if $1 \to \mathsf{N} \to \mathsf{T} \to \mathsf{Q} \to 1$ is a short exact sequence of crossed modules, we are looking for a natural transformation to a new sequence $1 \to \mathsf{LN} \to \mathsf{E} \to \mathsf{Q} \to 1$ where the morphism $\mathsf{N} \to \mathsf{LN}$ is the localization coaugmentation ℓ^{N} and $\mathsf{T} \to \mathsf{E}$ is inverted by L. In the case of groups [CD05], Casacuberta and Descheemaeker gave an explicit description of such a construction for any short exact sequence of groups by using the notions of actions and semi-direct products. With similar tools introduced by Norrie [Nor90], we thought it would be possible to adapt the construction for groups to the case of crossed modules. Surprisingly, this approach does not work even when we restrict our setting to the case of localization functors L for which the coaugmentation $\mathsf{T} \to \mathsf{LT}$ is a regular epimorphism for all crossed modules T. It turns out that fiberwise localization unexpectedly fails to exist in general.

Theorem 3.4. *Let* $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ *be a regular-epi localization functor. Let us consider the following exact sequence of crossed modules.*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$$

This exact sequence admits a fiberwise localization if and only if $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ *is a normal subcrossed module of* T*.*

This helps us to understand that even harmless looking nullification functors such as $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$, the functor that kills all copies of the crossed module $0 \to \mathbb{Z}$ concentrated in one degree, do not satisfy this normality condition, as we prove in Theorem 4.5. To our knowledge this is the first example of this kind. We show finally in Proposition 4.6 that the fiber of the same nullification functor, known as acyclization functors in other settings such as groups or spaces, is not acyclic in general. The two phenomena were known to be related (fiberwise nullification and acyclization), see [GS22], but again, this is the first concrete example we know of.

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1. The semi-abelian category of crossed modules

In this first section, we provide the basic definitions, notation, and constructions we use in the category of crossed modules. We describe in particular pushouts and cokernels for crossed modules. We follow Norrie [Nor90] and Brown-Higgins [BH78].

Definition 1.1. [Whi49] A *crossed module of groups* is given by a morphism of groups $\partial^{\mathsf{X}}\colon X_1 \to X_2$ endowed with an action by automorphisms of groups of X_2 on X_1, denoted by $X_2 \times X_1 \to X_1\colon (b,x) \mapsto {}^b x$, such that for any b in X_2 and any x, y in X_1,

$$\partial^{\mathsf{X}}({}^b x) = b\partial^{\mathsf{X}}(x)b^{-1}, \tag{1}$$

$$ {}^{\partial^{\mathsf{X}}(x)}y = xyx^{-1}. \tag{2}$$

Hence a crossed module is a triple $(X_1, X_2, \partial^{\mathsf{X}})$ and we will sometimes refer to ∂^X as the *connecting morphism*. For the sake of readability, we will also use the notation X for such a crossed module. We give several examples of this notion.

Example 1.2.
(1) The inclusion of a normal subgroup N of M is a crossed module where the action of M on N is given by the conjugation ${}^m n = mnm^{-1}$. As a particular example, the identity morphism $G = G$ provides a way to construct a crossed module from a single group.
(2) For any group G, the inclusion of the trivial group $1 \rightarrowtail G$ endowed with the trivial action is a crossed module.
(3) Let $g : A \to B$ be a surjective morphism of groups such that its kernel is included in the center of A, i.e. $ker(g) \subseteq Z(A) = \{x \in A \mid xa = ax,\ \forall a \in A\}$. There exists an action of B on A via $B \times A \to A\colon (b,a) \mapsto xax^{-1}$, where $g(x) = b$. This action is well defined since $ker(g) \subseteq Z(A)$. One can check that this gives to the morphism g the structure of a crossed module.

Definition 1.3. Let $\mathsf{N} := (N_1, N_2, \partial^{\mathsf{N}})$ and $\mathsf{M} := (M_1, M_2, \partial^{\mathsf{M}})$ be two crossed modules, a *morphism of crossed modules* is given by a pair of group homomorphisms $\alpha := (\alpha_1, \alpha_2) : (N_1 \to M_1, N_2 \to M_2)$ such that the two following diagrams commute

$$\begin{array}{ccc} N_1 & \xrightarrow{\alpha_1} & M_1 \\ {\scriptstyle \partial^{\mathsf{N}}}\downarrow & & \downarrow{\scriptstyle \partial^{\mathsf{M}}} \\ N_2 & \xrightarrow[\alpha_2]{} & M_2 \end{array} \qquad \begin{array}{ccc} N_2 \times N_1 & \longrightarrow & N_1 \\ {\scriptstyle (\alpha_2,\alpha_1)}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ M_2 \times M_1 & \longrightarrow & M_1. \end{array}$$

where the horizontal arrows in the right diagram are the respective group actions of the two crossed modules.

The two definitions above give rise to the category XMod of crossed modules of groups. We remark that there is an embedding of the category of groups in this category via two functors which have been introduced on the objects in Example 1.2. These functors are respectively left and right adjoint to the truncation functor $Tr\colon \mathsf{XMod} \to \mathsf{Grp}$ that sends a crossed module $\mathsf{T} := (T_1, T_2, \partial^{\mathsf{T}})$ to T_2. This will help us to import group theoretical results into XMod.

Lemma 1.4. *The functor* $\mathsf{X}\colon \mathsf{Grp} \to \mathsf{XMod}$ *which sends a group* G *to the crossed module* $\mathsf{X}G = (1, G, 1)$ *reduced to the group* G *at level* 2 *is left adjoint to the truncation functor* Tr. *The functor* $\mathsf{R}\colon \mathsf{Grp} \to \mathsf{XMod}\colon G \mapsto (G, G, Id_G)$ *is right adjoint to the truncation functor* Tr.

Proof. We have natural isomorphisms $Hom_{\mathsf{XMod}}(\mathsf{X}G, \mathsf{T}) \cong Hom_{\mathsf{Grp}}(G, T_2)$ for any crossed module T and likewise $Hom_{\mathsf{XMod}}(\mathsf{T}, \mathsf{R}G) \cong Hom_{\mathsf{Grp}}(T_2, G)$. □

There is an obvious notion of subcrossed module, see [Nor90]. One simply requires the subobject to be made levelwise of subgroups, the connecting homomorphism and the action are induced by the given connecting homomorphism and action. In other words, we have the following definition.

Definition 1.5. Let $(i_1, i_2)\colon \mathsf{N} \to \mathsf{T}$ be a morphism of crossed modules, if i_1 and i_2 are group inclusions we say that N is a *subcrossed module* of T.

