

# FLOYD'S MANIFOLD IS A CONJUGATION SPACE

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ABSTRACT. E. E. Floyd showed in 1973 that there exist only two non-trivial cobordism classes that contain manifolds with three cells, and that they lie in dimensions 10 and 5. We prove that there is an action of the cyclic group  $\mathbf{C}_2$  on the 10-dimensional Floyd manifold which turns it into a conjugation manifold in the sense of Hausmann, Holm, and Puppe. The submanifold of fixed points is the 5-dimensional Floyd manifold, whose cohomology is isomorphic to that of the large one, scaled down by dividing the cohomological degree by a factor two.

## 1. INTRODUCTION

In [1] Edwin E. Floyd showed, among many other things, that there exist only two non-trivial cobordism classes that contain manifolds with three cells, and that they lie in dimensions 10 and 5. He also produced representatives of these classes, which we name here  $F_{10}$  and  $F_5$ . His computations show that the mod 2 cohomology of the larger one has generators  $1, u_4, u_6, u_{10}$  in degrees 0, 4, 6, and 10 respectively, whereas the smaller one looks like a divided by 2 version with generators  $1, v_2, v_3, v_5$  in degrees 0, 2, 3, and 5 respectively. Moreover, the product of the two middle generators give the top one, so that the abstract “halving isomorphism” between  $H^*(F_{10}; \mathbb{F}_2)$  and  $H^*(F_5; \mathbb{F}_2)$  is actually compatible with the cup product. More is true since Floyd also identified the action of the Steenrod algebra and computed that  $Sq^2 u_4 = u_6$  and  $Sq^4 u_6 = u_{10}$ , an analogous result being true for  $F_5$ . Hence the halving isomorphism is also compatible with the action of the Steenrod algebra.

From the point of view of the theory of conjugation spaces introduced by Hausmann, Holm, and Puppe in [3], and further developments on the action of the Steenrod algebra by Franz and Puppe, [2], this remarkable behavior looks like an algebraic shadow of the fact that the larger manifold  $F_{10}$  could be a conjugation manifold.

We give here an affirmative answer to this conjecture, after several attempts. This problem appears as an open question in our previous article on the subject, [7], where this was only proved for the suspension spectrum in the stable homotopy category.

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The main problem is to endow  $F_{10}$  with an action by the group of two elements  $\mathbf{C}_2$  in such a way that it becomes a conjugation space with fixed points  $F_5$ .

**Theorem 4.7.** *The Floyd manifold  $F_{10}$  is a smooth conjugation manifold with fixed points  $F_5$ .*

The strategy is to analyze carefully Floyd's surgery construction and perform it equivariantly. For this first step we use Lück and Uribe's theory of equivariant principal bundles, [4]. This yields an action of  $\mathbf{C}_2$  on  $F_{10}$ . The second and final step consists then in verifying the "cohomological purity" in the genuine equivariant stable sense as explained in our joint work with Ricka, [6]. A key ingredient here is Clover May's splitting result, [5].

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## 2. CONSTRUCTION OF FLOYD'S MANIFOLDS

We will go through a very careful analysis of Floyd's construction [1, Section 3] to ascertain how it inherits a natural  $\mathbf{C}_2$ -action. Let  $S^3 \subset \mathbb{C} \oplus \mathbb{C}$  denote the unit sphere in the standard metric. It inherits two intertwined actions: one on the right by multiplication by elements in the unit circle  $S^1 \subset \mathbb{C}$  and the other on the left by  $\mathbf{C}_2$  is induced by conjugation on each factor. If  $\theta \in S^1$  and  $\tau$  denotes the non-trivial element in  $\mathbf{C}_2$ , then :

$$(\star) \quad \tau(x\theta) = \tau(x)\bar{\theta} \text{ for all } x \in S^3$$

As a  $\mathbf{C}_2$ -representation sphere  $S^3 = S^{2\sigma+1} = S^{2\rho-1}$  in the notation of [6], where  $\sigma$  stands for the sign representation and  $\rho$  for the regular representation.

Notice that because of  $(\star)$ , the action of  $S^1$  on  $S^3$  restricts to an action of  $\mathbf{C}_2 \cong \{\pm 1\} \subset S^1$  on  $(S^3)^{\mathbf{C}_2} = S^1$ , more precisely it restricts to the antipodal action on the fixed circle.