For N a subcrossed module of $\mathsf{T} = (T_1, T_2, \partial^{\mathsf{T}})$, we introduce a subgroup of T_1 denoted as follows

$$[N_2, T_1] := \langle\ {}^{n_2}t_1 t_1^{-1} \mid t_1 \in T_1, n_2 \in N_2 \rangle.$$

Let us notice that the commutator subgroup of T_1, $[T_1, T_1] = \langle t_1 t_1' t_1^{-1} t_1'^{-1} \mid t_1, t_1' \in T_1 \rangle$, is included in $[T_2, T_1]$. Indeed, we have the following inclusions via condition (2) of crossed modules:

$$[T_1, T_1] = [\partial(T_1), T_1] \subseteq [T_2, T_1].$$

What is less obvious maybe and the source of interesting phenomena in XMod that one cannot see within the category of groups is the notion of normality and thus of quotient or cokernel.

Definition 1.6. A subcrossed module $\mathsf{N} := (N_1, N_2, \partial^{\mathsf{N}})$ of $\mathsf{T} := (T_1, T_2, \partial^{\mathsf{T}})$ is a *normal subcrossed module* if the following conditions hold

(1) N_2 is a normal subgroup of T_2;
(2) For any $t_2 \in T_2$ and $n_1 \in N_1$; ${}^{t_2}n_1 \in N_1$.
(3) $[N_2, T_1] \subseteq N_1$

We recall from [BH78; LG94] the construction of the quotient. It illustrates the fact that colimits are not straightforward to construct in the category of crossed modules, but in the case of cokernels we have an explicit formula that will be very useful in concrete computations. Let $G \times H \to H$ be an action of groups and S a subgroup of H, we denote by S_G the closure of S via the action of G:

$$S_G := \langle\ {}^{g}s \mid g \in G, s \in S \rangle.$$

Definition 1.7. Let $f\colon \mathsf{H} \to \mathsf{T}$ be a morphism of crossed modules. The *cokernel* of f is the crossed module $\mathsf{coker}f$ given by the following morphism of crossed modules

$$\begin{array}{ccc} T_1 & \longrightarrow & T_1/\big(f_1(H_1)_{T_2}[f_2(H_2)_{T_2}, T_1]\big) \\ {\scriptstyle \partial^{\mathsf{T}}}\downarrow & & \downarrow{\scriptstyle \tilde{\partial}^{\mathsf{T}}} \\ T_2 & \longrightarrow & coker(f_2) \end{array}$$

where $\tilde{\partial}^{\mathsf{T}}$ is induced by the universal property of the cokernel of groups.

Remark 1.8. Note that when H is a normal subcrossed module of T the above definition of cokernel coincides simply with the levelwise quotient by the normal subgroups $H_1 \triangleleft T_1$ and $H_2 \triangleleft T_2$.

The kernel of a morphism of crossed modules is defined "component-wise" as in the category of groups. More precisely, let $(\alpha_1, \alpha_2)\colon \mathsf{A} \to \mathsf{B}$ be a morphism of crossed modules. The kernel of this morphism denoted by $\mathsf{ker}(\alpha_1, \alpha_2)$ is given by

$$(ker(\alpha_1), ker(\alpha_2), \hat{\partial}^{\mathsf{A}}),$$

where $\hat{\partial}^{\mathsf{A}}$ is induced by the universal property of $ker(\alpha_2)$.

Remark 1.9. We recall that in XMod it is equivalent to being a normal subcrossed module or the kernel of some morphism (a normal monomorphism) [Nor90].

More generally, all limits are computed "component-wise" as in the category of groups. For example, pullbacks in XMod are built as follows [LG94]. Let $f\colon \mathsf{T} \to \mathsf{Q}$ and $g\colon \mathsf{Q}' \to \mathsf{Q}$ be two morphisms of crossed modules. Then the pullback of f along g is given by the following square

$$\begin{array}{ccc} \mathsf{T}' & \xrightarrow{\pi_{\mathsf{Q}'}} & \mathsf{Q}' \\ {\scriptstyle \pi_{\mathsf{T}}}\downarrow & & \downarrow{\scriptstyle g} \\ \mathsf{T} & \xrightarrow[f]{} & \mathsf{Q} \end{array}$$

The object part T' of the pullback is built component-wise as in the case of groups

$$(T_1 \times_{Q_1} Q'_1, T_2 \times_{Q_2} Q'_2, \partial'),$$

where ∂' and the action are induced by the universal property of the pullbacks in Grp. The projections are the natural ones, given also component-wise.

In contrast to the limits, which are built component-wise, colimits are not. In particular, the construction of cokernels is not as straightforward as the case of groups, as we saw in Definition 1.7. We refer to [BH78, Proposition 11] for the description of pushouts in XMod. Note that thanks to the adjunction of Lemma 1.4, the "second level" of the pushout of crossed modules is always constructed as in Grp.

The category of crossed modules is semi-abelian, as shown in [JMT02]. This notion has been introduced by Janelidze, Márki, and Tholen in [JMT02].

Semi-abelian categories enjoy many nice properties we will use in the following sections, such as the traditional homological lemmas, [BB04], the Split Short Five Lemma, [Bou06], the Noether Isomorphism Theorems, [BB04], and that one can recognize pullbacks by looking at kernels or cokernels, [BB04, Lemmas 4.2.4 and 4.2.5]. For the sake of completeness, we recall Lemma 4.2.4 in [BB04], which will be useful several times in this article.

Proposition 1.10 (Lemma 4.2.4 [BB04])**.** *Let $\mathcal{C}$ be a pointed category. We consider the following diagram where κ is the kernel of α*

$$\begin{array}{ccccc} N' & \xrightarrow{\kappa'} & T' & \xrightarrow{\alpha'} & Q' \\ {\scriptstyle u}\downarrow & (1) & {\scriptstyle v}\downarrow & (2) & \downarrow{\scriptstyle w} \\ N & \xrightarrow{\kappa} & T & \xrightarrow{\alpha} & Q \end{array}$$

(1) *If w is a monomorphism then $\kappa' = ker(\alpha')$ if and only if (1) is a pullback;*
(2) *When (2) is a pullback and $\alpha' \circ \kappa'$ is the zero morphism, κ' is the kernel of α' if and only if u is an isomorphism.*

Remark 1.11. The relevant categorical notion of epimorphism in this context is that of regular epimorphism (a coequalizer of a pair of parallel arrows). In the category of crossed modules, a morphism $f = (f_1, f_2)$ is a regular epimorphism if and only if it is surjective on each component, i.e. f_1 and f_2 are surjective group homomorphisms [LLR04, Proposition 2.2]. Moreover, we note that each surjective morphism is an epimorphism but there exist epimorphisms that are not surjective . Since XMod is a pointed protomodular category, regular epimorphisms and normal epimorphisms (the cokernel of some morphism) coincide.