Let us define the projective planes  $\mathbb{C}P^2$  and  $\mathbb{H}P^2$  as the set of *right* complex lines in  $\mathbb{C}^3$  (resp. quaternionic lines in  $\mathbb{H}^3$ ). To fix notations, writing  $\mathbb{K}$  for  $\mathbb{C}$  or  $\mathbb{H}$ , we have an element

$$[x : y : z] = \{(x\lambda, y\lambda, z\lambda) \mid \lambda \in \mathbb{K}^*\} \in \mathbb{K}P^2$$

for any  $(x, y, z) \in \mathbb{K}^3 \setminus \{0, 0, 0\}$ .

Decompose  $\mathbb{H}$  as  $\mathbb{C} \oplus \mathbb{C}j$ . Then  $\mathbb{H}$  also has two intertwined actions, one by *left* multiplication by elements in the unit circle  $S^1$  on the second  $\mathbb{C}$ -copy, and the other, also on the left, by  $\mathbf{C}_2$  and induced by conjugation on each  $\mathbb{C}$ -copy above. Both

actions induce actions on the quaternionic projective plane and these two actions satisfy:

$$(\star\star) \quad \tau(\theta x) = \bar{\theta}\tau(x) \text{ for all } x \in \mathbb{H}P^2$$

To analyze the  $\mathbf{C}_2$ -fixed points and the inherited action from  $S^1$ , assume that we have a point  $[x : y : z]$  with  $z \neq 0$  which is fixed (the two other cases  $x \neq 0$  and  $y \neq 0$  are similar). Let us pick the representative of this class of the form  $(a, b, 1) \in \mathbb{H}^3$ . Then

$$\tau \cdot [a : b : 1] = [\bar{a} : \bar{b} : 1]$$

Since  $(a, b, 1)$  and  $(\bar{a}, \bar{b}, 1)$  are in the same right quaternionic line if and only if  $a, b \in \mathbb{R} \oplus \mathbb{R}j$ , this shows that  $(\mathbb{H}P^2)^{\mathbf{C}_2} \cong \mathbb{C}P^2$ , where the complex numbers lie “diagonally” in  $\mathbb{H}$ . Moreover, since  $\theta \cdot 1 = 1$  for  $\theta \in S^1$ , the  $S^1$ -action on  $\mathbb{H}P^2$  restricts to a  $\{\pm 1\}$ -action on the  $\mathbf{C}_2$ -fixed points  $\mathbb{C}P^2$ , and this action coincides with the one induced by conjugation on  $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}j$  by definition of the  $S^1$ -action (on the second coordinate only).

Let us consider  $\widetilde{F}_{10} = S^3 \times_{S^1} \mathbb{H}P^2$  just like Floyd did in [1, Lemma 3.5]. Because of the intertwined  $S^1$ - and  $\mathbf{C}_2$ -actions on each factor,  $\widetilde{F}_{10}$  has an induced  $\mathbf{C}_2$ -action and by the previous computations:

$$(\widetilde{F}_{10})^{\mathbf{C}_2} = S^1 \times_{\mathbf{C}_2} \mathbb{C}P^2 = \widetilde{F}_5$$

Notice that both manifolds  $\widetilde{F}_{10}$  and  $\widetilde{F}_5$  fit into fibration sequences by projecting on the first factors, where the right one is acted upon by  $\mathbf{C}_2$  and has as fixed points the left one:

$$\begin{array}{ccc} \mathbb{C}P^2 & \hookrightarrow & \mathbb{H}P^2 \\ \downarrow & & \downarrow \\ S^1 \times_{\mathbf{C}_2} \mathbb{C}P^2 & \hookrightarrow & S^3 \times_{S^1} \mathbb{H}P^2 \\ \downarrow & & \downarrow \\ S^1 & \hookrightarrow & S^\rho \end{array}$$

The base spaces are respectively  $S^1 = \mathbb{R}P^1 = S^1/\mathbf{C}_2$  and  $S^\rho = \mathbb{C}P^1 = S^3/S^1$ . It is immediate to check that the  $\mathbf{C}_2$ -action is smooth, and we have in fact a smooth fibration. As a direct consequence of [3, Theorem 5.3]:

**Proposition 2.1.** *The manifold  $\widetilde{F}_{10}$  is a smooth conjugation manifold with fixed point set  $\widetilde{F}_5$ .  $\square$*