2. Localization functors

We recall the definition of localization functors, which we describe for crossed modules.

Definition 2.1. A *localization* functor in the category of crossed modules is a coaugmented idempotent functor $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$. The coaugmentation is a natural transformation $\ell\colon \mathsf{Id} \to \mathsf{L}$. Both ℓ^{LX} and $\mathsf{L}\ell^{\mathsf{X}}$ are isomorphisms and in particular we have $\ell^{\mathsf{LX}} = \mathsf{L}\ell^{\mathsf{X}}$, see [Cas00, Proposition 1.1].

Definition 2.2. Let L be a localization functor. A crossed module T is L-*local* if the coaugmentation morphism $\ell^{\mathsf{T}}\colon \mathsf{T} \to \mathsf{LT}$ is an isomorphism. A morphism $f\colon \mathsf{N} \to \mathsf{M}$ is an L-*equivalence* if $\mathsf{L}f$ is an isomorphism.

Here are a few basic and useful properties of L-equivalences.

Lemma 2.3. (1) *The pushout of an L-equivalence is an L-equivalence.*
(2) *The composition of L-equivalences is an L-equivalence.*
(3) *A κ-filtered colimit of a diagram T_β of L-equivalences $\mathsf{T}_\beta \to \mathsf{T}_{\beta+1}$ for all successor ordinals $\beta + 1 < \kappa$ yields an L-equivalence $\mathsf{T}_0 \to \mathsf{T}_\kappa = \mathsf{colim}_{\beta<\kappa}\mathsf{T}_\beta$.*
(4) *Let F be an I-indexed diagram of L-equivalences in the category of morphisms of crossed modules. Then the colimit $\mathsf{colim}_{\mathsf{I}}\mathsf{F}$ is an L-equivalence.*

Proof. Property (4) is [Cas00, Proposition 1.3]. Properties (1), (2) and (3) follow. □

Sometimes one comes across a localization functor by finding a full reflexive subcategory $\mathcal{L}$ of XMod (for example of all abelian crossed modules). The pair of adjoint functors $\mathsf{U}\colon \mathcal{L} \leftrightarrows \mathsf{XMod}\colon \mathsf{F}$ yields a localization functor $\mathsf{L} = \mathsf{FU}$. This is the approach taken by Cassidy, Hébert, and Kelly in [CHK85]. There are other situations where one constructs localization functors by fixing a set of morphisms which are required to become isomorphisms after localization. In this way any set of morphisms S defines a localization functor L_S inverting the elements of S. By letting f be the coproduct of all morphisms in S it is good enough to

study localization functors of the form L_f. The existence and construction of such functors follow from very general results, see for example Bousfield's foundational work [Bou76] or the more recent account by Hirschhorn [Hir03] in a model categorical setting.

Definition 2.4. Let f be a morphism of crossed modules. A crossed module T is L_f-*local* if $Hom(f, \mathsf{T})$ is an isomorphism. A morphism g in XMod is an L_f-*equivalence* if $Hom(g, \mathsf{T})$ is an isomorphism for any L_f-local T.

In this context, local objects and local equivalences coincide with the notion introduced previously in Definition 2.2. The properties we stated in Proposition 2.3 are then analogous to [Hir03, Proposition 1.2.21] for the first claim, by the universal property of a pushout, and the second statement follows by induction as [Hir03, Proposition 1.2.20]. Note that L-equivalences and L-local objects defined in Definition 2.2 can also be expressed via their universal properties as in Definition 2.4.

Proposition 2.5. *A crossed module* T *is* L-local *if* $Hom(h, \mathsf{T})$ *is an isomorphism for any* L-*equivalence* h*. A morphism* g *in* XMod *is an* L-equivalence *if* $Hom(g, \mathsf{T})$ *is an isomorphism for any* L-*local* T*.*

Definition 2.6. Let f be a morphism of the form $\mathsf{A} \to 1$, where A is a crossed module. The localization functor L_f is written P_A and is called a *nullification* functor. One calls the local objects A-*null* or A-*local* and a crossed module T such that $\mathsf{P}_\mathsf{A}\mathsf{T} = 1$ is called A-*acyclic*.

In this article localization functors having the property that the coaugmentation morphism is always a regular epimorphism will play a central role.

Definition 2.7. A localization functor is called a *regular-epi localization* if for any crossed module T the morphism $\ell^\mathsf{T} \colon \mathsf{T} \to \mathsf{LT}$ is a regular epimorphism.

To prove that any nullification functor is a regular-epi localization we will need to describe in a precise way how $\mathsf{P}_\mathsf{A}\mathsf{T}$ is constructed by successively killing all maps from A to T.

Proposition 2.8. *Let* A *be a crossed module. Then* P_A *is a regular-epi localization.*

Proof. Let T be a crossed module. Consider the coproduct $\coprod \mathsf{A}$ taken over $Hom(\mathsf{A}, \mathsf{T})$ and form an evaluation morphism $ev \colon \coprod \mathsf{A} \to \mathsf{T}$ where the component map $\mathsf{A} \to \mathsf{T}$ indexed by the morphism g is precisely g. We then define $\mathsf{T}_0 = \mathsf{T}$ and T_1 is the pushout of the diagram

$$1 \leftarrow \coprod \mathsf{A} \xrightarrow{\mathsf{ev}} \mathsf{T}_0$$

In other words, T_1 is the cokernel of the evaluation map as described in Definition 1.7, so $\mathsf{T}_0 \to \mathsf{T}_1$ is a regular epimorphism. It is also a P_A-equivalence, being the pushout of the P_A-equivalence $\coprod \mathsf{A} \to 1$ (see Lemma 2.3).

For each successor ordinal $\beta + 1$, the crossed module $\mathsf{T}_{\beta+1}$ is defined as above, $\mathsf{T}_{\beta+1} = (\mathsf{T}_\beta)_1$, and for a limit ordinal β, we define $\mathsf{T}_\beta = \mathsf{colim}_{\gamma<\beta}\mathsf{T}_\gamma$. As a composition of regular epimorphisms is again a regular epimorphism we obtain by induction that the morphism $\mathsf{T} \to \mathsf{T}_{\beta+1}$ is a regular epimorphism. The case of limit ordinals is taken care of by the fact that regular epimorphisms are coequalizers, which are preserved under colimits.

To finish the proof we have to explain why this process stops, which follows from Quillen's small object argument, [Hir03, Proposition 10.5.16] or [Qui67], as is well-known.

We choose λ to be the first infinite ordinal greater than the number of chosen generators of A_1 and A_2. This implies that a morphism out of A is determined by strictly less than λ

images of elements so that A is λ-small with respect to $\mathsf{T}_\lambda = \mathsf{colim}_{\beta<\lambda}\mathsf{T}_\beta$, i.e. any morphism $g\colon \mathsf{A} \to \mathsf{T}_\lambda$ factors through some intermediate stage T_β for a certain $\beta < \lambda$. This comes from the fact that filtered colimits are created in sets and every chosen generator t must be sent to some T_{α_t}. The ordinal β is then the union of all α_t's.

Therefore g becomes trivial in $\mathsf{T}_{\beta+1}$, which shows that the crossed module $\mathsf{P}_\mathsf{A}\mathsf{T} := \mathsf{T}_\lambda$ is A-null. The map $\mathsf{T} \to \mathsf{P}_\mathsf{A}\mathsf{T}$ is an A-equivalence by Lemma 2.3 since it is obtained by iterating pushouts along P_A-equivalences. □

Sometimes it is handy to rely on our group theoretical knowledge to construct simple examples of localization functors and how they behave on crossed modules. Recall the functor X from Lemma 1.4.

Proposition 2.9. *Let $\varphi\colon G \to H$ be a group homomorphism. The localization functor $\mathsf{L}_{\mathsf{X}\varphi}$ verifies $\mathsf{L}_{\mathsf{X}\varphi}\mathsf{X}T \cong \mathsf{X}\mathsf{L}_\varphi T$ for any group T.*

Proof. The adjunction in Lemma 1.4 tells us that a crossed module A is $\mathsf{X}\varphi$-local if and only if A_2 is a φ-local group. In particular, $\mathsf{X}\mathsf{L}_\varphi T$ is $\mathsf{X}\varphi$-local. Moreover, X sends L_φ-equivalences to $\mathsf{L}_{\mathsf{X}\varphi}$-equivalences. □

In this situation, it is thus easy to localize a crossed module that is reduced to a group (granted that we know how to localize groups), but the effect on arbitrary crossed modules can be more surprising because the groups at level one are linked to the groups at level two via the connecting homomorphism.

The end of this section is devoted to illustrating the notion of localization functor by a handful of natural examples. We give a non-exhaustive list of examples of localization functors of crossed modules: some of them are obtained by using the construction of nullification functors (Example 2.10, Example 2.11), some are built using well-known constructions (Example 2.12, Example 2.13), and some are induced by an adjunction (Example 2.15). It is interesting to notice that some of the following examples already appear in the literature. In particular, the subcategories induced by the local objects of Example 2.10 and Example 2.11 form a hereditary torsion theory [BG06]. We start with an important functor as it will be the key player in our counter-examples.

Example 2.10. The nullification functor $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ with respect to the crossed module $\mathsf{X}\mathbb{Z}$ is described as follows

$$\mathsf{P}_{\mathsf{X}\mathbb{Z}}\begin{pmatrix} N_1 \\ {\scriptstyle\partial}\downarrow \\ N_2 \end{pmatrix} = \begin{matrix} N_1/[N_2,N_1] \\ \downarrow \\ 1 \end{matrix}$$

In this example, the construction detailed in Proposition 2.8 can be done in a single step, since the coproduct of all morphisms $\mathsf{X}\mathbb{Z} \to \mathsf{N}$ comes from a surjective group homomorphism $F \to N_2$, where F is a free group. Hence, in the first step of the construction we construct the quotient of $\mathsf{X}F \to \mathsf{N}$ by killing its normal closure as introduced in Definition 1.7. This kills obviously N_2 and quotients N_1 out by $inc(1)_{N_2}[Id(N_2)_{N_2}, N_1] = [N_2, N_1]$.

The map from N to this quotient is an $\mathsf{X}\mathbb{Z}$-equivalence being the pushout of an equivalence and this quotient is local (the bottom group is trivial). From the point of view of reflexive subcategories, this localization functor corresponds to the reflector associated with the subcategory of crossed modules of the form $(A, 1)$ where A is any abelian group and the connecting homomorphism is the trivial homomorphism.

Example 2.11. We give the explicit description of the functor of nullification respectively to the crossed module $\mathbb{Z} \to 0$. The functor is then defined as follows:

$$\mathsf{P}_{\mathbb{Z}\to 0}\begin{pmatrix} N_1 \\ {\scriptstyle \partial^{\mathsf{N}}}\downarrow \\ N_2 \end{pmatrix} = \begin{matrix} \partial^{\mathsf{N}}(N_1) \\ {\scriptstyle inc}\downarrow \\ N_2 \end{matrix}$$

Let us notice that the objects of the subcategory of $P_{\mathbb{Z}\to 0}$-local objects are inclusions of normal subgroups as in Example 1.2. In terms of internal groupoids, this construction can be seen as the reflector of the category of internal groupoids to the category of internal equivalence relations (see example 2.5 in [GE13]).

The next two examples already appeared in [Nor87].

Example 2.12. Another localization functor is given by the abelianization functor. We denote this functor by $\mathsf{Ab}\colon \mathsf{XMod} \to \mathsf{XMod}$ and it is defined as follows

$$\mathsf{Ab}\begin{pmatrix} N_1 \\ {\scriptstyle \partial}\downarrow \\ N_2 \end{pmatrix} = \begin{matrix} N_1/[N_2, N_1] \\ {\scriptstyle \tilde{\partial}}\downarrow \\ N_2/[N_2, N_2] \end{matrix}$$

Example 2.13. The abelianization functor can be generalized and we can define the nilpotent functors. Indeed, in [Nor87], Norrie defined the notion of lower central series. Hence, we can quotient any crossed module by the k-th term in its lower central series and obtain a functor

$$\mathsf{Nil}_{\mathsf{k}}\colon \mathsf{XMod} \to \mathsf{XMod}\colon \mathsf{G} \mapsto \mathsf{G}/\Gamma_{\mathsf{k}}(\mathsf{G})$$

where $\Gamma_{\mathsf{k}}(\mathsf{G})$ is the k-th term in the lower central series of G. We give an explicit description of the functor $\mathsf{Nil}_2\colon \mathsf{XMod} \to \mathsf{XMod}$.