Since the  $S^1$ -action on  $\mathbb{H}P^2$  has a fixed point, which moreover can be chosen to lie on  $\widetilde{F}_5$ , both fibrations have compatible sections:

$$\begin{array}{ccc}
 & S^1 \times_{\mathbf{C}_2} \mathbb{C}P^2 & \hookrightarrow S^3 \times_{S^1} \mathbb{H}P^2 \\
 S^1 \hookrightarrow & \nearrow & \nearrow \\
 & \downarrow & \downarrow \\
 & S^1 & \hookrightarrow S^\rho
 \end{array}$$

where again, the right hand side diagram has a  $\mathbf{C}_2$ -action with fixed points the left hand side diagram.

To construct his manifolds  $F_{10}$  and  $F_5$ , Floyd proceeds by surgering these split spheres in  $\widetilde{F}_{10}$  and  $\widetilde{F}_5$  and we want to do the same equivariantly on  $\widetilde{F}_{10}$  and simultaneously on  $\widetilde{F}_5$ .

### 3. THE SURGERY STEP

We will use the framework developed by Lück and Uribe in [4] to prove that the normal bundle to  $S^\rho \hookrightarrow \widetilde{F}_{10}$  is equivariantly trivial. We will work with the associated frame bundle, as it is a principal  $GL_8(\mathbb{R})$ -bundle whose triviality is equivalent to that of the normal bundle.

Because we have split the fibration  $\widetilde{F}_{10} \rightarrow S^\rho$  by choosing an  $S^1$ -fixed point in  $\mathbb{H}P^2$ , the normal bundle to the splitting  $S^\rho \hookrightarrow \widetilde{F}_{10}$  is given by the vertical tangent bundle in the fibration, that is, the tangent bundle to the fiber  $\mathbb{H}P^2$ . So, if  $T\widetilde{F}_{10}$  is the tangent bundle, our model for the normal bundle is  $T_V\widetilde{F}_{10}$ , the rank 8 vertical bundle, which fiberwise is the kernel of the projection to  $S^\rho$ . The group  $\mathbf{C}_2$  acts by diffeomorphisms on  $\widetilde{F}_{10}$ , and by equivariance, preserves this bundle: for  $\gamma \in \mathbf{C}_2$ , the tangent map  $T\gamma$  acts on the left on  $T_V\widetilde{F}_{10}$ . The associated frame bundle  $Fr(T_V\widetilde{F}_{10})$  has as model the bundle one obtains by considering fiberwise, over a point  $x \in S^\rho$ ,

$$\text{Isom}(\mathbb{R}^8, T_{xV}\widetilde{F}_{10}),$$

the space of linear isomorphisms out of  $\mathbb{R}^8$ . The canonical left action of  $GL_8(\mathbb{R})$  induces then a right action by contravariance on  $Fr(T_V\widetilde{F}_{10})$ . Precomposition commutes with the postcomposition  $\mathbf{C}_2$ -action induced by the tangent action on  $T_{xV}\widetilde{F}_{10}$ . Hence the associated frame bundle is an equivariant bundle without intertwine.

We first need to fix a set  $\mathcal{R}$  of local isotropy representations, satisfying Condition (H) of Lück-Uribe, [4, Definition 6.1]. This ensures that the classification problem of the normal bundle is a purely homotopical one by [4, Theorem 10.1], which means that the equivariant triviality of the bundle will be a consequence of the null-homotopy of the classifying map.

Locally, in a neighborhood of the fixed point that was chosen to split the fibration, the vertical direction looks like a quadruple of complex numbers  $(z_1, z_2, z_3, z_4)$  on which  $\mathbf{C}_2$  acts by complex conjugation as described in the previous section. Hence the representation of  $\mathbf{C}_2$  is given by a homomorphism  $\alpha: \mathbf{C}_2 \rightarrow GL_8(\mathbb{R})$  conjugate to  $I_4 \oplus (-I_4)$ .