$$\mathsf{Nil}_2\begin{pmatrix} N_1 \\ {\scriptstyle \partial}\downarrow \\ N_2 \end{pmatrix} = \begin{matrix} N_1/{<}\,[[N_2, N_2], N_1], [[N_2, [N_2, N_1]]\,{>} \\ {\scriptstyle \tilde{\partial}}\downarrow \\ N_2/[[N_2, N_2], N_2] \end{matrix}$$

Example 2.14. We can also consider the localization functor $\mathsf{C}\colon \mathsf{XMod} \to \mathsf{XMod}$ defined by

$$\mathsf{C}\begin{pmatrix} N_1 \\ {\scriptstyle \partial^{\mathsf{N}}}\downarrow \\ N_2 \end{pmatrix} = \begin{matrix} 1 \\ \downarrow \\ N_2/\partial^{\mathsf{N}}(N_1) \end{matrix}$$

In fact, the localization functor C is exactly the nullification functor $\mathsf{P}_{\mathbb{Z}\xrightarrow{id}\mathbb{Z}}$.

Example 2.15. We give a final example of a functor of localization of crossed modules $\mathsf{I}\colon \mathsf{XMod} \to \mathsf{XMod}$

$$\mathsf{I}\begin{pmatrix} N_1 \\ {\scriptstyle \partial^{\mathsf{N}}}\downarrow \\ N_2 \end{pmatrix} = \begin{matrix} N_2 \\ {\scriptstyle Id}\downarrow \\ N_2 \end{matrix}$$

This localization functor is induced by the adjunction between the truncation functor, introduced in Lemma 1.4, $Tr\colon \mathsf{XMod} \to \mathsf{Grp}\colon (T_1, T_2, \partial^{\mathsf{T}}) \mapsto T_2$ and the functor $\mathsf{R}\colon \mathsf{Grp} \to \mathsf{XMod}$ sending T to (T, T, Id), which plays here the role of right adjoint of Tr.

Remark 2.16. The functors considered in Examples 2.10, 2.11, 2.12, 2.13 and 2.14 are regular-epi localizations. In particular, every nullification functor is a regular-epi localization. However, the converse is not true as illustrated by the functor Ab of Example 2.12. There is a large collection of regular-epi localization functors. Indeed, similarly to the observation in [Cas95], if f is a regular epimorphism, then the functor L_f is a regular-epi localization functor.

From now on, every localization functor that we consider is a regular-epi localization.

3. Construction of fiberwise localization

In this section, we study the concept of fiberwise localization for an exact sequence of crossed modules. We show how to construct such a fiberwise localization when the exact sequence satisfies a certain normality condition, reminiscent of condition (N) in [GE13]. The authors introduced this condition (N) to obtain a weaker context (than abelian categories), in which a torsion theory gives rise to a monotone-light factorization system. At the end of the section, we investigate this condition in detail and show it is necessary to obtain a fiberwise localization.

Definition 3.1. Let $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ be a localization functor. An exact sequence

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$$

admits a fiberwise localization if there exists such a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathsf{N} & \xrightarrow{\kappa} & \mathsf{T} & \xrightarrow{\alpha} & \mathsf{Q} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \ell^{\mathsf{N}}} & & \downarrow{\scriptstyle g} & & \| & & \\ 1 & \longrightarrow & \mathsf{LN} & \xrightarrow{j} & \mathsf{E} & \xrightarrow{p} & \mathsf{Q} & \longrightarrow & 1 \end{array}$$

where g is an L-equivalence.

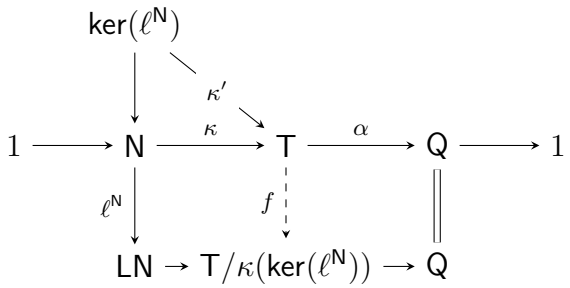
We give a sufficient condition on this exact sequence to construct a fiberwise localization.

Proposition 3.2. *Let $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ be a regular-epi localization functor. Any exact sequence of crossed modules such that $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is a normal subcrossed module of T,*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$$

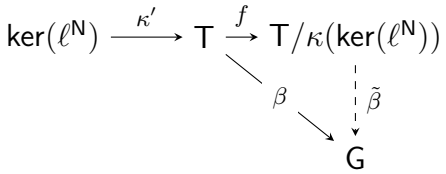
admits a fiberwise localization.

Proof. By assuming that $\ell^{\mathsf{N}}\colon \mathsf{N} \to \mathsf{LN}$ is a regular epimorphism and that $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is a normal crossed module of T, we can construct the following diagram

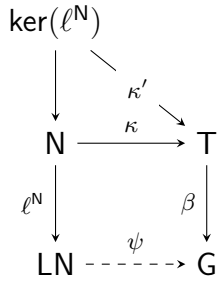


where f is the cokernel of the normal morphism $\kappa' = \kappa\,|_{\mathsf{ker}(\ell^{\mathsf{N}})} \colon \mathsf{ker}(\ell^{\mathsf{N}}) \to \mathsf{T}$. The lower sequence is exact via the first and second isomorphism theorems for crossed modules (Theorems 2.1 and 2.2 in [Nor87]).

To end this proof, we need to show that $f \colon \mathsf{T} \to \mathsf{T}/\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is an L-equivalence. Let G be a local object in XMod and $\beta \colon \mathsf{T} \to \mathsf{G}$ be a morphism of crossed modules. We need to prove that there exists a unique morphism of crossed modules $\tilde{\beta}$ from $\mathsf{T}/\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ to G (Proposition 2.5). This morphism is induced by the universal property of the cokernel f and the universal property of the localization. To use the universal property of the cokernel f



we need to prove that $\beta \circ \kappa'$ is the zero morphism. This can be deduced from the commutativity of the following diagram



where ψ is induced by the universal property of the localization. So we can conclude that f is an L-equivalence. Hence we built a fiberwise localization. □

In the previous proposition, we gave a condition on the exact sequence of crossed modules ensuring the existence of a fiberwise localization. Now, we prove that this condition is actually mandatory.

Proposition 3.3. *Let* $\mathsf{L} \colon \mathsf{XMod} \to \mathsf{XMod}$ *be a regular-epi localization functor. If the following exact sequence of crossed modules*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1 \tag{3}$$

admits a fiberwise localization, then the kernel $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ *is a normal subcrossed module of* T.

Proof. Suppose we have a fiberwise localization for the exact sequence (3) with ℓ^{N} a regular epimorphism. It means that there exists $\mathsf{E} \in \mathsf{XMod}$ and a diagram

$$\begin{array}{ccccccccc}
 & & & & \mathsf{ker}(\ell^{\mathsf{N}}) & \longrightarrow & \mathsf{ker}(f) & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathsf{N} & \xrightarrow{\kappa} & \mathsf{T} & \xrightarrow{\alpha} & \mathsf{Q} & \longrightarrow & 1 \\
 & & \ell^{\mathsf{N}} \downarrow & & (1) & & f \downarrow & & \| \\
1 & \longrightarrow & \mathsf{LN} & \xrightarrow{j} & \mathsf{E} & \xrightarrow{p} & \mathsf{Q} & \longrightarrow & 1
\end{array}$$

We use Proposition 1.10 to observe that (1) is a pullback and that $\mathsf{ker}(\ell^{\mathsf{N}})$ is isomorphic to $\mathsf{ker}(f)$. Hence, we can conclude that $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is a normal subcrossed module of T. □

Thanks to the two previous propositions we can now state the following theorem.

Theorem 3.4. *Let* $\mathsf{L}\colon \mathsf{XMod} \to \mathsf{XMod}$ *be a regular-epi localization functor. An exact sequence of crossed modules*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1 \tag{4}$$

admits a fiberwise localization if and only if $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ *is a normal subcrossed module of* T. □

Remark 3.5. This theorem can be generalized and holds in any semi-abelian category, [GS22]. Our restricted setup, namely that of crossed modules allows us to produce concrete examples where one can actually check the normality condition.

We emphasize that the normality condition for $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ in Theorem 3.4 is actually the same as the one called (N) in [GE13]. It is interesting to notice that this condition appears in completely different contexts and for different purposes. In our case, it is the adequate condition to obtain a fiberwise localization. In [GE13], they require that a torsion theory (in a normal category) satisfies this condition to obtain a monotone-light factorization system. We investigate this condition of normality in the category of crossed modules and reexpress it as an easier statement to verify. Indeed, we prove that several items in the definition of normal subcrossed modules always hold.

Proposition 3.6. *Let* $\kappa\colon N := (N_1, N_2, \partial^{\mathsf{N}}) \to \mathsf{T} := (T_1, T_2, \partial^{\mathsf{T}})$ *be a normal monomorphism of crossed modules, then* $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ *is a subcrossed module of* T *and we have the two following properties:*

(i) $\kappa_2(ker(\ell_2^{\mathsf{N}}))$ *is a normal subgroup of* T_2.
(ii) For any $t_2 \in T_2$ *and* $n_1 \in ker(\ell_1^{\mathsf{N}})$ *then* ${}^{t_2}\kappa_1(n_1) \in \kappa_1(ker(\ell_1^{\mathsf{N}}))$.

Proof. Using Remark 1.9 we identify N with the normal subcrossed module $\kappa(\mathsf{N})$ and thus omit the use of κ in this proof. It is straightforward to see that $\mathsf{ker}(\ell^{\mathsf{N}})$ is a subcrossed module of T via Definition 1.5. To show properties (i) and (ii) of the lemma we will use the following construction. Since N is a normal subcrossed module of T we have an induced action of T on N as explained in Definition 1.3.5 in [LG94]. In terms of crossed modules, it implies that we have a "conjugation" morphism of crossed modules $c_{t_2} := (\theta_{t_2}, \sigma_{t_2}) \in Aut(N_1, N_2, \partial^{\mathsf{N}})$ depending on an element $t_2 \in T_2$ defined by

$$\begin{array}{ll} \theta_{t_2} : N_1 \to N_1 & \sigma_{t_2} : N_2 \to N_2 \\ n_1 \mapsto {}^{t_2}n_1 & n_2 \mapsto t_2 n_2 t_2^{-1} \end{array}$$

If we consider the morphism $c_{t_2} \colon \mathsf{N} \to \mathsf{N}$ in XMod we can construct the following diagram

$$\begin{array}{ccccc} \mathsf{ker}(\ell^{\mathsf{N}}) & \rightarrowtail & \mathsf{N} & \xrightarrow{\ell^{\mathsf{N}}} & \mathsf{LN} \\ {\scriptstyle c_{t_2}|_{\mathsf{ker}(\ell^{\mathsf{N}})}}\downarrow & & \downarrow{\scriptstyle c_{t_2}} & & \downarrow{\scriptstyle \mathsf{L}c_{t_2}} \\ \mathsf{ker}(\ell^{\mathsf{N}}) & \rightarrowtail & \mathsf{N} & \xrightarrow[\ell^{\mathsf{N}}]{} & \mathsf{LN} \end{array}$$

By definition of the kernel and its universal property, c_{t_2} restricts to the kernel, which implies that properties (i) and (ii) hold. □

This implies that the normality condition of $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ in the crossed module T can be expressed as follows.

Corollary 3.7. *Let $\kappa \colon N := (N_1, N_2, \partial^{\mathsf{N}}) \to \mathsf{T} := (T_1, T_2, \partial^{\mathsf{T}})$ be a normal monomorphism of crossed modules, then $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is a normal subcrossed module of T if and only if we have the following inclusion*

$$[\kappa_2(ker(\ell_2^{\mathsf{N}})), T_1] \subseteq \kappa_1(ker(\ell_1^{\mathsf{N}})) \tag{5}$$

4. Examples and counter-examples for fiberwise localizations

In this section, we illustrate the construction of fiberwise localization in XMod by using the normality condition. It is first interesting to notice that for some particular localization functors, condition (5) is always satisfied. For instance, it is the case if the functor $\mathsf{L} \colon \mathsf{XMod} \to \mathsf{XMod}$ preserves monomorphisms. Surprisingly, there exist examples for which the normality condition does not hold. In contrast to the case of groups or topological spaces, in XMod fiberwise localization does not always exist.

Lemma 4.1. *Let $\mathsf{L} \colon \mathsf{XMod} \to \mathsf{XMod}$ be a regular-epi localization functor that preserves monomorphisms and*

$$1 \longrightarrow \mathsf{N} \xrightarrow{\kappa} \mathsf{T} \xrightarrow{\alpha} \mathsf{Q} \longrightarrow 1$$

be an exact sequence of crossed modules. Then $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is a normal subcrossed module of T.