At a non-fixed point, the isotropy subgroup is per force the trivial group and the representation the trivial one. We thus set  $\mathcal{R}' = \{(1_{\mathbf{C}_2}, \text{triv}), (\mathbf{C}_2, \alpha)\}$ , and close it under conjugation by elements in  $GL_8(\mathbb{R})$  to form a family  $\mathcal{R}$ . This is, in the terminology of [4, Definition 3.4], the family of representations associated to  $S^\rho$ . Notice that the passage from pre-families to families is here effortless: the group  $\mathbf{C}_2$  is so small that two representations thereon either coincide or coincide only on the trivial element. Because  $\mathbf{C}_2$  is finite and  $GL_8(\mathbb{R})$  is a quasi-connected Lie group, by [4, Theorem 6.3], condition (H) is satisfied for the family  $\mathcal{R}$ .

These choices define a classifying space  $B(\mathbf{C}_2, GL_8(\mathbb{R}), \mathcal{R})$ , see [4, Theorem 11.4], which we write  $B(\mathbf{C}_2, GL_8(\mathbb{R}))$  for short. The isomorphism classes of equivariant bundles over  $S^\rho$  with isotropy representations in  $\mathcal{R}$  are in bijection with the set of equivariant homotopy classes into  $B(\mathbf{C}_2, GL_8(\mathbb{R}))$ . Hence, to our bundle corresponds a (homotopy class of)  $\mathbf{C}_2$ -equivariant map  $f: S^\rho \rightarrow B(\mathbf{C}_2, GL_8(\mathbb{R}))$ . We analyze this map by decomposing  $S^\rho$  into the fixed equator  $S^1$  and two 2-cells, the hemispheres, that are permuted. Our aim is to extend this map to a map  $F$  defined on the whole ball  $B^\rho$  whose boundary is  $S^\rho$ . We will do this by first extending over the fixed equatorial disk, and then filling in the two remaining spheres. Since we are only interested in the triviality of our bundle, we will make our computations in the reduced setting, i.e., we consider  $f$  as an element in the set of pointed homotopy classes:

$$[S^\rho, B(\mathbf{C}_2, GL_8(\mathbb{R}))]^{\mathbf{C}_2}$$

Our first step is to restrict our equivariant bundle to the fixed circle and show that it is trivial.

**Lemma 3.1.** *The classifying map  $f$  extends to a map  $f': S^\rho \cup D^2 \rightarrow B(\mathbf{C}_2, GL_8(\mathbb{R}))$ .*

*Proof.* By Lück-Urbe's results, the restriction of  $f$  to  $S^1$  amounts to having a non-equivariant bundle on  $S^1$  but with a reduced structural group. More precisely, since  $S^1$  is fixed under the  $\mathbf{C}_2$ -action, its image lies in the fixed points of the equivariant classifying space. Since up to conjugacy we only have one representation of  $\mathbf{C}_2$  in  $\mathcal{R}$ , namely  $\alpha$ , we identify these fixed points by [4, Theorem 13.1]:

$$B(\mathbf{C}_2, GL_8(\mathbb{R}))^{\mathbf{C}_2} \simeq BC_{GL_8(\mathbb{R})}(\alpha)$$

Let us fix as a representative for the representation  $\alpha$  the one that sends  $\tau \in \mathbf{C}_2$  to  $\begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}$ . The centralizer is  $C_{GL_8(\mathbb{R})}(\alpha) = GL_4(\mathbb{R}) \times GL_4(\mathbb{R})$  by direct computation

so that the trivial  $\dashv$  fixed points adjunction yields an isomorphism

$$[S^1, B(\mathbf{C}_2, GL_8(\mathbb{R}))]^{\mathbf{C}_2} \cong [S^1, B(\mathbf{C}_2, GL_8(\mathbb{R}))^{\mathbf{C}_2}] \cong \pi_1(B(GL_4(\mathbb{R}) \times GL_4(\mathbb{R}))) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

To analyze our restricted bundle on  $S^1$ , recall from Floyd's work that the bundle on  $S^\rho$  is non-equivariantly two times some bundle. So let  $g : S^2 \rightarrow BGL_4(\mathbb{R})$  be a classifying map of the latter bundle. This means that a classifying map for the double is given by the following composition:

$$G : S^2 \xrightarrow{\Delta} S^2 \times S^2 \xrightarrow{g \times g} BGL_4(\mathbb{R}) \times BGL_4(\mathbb{R}) \xrightarrow{j} BGL_8(\mathbb{R})$$

Here again, by definition of the Whitney sum, the map  $j$  is induced by the diagonal embedding of the two blocks, i.e., by the inclusion  $C_{GL_8(\mathbb{R})}(\alpha) \subset GL_8(\mathbb{R})!$  The equivariant classifying map  $B(\mathbf{C}_2, GL_8(\mathbb{R}))$  has a canonical forgetful map to  $BGL_8(\mathbb{R})$ , and Lück-Urbe's argument explains that the composition

$$BC_{GL_8(\mathbb{R})}(\alpha) \rightarrow B(\mathbf{C}_2, GL_8(\mathbb{R})) \rightarrow BGL_8(\mathbb{R})$$

is given by the same map  $j$ , induced by the canonical inclusion. This means that a classifying map for our restricted bundle is given precisely by the composition:

$$G|_{S^1} : S^1 \longrightarrow S^2 \xrightarrow{\Delta} S^2 \times S^2 \xrightarrow{g \times g} BGL_4(\mathbb{R}) \times BGL_4(\mathbb{R})$$

As the first map is null-homotopic, so is the composite and we can extend  $G|_{S^1}$  to a map on the disc.  $\square$

Let us think about the added disk  $D^2$  above as an equatorial disk in  $B^\rho$ , fixed by the  $\mathbf{C}_2$ -action. To continue our argument, we choose as an extension of the map  $G|_{S^1}$  to  $D^2$  the one provided by the map  $G$  itself on the southern hemisphere of its source  $S^2$ . The following lemma formalizes the fact that we can fill the northern hemisphere with the non-equivariant trivialization obtained by Floyd and extend to the southern half-ball by applying the  $\mathbf{C}_2$ -action.

**Lemma 3.2.** *The classifying map  $f$  extends to a map  $F : B^\rho \rightarrow B(\mathbf{C}_2, GL_8(\mathbb{R}))$ .*

*Proof.* Observe that  $S^\rho \cup D^2$  is equivariantly homotopic to  $S^2 \vee S^2$  with the flip action, and this a free pointed  $\mathbf{C}_2$ -space. By the free  $\dashv$  forgetful adjunction:

$$[S^2 \vee S^2, B(\mathbf{C}_2, GL_8(\mathbb{R}))]^{\mathbf{C}_2} \cong [S^2, B(\mathbf{C}_2, GL_8(\mathbb{R}))^e] \cong [S^2, BGL_8(\mathbb{R})]$$

where the second isomorphism comes again from [4, Theorem 13.1]. The second homotopy group of  $BGL_8(\mathbb{R})$  is isomorphic to  $\pi_1 O(8) \cong \mathbb{Z}/2$ .

Through this isomorphism  $f'$  corresponds to its restriction to the union of the northern hemisphere of  $S^\rho$  and the equatorial disc. However, by construction, a classifying map for this bundle is given by  $G$ : we had it on the northern hemisphere and chose the southern part on the added disc  $D^2$ . In particular, this is a classifying map of twice a bundle.

We use now Floyd's original argument to see that this bundle is trivial (it is twice something in  $\mathbb{Z}/2$ ). Therefore  $f'$  is null-homotopic and thus extends to  $B^\rho$ .  $\square$

As an immediate consequence we have our desired result that will be used to do the surgery step:

**Theorem 3.3.** *The equivariant normal bundle to  $S^\rho \hookrightarrow \widetilde{F}_{10}$  is equivariantly trivial, i.e., the total space of the fibration is equivariantly homeomorphic to  $S^\rho \times \mathbb{R}^{4\rho}$ .  $\square$*

#### 4. THE FLOYD MANIFOLD $F_{10}$ IS A CONJUGATION MANIFOLD

We are now ready to perform Floyd's construction of the manifolds  $F_5$  and  $F_{10}$  simultaneously and equivariantly on the larger one. By Theorem 3.3, we learn that the tubular neighborhood of  $S^\rho$  is equivariantly homeomorphic to  $S^\rho \times D(4\rho)$  with boundary  $S^\rho \times S^{4\rho-1}$ . We now remove the interior and glue back a copy of  $D(\rho+1) \times S^{4\rho-1}$ , thus killing the sphere  $S^\rho$  to get  $F_{10}$  with a  $\mathbf{C}_2$ -action. Observe that on the fixed point subset, we replaced  $S^1 \times D(4)$ , a tubular neighbourhood of  $S^1 \hookrightarrow \widetilde{F}_5$ , by  $D(2) \times S^3$ , i.e. we performed a standard non-equivariant surgery and get by Floyd's argument his manifold  $F_5$ . In particular, by construction, we have:

$$(F_{10})^{\mathbf{C}_2} = F_5$$

To prove our main result now, we will verify the homotopical criterion for being a conjugation space from [6]. Let  $\mathbf{H}\underline{\mathbf{F}}$  denote the genuine  $\mathbf{C}_2$ -equivariant spectrum associated to the constant Mackey functor with values  $\mathbf{F}_2$ , the field of two elements.