Proof. Consider the following diagram where $\mathsf{L}\kappa$ is a monomorphism.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathsf{ker}(\ell^{\mathsf{N}}) & \rightarrowtail & \mathsf{N} & \xrightarrow{\ell^{\mathsf{N}}} & \mathsf{LN} \\ & & \downarrow & (1) & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \mathsf{L}\kappa} \\ 1 & \longrightarrow & \mathsf{ker}(\ell^{\mathsf{T}}) & \rightarrowtail & \mathsf{T} & \xrightarrow[\ell^{\mathsf{T}}]{} & \mathsf{LT} \end{array}$$

Since $\mathsf{L}\kappa$ is a monomorphism then (1) is a pullback by Proposition 1.10. It implies that $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is a normal subcrossed module of T as it can be seen as the intersection of the normal subcrossed modules N and $\mathsf{ker}(\ell^{\mathsf{T}})$ of T. □

We now give some examples of localization functors and exact sequences for which we can apply the fiberwise localization construction.

Example 4.2. Let us consider the nullification functor with respect to the crossed module $\mathbb{Z} \to 0$ as defined in Example 2.11.

$$\mathsf{P}_{\mathbb{Z}\to 0}\begin{pmatrix} N_1 \\ {\scriptstyle \partial^{\mathsf{N}}}\downarrow \\ N_2 \end{pmatrix} = \begin{matrix} \partial^{\mathsf{N}}(N_1) \\ {\scriptstyle inc}\downarrow \\ N_2 \end{matrix}$$

Monomorphisms can be described "component-wise" as two monomorphisms in the category of groups. We can see that if N is a subcrossed module of M, then $\partial^{\mathsf{N}}(N_1)$ is included in $\partial^{\mathsf{M}}(M_1)$ since ∂^{N} is the restriction of ∂^{M}. The other conditions are trivial so we can conclude that $\mathsf{P}_{\mathbb{Z}\to 0}$ preserves monomorphisms. This observation was also made in Example 2.5 in [GE13], where they noticed that it implies that the condition that they called (N) holds. In our case, condition (N) is the necessary and sufficient condition to obtain fiberwise localization (see Theorem 3.4). It implies that for any exact sequence of crossed modules there exists a fiberwise localization for the functor $\mathsf{P}_{\mathbb{Z}\to 0}$.

The functor $\mathsf{C}\colon \mathsf{XMod} \to \mathsf{XMod}$ introduced in Example 2.14 is also an example of a localization functor that satisfies always the condition of Theorem 3.4.

Example 4.3. Let

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \xrightarrow{\kappa_1} & T_1 & \xrightarrow{\alpha_1} & Q_1 & \longrightarrow & 1 \\ & & {\scriptstyle \partial^{\mathsf{N}}}\downarrow & & {\scriptstyle \partial^{\mathsf{T}}}\downarrow & & \downarrow{\scriptstyle \partial^{\mathsf{Q}}} & & \\ 1 & \longrightarrow & N_2 & \xrightarrow{\kappa_2} & T_2 & \xrightarrow{\alpha_2} & Q_2 & \longrightarrow & 1 \end{array}$$

be an exact sequence of crossed modules. We consider the functor $\mathsf{C}\colon \mathsf{XMod} \to \mathsf{XMod}$ and prove that $\kappa(\mathsf{ker}(\ell^{\mathsf{N}}))$ is always normal in T. Indeed, we just need to verify (5) for $\kappa(\mathsf{ker}(\ell^{\mathsf{N}})) = (N_1, \partial^{\mathsf{N}}(N_1)), \tilde{\partial}^{\mathsf{N}})$. We have the following inclusions

$$[\partial^{\mathsf{N}}(N_1), T_1] \subseteq [N_2, T_1] \subseteq N_1$$

since N normal in T and we conclude that for the localization functor C there always exists a fiberwise localization. It is interesting to notice that even if this functor admits a fiberwise localization it does not necessarily preserve monomorphisms. We illustrate this statement via the following example where S_4 is the symmetric group of order 4 and A_4 the alternating group.

$$\mathsf{C}\begin{pmatrix} A_4 & \longrightarrow & S_4 \\ \downarrow & & \| \\ S_4 & = & S_4 \end{pmatrix} = \begin{matrix} 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \end{matrix}$$

Example 4.4. We consider the following two crossed modules $\mathsf{R}A_4$ and $\mathsf{R}S_4$ where R is the functor defined in Lemma 1.4 . We can verify that $\mathsf{R}A_4$ is a normal subcrossed module of $\mathsf{R}S_4$. So we have the following exact sequence.

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & A_4 & \xrightarrow{\kappa} & S_4 & \xrightarrow{\pi} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\
 & & \| & & \| & & \| & & \\
1 & \longrightarrow & A_4 & \xrightarrow{\kappa} & S_4 & \xrightarrow{\pi} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1
\end{array}
\tag{6}
$$

By considering the abelianization functor, as defined in Example 2.12, we wonder if it is possible to construct for this exact sequence a fiberwise localization. First, we give an explicit description of the functor Ab applied to $\mathsf{R}A_4$.

$$
\mathsf{Ab}\left(\begin{array}{c} A_4 \\ \| \\ A_4 \end{array}\right) = \begin{array}{c} A_4/[A_4,A_4] \\ \| \\ A_4/[A_4,A_4] \end{array} = \begin{array}{c} A_4/V_4 \\ \| \\ A_4/V_4 \end{array} \cong \begin{array}{c} \mathbb{Z}/3\mathbb{Z} \\ \| \\ \mathbb{Z}/3\mathbb{Z} \end{array}
$$

where V_4 denotes the Klein four-group. To be able to apply the construction of the fiberwise localization, we need that $\kappa(\mathsf{ker}(\ell^{\mathsf{R}A_4}))$ is a normal subcrossed module of $\mathsf{R}S_4$. As proved in Corollary 3.7, we only need to verify condition (5), i.e. $[V_4, S_4]$ is included in V_4. Since V_4 is a normal subgroup of S_4, the equality $[V_4, S_4] \subseteq V_4$ holds and $\mathsf{R}V_4$ is a normal subcrossed module of $\mathsf{R}S_4$. Hence, we have the following construction of the fiberwise localization

$$
\begin{array}{ccccccccc}
 & & & & \mathsf{R}V_4 & & & & \\
 & & & \swarrow & \downarrow & \searrow & & & \\
1 & \longrightarrow & \mathsf{R}A_4 & & \longrightarrow & & \mathsf{R}S_4 & \xrightarrow{\alpha} & \mathsf{R}(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 1 \\
 & & {\scriptstyle \ell^{\mathsf{R}A_4}}\downarrow & & & & {\scriptstyle f}\downarrow & & \| \\
1 & \longrightarrow & \mathsf{R}(\mathbb{Z}/3\mathbb{Z}) & & \longrightarrow & & \mathsf{R}S_3 & \longrightarrow & \mathsf{R}(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 1
\end{array}
$$

Now, we would like to emphasize the fact that condition (5) is not trivially satisfied. The counter-example we propose is very similar to the previous example where fiberwise abelianization was shown to exist.

Theorem 4.5. *There exist regular-epi localization functors* L *and exact sequences in* Xmod *for which the fiberwise localization does not exist.*

Proof. Let us consider the following exact sequence

$$
1 \longrightarrow \mathsf{R}A_4 \longrightarrow \mathsf{R}S_4 \longrightarrow \mathsf{R}(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 1
$$

and apply the functor $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ defined in Example 2.10, to $\mathsf{R}A_4$.

$$
\mathsf{P}_{\mathsf{X}\mathbb{Z}}\left(\begin{array}{c} A_4 \\ \| \\ A_4 \end{array}\right) = \begin{array}{c} A_4/[A_4,A_4] \\ \downarrow \\ 1 \end{array} = \begin{array}{c} A_4/V_4 \\ \downarrow \\ 1 \end{array}
$$

where $V_4 = [A_4, A_4]$ is the Klein four-group. The kernel of $\ell^{\mathsf{R}A_4}$ is given by the crossed module $V_4 \hookrightarrow A_4$. To be able to construct a fiberwise localization for the exact sequence (6), we need the image of this kernel to be a normal subcrossed module of $\mathsf{R}S_4$. This condition is not

satisfied since (5) does not hold. Indeed, the action of S_4 on A_4 is given by conjugation, hence, condition (5) means that the commutator $[S_4, A_4]$ has to be included in V_4. But we have the following equality $[S_4, A_4] = A_4$, which implies that $V_4 \hookrightarrow A_4$ is not a normal subcrossed module of $\mathsf{R}S_4$. Therefore, fiberwise localization does not exist for this nullification functor and this exact sequence. □

We have understood at this point that localization functors of crossed modules do not behave like localization functors of groups. One other major difference is illustrated by the behavior of the kernel of a nullification functor. In the category of groups (the homotopical analog is also true for topological spaces), the kernels of the nullification morphisms are acyclic. In fact, $L(ker(\ell^G))$ is trivial for any group G if and only if L is a nullification functor, [FS15, Lemma 5.1]. In the context of crossed modules, such a characterization of nullification functors fails.

Proposition 4.6. *There are nullification functors* P_A *in* XMod *such that the kernels of their localization morphisms are not* A*-acyclic in general.*

Proof. Let us consider the nullification functor with respect to the crossed module $\mathsf{X}\mathbb{Z}$ as in Example 2.11. Let us apply this functor to the crossed module (D_8, D_8, Id_{D_8}), where D_8 is the dihedral group of order eight.

$$\mathsf{P}_{\mathsf{X}\mathbb{Z}}\begin{pmatrix} D_8 \\ \| \\ D_8 \end{pmatrix} = \begin{matrix} \frac{D_8}{[D_8,D_8]} \\ \downarrow \\ 1 \end{matrix} = \begin{matrix} \frac{D_8}{C_2} \\ \downarrow \\ 1 \end{matrix}$$

where $[D_8, D_8] = C_2$, which is actually the center of D_8. Hence, the kernel of this construction is given by the crossed module (C_2, D_8, inc). By applying the functor $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ on this crossed module we obtain the following crossed module

$$\mathsf{P}_{\mathsf{X}\mathbb{Z}}\begin{pmatrix} C_2 \\ {\scriptstyle inc}\downarrow \\ D_8 \end{pmatrix} = \begin{matrix} \frac{C_2}{[C_2,D_8]} \\ \downarrow \\ 1 \end{matrix} = \begin{matrix} \frac{C_2}{1} \\ \downarrow \\ 1 \end{matrix} = \begin{matrix} C_2 \\ \downarrow \\ 1 \end{matrix}$$

which is not the trivial crossed module. □

Remark 4.7. Thanks to a general result for any semi-abelian category of [GS22], we know that if a nullification functor P_A admits a fiberwise localization, then the kernels of their localization morphisms are A-acyclic.

Conclusion

In this article, we studied fiberwise localization for regular-epi localization functors of crossed modules of groups. To sum up, we found an adequate normality condition on a short exact sequence and a regular-epi localization functor that guarantees the existence of a fiberwise localization. We proved that this condition can be expressed easily in XMod. This simple statement allowed us to obtain examples of fiberwise localizations and unexpectedly also counter-examples: the nullification functor $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ does not admit a fiberwise localization in general. This observation highlighted an interesting difference between the fiberwise localizations of crossed modules and groups, which can be tracked down to the notion

of characteristic subobject, namely one which is not only normal, but invariant under all automorphisms. If we have groups $K \leq N \leq T$ such that K is a characteristic subgroup of N and N is a normal subgroup of T, then K is a normal subgroup of T. However the corresponding statement does not hold for crossed modules, as we have seen in Theorem 4.5 (note that $\mathsf{ker}(\ell^{\mathsf{N}})$ is a characteristic subcrossed module of N).

We observed another important difference between nullification functors of crossed modules and groups. In the category of groups, nullification functors P_{A} are characterized by the fact that the kernels of their localization morphisms are A-acyclic. In XMod, $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ does not satisfy this property. It is interesting to notice that even if the functor of nullification $\mathsf{P}_{\mathsf{X}\mathbb{Z}}$ fails to have a fiberwise localization or to have $\mathsf{X}\mathbb{Z}$-acyclic kernels, its local objects form a Birkhoff subcategory.

Fiberwise localization is an important tool in the study of conditional flatness: in [GS22] and [FS15], the authors used functorial fiberwise localization to study conditionally flat localizations. Since we are not always able to construct a fiberwise localization for regular-epi localization functors of crossed modules, it is interesting to investigate conditional flatness for such functors, which we do in the forthcoming paper [MSS22]. We study how we can simplify this property by following the strategy of [FS15; GS22]. We prove that a regular-epi localization functor is conditionally flat if and only if it is admissible (in the sense of Galois) for the class of regular epimorphisms. We also prove that nullification functors are conditionally flat even if they fail to be characterized by an acyclicity condition.

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