**Theorem 4.1.** [6, Theorem 1.3] *A compact equivariant space  $X$  is a conjugation space if and only if there exist finitely many non-negative integers  $(m_i)_{i \in I}$  such that*

$$X \wedge \mathbf{H}\underline{\mathbf{F}} \simeq \bigvee_{i \in I} S^{n_i \rho} \wedge \mathbf{H}\underline{\mathbf{F}}$$

The starting point to prove this is the very nice structural theorem of C. May [5]. Recall that  $\sigma$  denotes the sign representation.

**Theorem 4.2.** [5, Theorem 6.13] *Let  $X$  be any pointed compact equivariant space. Denote by  $S_{a+}^k$  the  $k$ -dimensional sphere with the antipodal action and a disjoint base-point. Then there exist two families of finitely many non-negative integers  $(m_i, n_i)_{i \in I}$   $(s_j)_{j \in J}$  and an equivariant equivalence:*

$$X \wedge \mathbf{H}\underline{\mathbf{F}} \simeq \bigvee_{i \in I} S^{m_i + n_i \sigma} \wedge \mathbf{H}\underline{\mathbf{F}} \vee \bigvee_{j \in J} S_{a+}^{s_j} \wedge \mathbf{H}\underline{\mathbf{F}}.$$

To apply May's result, we essentially have to perform a computation in equivariant homology. For this we will analyze the two homotopy push-outs that summarize the surgery step. Let  $M$  denote the complement of an equivariant tubular neighborhood

of the split representation sphere  $S^\rho \subset \widetilde{F}_{10}$ . Then we have two homotopy push-out squares, that we complete by adding the cofibers (where the subscripts only indicate the horizontal or vertical direction):

$$\begin{array}{ccccc}
 S^\rho \times S^{4\rho-1} & \hookrightarrow & M & \longrightarrow & \text{Cof}_h \\
 \downarrow & & \downarrow & & \parallel \\
 S^\rho \times D(4\rho) & \hookrightarrow & \widetilde{F}_{10} & \longrightarrow & \text{Cof}_h \\
 \downarrow & & \downarrow & & \\
 \text{Cof}_{vA} & \xlongequal{\quad} & \text{Cof}_{vA} & & 
 \end{array}$$
  

$$\begin{array}{ccccc}
 S^\rho \times S^{4\rho-1} & \hookrightarrow & M & \longrightarrow & \text{Cof}_h \\
 \downarrow & & \downarrow & & \parallel \\
 D(\rho+1) \times S^{4\rho-1} & \hookrightarrow & F_{10} & \longrightarrow & \text{Cof}_h \\
 \downarrow & & \downarrow & & \\
 \text{Cof}_{vB} & \xlongequal{\quad} & \text{Cof}_{vB} & & 
 \end{array}$$

Notice that the horizontal cofibers in both diagrams are the same since they share the same top horizontal cofibration.

We will now analyze the long exact sequences for both squares, in both equivariant and non-equivariant homology. To simplify the notation, since all spaces will have free homology, we will write  $\mathbf{H}\mathbf{F}[V_1, V_2, \dots, V_n]$  for a smash product  $\mathbf{H}\mathbf{F} \wedge X$  equivalent to a wedge  $\mathbf{H}\mathbf{F} \wedge S^{V_1} \vee \mathbf{H}\mathbf{F} \wedge S^{V_2} \vee \dots \vee \mathbf{H}\mathbf{F} \wedge S^{V_n}$ .

There are three key tools to carry out the computations. Firstly, maps out of free  $\mathbf{H}\mathbf{F}$ -modules, like  $\mathbf{H}\mathbf{F} \wedge S^{n\sigma+m}$  are determined by the value on one generator, secondly for free modules of the form  $\mathbf{H}\mathbf{F} \wedge S^{n\sigma+m}$  we have a preferred generator, namely the only class of minimal degree that survives to a non-trivial class when forgetting the actions, and thirdly we know the answer in the non-equivariant setting, as these elementary computations have been carried out by Floyd [1].

To understand the vertical cofibers in both diagrams, let us recall:

**Definition 4.3.** The *half-smash product*  $A \ltimes B$  of two pointed spaces  $A$  and  $B$  is the cofiber of the inclusion  $A \times * \hookrightarrow A \times B$ .

When  $A$  and  $B$  are  $\mathbf{C}_2$ -equivariant spaces and the base point in  $B$  is a fixed point, the half-smash inherits an action of  $\mathbf{C}_2$ .

**Lemma 4.4.** *The homotopy cofiber of the projection  $A \times B \rightarrow A$  is homotopy equivalent to  $\Sigma(A \ltimes B) \simeq \Sigma(A \wedge B) \vee \Sigma B$  and the map from  $A$  is null-homotopic.*



*Proof.* Factor the identity as  $A \hookrightarrow A \times B \rightarrow A$ . The associated cofibration sequence  $A \times B \rightarrow \text{Cof}(Id_A) \rightarrow \text{Cof}(A \times B \rightarrow A)$  yields the desired result. The decomposition as a wedge is a consequence of the splitting of the suspension of any product and that the map  $A \rightarrow \Sigma(A \times B)$  is null-homotopic follows from the fact that the projection map has a retraction.  $\square$

This applies again to the equivariant setting when the base point of the second factor is a fixed point (and the maps we consider are thus equivariant). Our interest lies in the half-smash product of representation spheres.

**Corollary 4.5.** *Let  $V_1$  and  $V_2$  be two orthogonal representations of  $\mathbf{C}_2$ . The homotopy cofiber of the projection  $S^{V_1} \times S^{V_2} \rightarrow S^{V_1}$  is then equivalent to  $S^{V_1+V_2+1} \vee S^{V_2+1}$ .*

In particular we have a stable equivariant splitting

$$\mathbf{HF} \wedge (S^{V_1} \times S^{V_2}) \simeq \mathbf{HF} \wedge S^{V_1+1} \vee \mathbf{HF} \wedge S^{V_1+V_2+1} = \mathbf{HF}[V_1 + 1, V_1 + V_2 + 1]$$

Smashing the first homotopy pushout with the spectrum  $\mathbf{HF}$ , we get a homotopy pushout of equivariant spectra:

$$\begin{array}{ccccc} \mathbf{HF}[\rho, 4\rho - 1, 5\rho - 1] & \longrightarrow & \mathbf{HF} \wedge M & \longrightarrow & \mathbf{HF} \wedge \text{Cof}_h \\ \downarrow & & \downarrow & & \parallel \\ \mathbf{HF}[\rho] & \xrightarrow{\textcircled{1}} & \mathbf{HF}[\rho, 2\rho, 3\rho, 4\rho, 5\rho] & \xrightarrow{\textcircled{2}} & \mathbf{HF} \wedge \text{Cof}_h \\ \downarrow 0 & & \downarrow & & \\ \mathbf{HF}[4\rho, 5\rho] & \xlongequal{\quad\quad\quad} & \mathbf{HF}[4\rho, 5\rho] & & \end{array}$$

where we have applied Corollary 4.5 and May's Theorem 4.2 to identify the spectra. The map labelled with a zero is null-homotopic, see Lemma 4.4. We proceed to identify the two arrows labeled  $\textcircled{1}$  and  $\textcircled{2}$ .

**Lemma 4.6.** *The map  $\mathbf{HF} \wedge M \rightarrow \mathbf{HF} \wedge \text{Cof}_h$  is the unique map of  $\mathbf{HF}$ -modules  $\mathbf{HF}[\rho, 2\rho, 3\rho] \rightarrow \mathbf{HF}[2\rho, 3\rho, 4\rho, 5\rho]$  that collapses the  $\mathbf{HF}[\rho]$  factor and identifies the  $\mathbf{HF}[2\rho, 3\rho]$  factor with the corresponding part in the target.*

*Proof.* Because the fibration  $\widetilde{F}_{10} \rightarrow S^\rho$  splits, and the arrow  $\textcircled{1}$  is induced by an arbitrary splitting (they are all homotopic, since the splittings we consider come from the choice of a fixed point in the sphere  $S^{4\rho-1}$ , and there the fixed points form a connected space), we get that  $\textcircled{1}$  is a split monomorphism, hence from the triangulated structure for  $\mathbf{HF}$ -modules in the equivariant stable homotopy category,  $\textcircled{2}$  is a split epimorphism.

We can now construct a commutative solid diagram comparing the wedge inclusion with the map ①:

$$\begin{array}{ccccc}
\mathbf{HF}[\rho] & \longrightarrow & \mathbf{HF}[\rho] \vee \mathbf{HF}[2\rho, 3\rho, 4\rho, 5\rho] & \longrightarrow & \mathbf{HF}[2\rho, 3\rho, 4\rho, 5\rho] \\
\parallel & & \downarrow \textcircled{1} \vee \textit{Can} & & \downarrow \text{dotted} \\
\mathbf{HF}[\rho] & \longrightarrow & \mathbf{HF}[\rho, 2\rho, 3\rho, 4\rho, 5\rho] & \longrightarrow & \mathbf{HF} \wedge \textit{Cof}_h
\end{array}$$

where  $\textit{Can}$  is the canonical inclusion. By the triangulated structure, the dotted arrow exists and enhances the diagram to a morphism of triangles. By construction the middle arrow is an epimorphism in homotopy and an  $\mathbf{HF}_*$ -module map. Because this ring is local, by Nakayama's Lemma, the map is an isomorphism. The five lemma shows then that the dotted arrow induces an isomorphism as well, and by identifying  $\mathbf{HF} \wedge \textit{Cof}_h$  via this isomorphism we have as desired that  $\mathbf{HF} \wedge \textit{Cof}_h \simeq \mathbf{HF}[2\rho, 3\rho, 4\rho, 5\rho]$ . Likewise the vertical map  $\mathbf{HF}[\rho, 2\rho, 3\rho, 4\rho, 5\rho] \rightarrow \mathbf{HF}[4\rho, 5\rho]$  is the projection, which allows us to identify  $\mathbf{HF} \wedge M$  with  $\mathbf{HF}[\rho, 2\rho, 3\rho]$ : This determines the map  $\mathbf{HF} \wedge M \rightarrow \mathbf{HF} \wedge \textit{Cof}_h$ .  $\square$

We finally come to our main result. We use now our second pushout diagram in which we plug in the previous identification.

**Theorem 4.7.** *The manifold  $F_{10}$  is a smooth conjugation manifold with fixed points  $F_5$ .*

*Proof.* We use Lemma 4.6 and apply the same strategy to the second pushout diagram smashed with  $\mathbf{HF}$ , rotating however the horizontal triangles to the left to simplify the final computation. The identification of the vertical cofiber comes again from Corollary 4.5.

$$\begin{array}{ccccc}
\mathbf{HF}[2\rho - 1, 3\rho - 1, 4\rho - 1, 5\rho - 1] & \xrightarrow{\textcircled{1}} & \mathbf{HF}[\rho, 4\rho - 1, 5\rho - 1] & \longrightarrow & \mathbf{HF}[\rho, 2\rho, 3\rho] \\
\parallel & & \downarrow \textcircled{2} & & \downarrow \\
\mathbf{HF}[2\rho - 1, 3\rho - 1, 4\rho - 1, 5\rho - 1] & \xrightarrow{\textcircled{3}} & \mathbf{HF}[4\rho - 1] & \longrightarrow & \mathbf{HF} \wedge F_{10} \\
& & \downarrow 0 & & \downarrow \\
& & \mathbf{HF}[\rho + 1, 5\rho] & \xlongequal{\quad} & \mathbf{HF}[\rho + 1, 5\rho]
\end{array}$$

We know from Lemma 4.6 that the map ① maps  $4\rho - 1 \mapsto 4\rho - 1$  and that ② is the canonical projection, hence ③ is the split epimorphism projection onto the  $4\rho - 1$  component. Therefore  $\mathbf{HF} \wedge F_{10} \simeq \mathbf{HF}[2\rho, 3\rho, 5\rho]$  and we conclude by Theorem 4.1 that  $F_{10}$  is a conjugation space. The fixed points have already been identified earlier as  $F_5$  and we have constructed the larger manifold as a smooth  $\mathbf{C}_2$ -equivariant manifold.  $\square$

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