A Step towards Non-Presentable Models of Homotopy Type Theory

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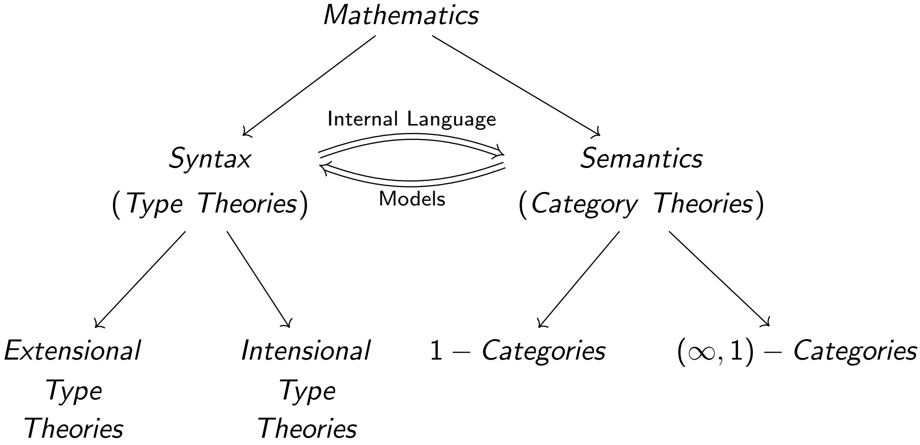
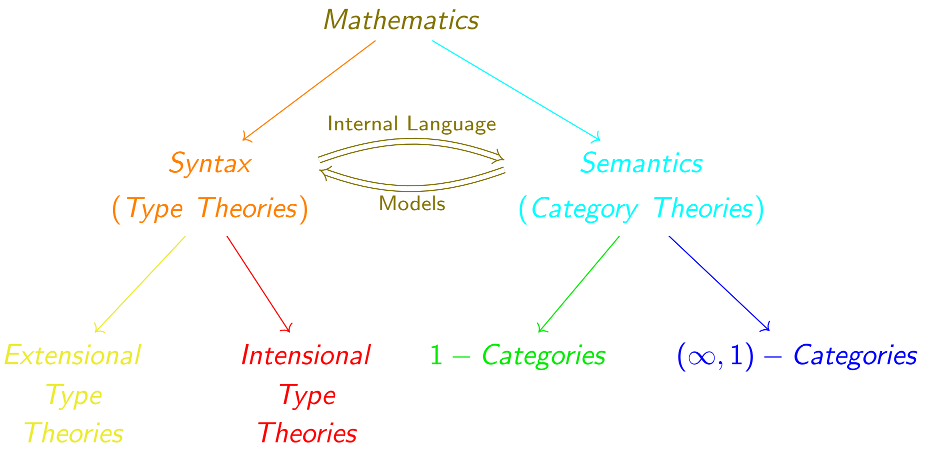
Syntax vs. Semantics



Syntax vs. Semantics



Extensional Type Theories vs. 1-Categories

Type Theory	Internal Language → / ← Models	Category Theory
$\lambda-$ Calculus	*Lambek* $\Longleftrightarrow$	Cartesian closed $1-$ categories
$\cup\vert$		$\cup\vert$
Extensional Martin-Löf Type Theory	*Lambek,Scott* $\Longleftrightarrow$	Locally Cartesian closed $1-$ categories
$\cup\vert$		$\cup\vert$
Higher order type theory	*Lambek,Scott* $\Longleftrightarrow$	Elementary $1-$ topoi

Lambek, *Cartesian Closed Categories and Typed Lambda-calculi*. Combinators and Func. Prog. Lang. (1985)

Lambek, Scott, *Introduction to higher order categorical logic* (1988)

Intensional Type Theories vs. $(\infty,1)$-Categories

Type Theory		Category Theory
Type Theory	Internal Language $\rightarrow$ / $\leftarrow$ Models	Category Theory
Type Theory with dependent sums and intensional identity types	$\Longleftrightarrow$ *Kapulkin,Szumilo*	finitely complete $(\infty,1)-$ categories
$\cup\!\mid$		$\cup\!\mid$
Intensional Martin-Löf Type Theory with $\prod,\sum,\mathrm{id}-$ types	$\Longrightarrow$ *Kapulkin*	Locally Cartesian closed $(\infty,1)-$ categories
$\cup\!\mid$		$\cup\!\mid$
Homotopy type theories	$\Longleftrightarrow$???	???

Kapulkin, Szumilo, *Internal languages of finitely complete $(\infty,1)$-categories* Selecta Math. (N.S.) 25 (2019)

Kapulkin, *Locally cartesian closed quasi-categories from type theory*, J. Topol. 10 (2017)

Is there anything we can say about models of intensional type theory?

If we add one non-elementary condition to the $(\infty, 1)$-category side, namely *presentability*, we do get interesting models:

1. *Presentable locally Cartesian closed $(\infty, 1)$-categories* are models of Intensional Martin-Löf Type Theory with $\prod$-, $\sum$-, and id-types. [Gepner-Kock, 2017], [Lumsdaine-Warren, 2015], [Shulman, 2015].
2. *Grothendieck $(\infty, 1)$-topoi* (presentable locally Cartesian closed $(\infty, 1)$-categories that satisfy *descent*) are a model for homotopy type theory [Shulman, 2019].

Intensional Type Theories vs. $(\infty,1)$-Categories

Type Theory	Internal Language ⇒ / ⇐ Models	Category Theory
Type Theory with dependent sums and intensional identity types	⇔ *Kapulkin,Szumilo*	finitely complete $(\infty,1)-$ categories
$\cup\vert$		$\cup\vert$
Intensional Martin-Löf Type Theory with $\prod, \sum, \mathrm{id}-$ types	*Kapulkin* ⇒ / ⇐ if Presentable	Locally Cartesian closed $(\infty,1)-$ categories
$\cup\vert$		$\cup\vert$
Homotopy type theories	⇐ if Grothendieck	???

Intensional Type Theories vs. $(\infty, 1)$-Categories

Type Theory		Category Theory
Type Theory	Internal Language ⇒ / ⇐ Models	Category Theory
Type Theory with dependent sums and intensional identity types	⇔ *Kapulkin,Szumilo*	finitely complete $(\infty, 1)-$categories
∪|		∪|
Intensional Martin-Löf Type Theory with $\prod, \sum, \mathrm{id}-$types	⇒ *Kapulkin*; ⇐ if Presentable	Locally Cartesian closed $(\infty, 1)-$categories
∪|		∪|
Homotopy type theories	⇐ if Grothendieck	Elementary $(\infty, 1)-$topos ???

Elementary $(\infty, 1)$-topoi are the Answer ...

This suggests that we should develop

Elementary $(\infty, 1)$-Topos Theory

and prove a result analogous to the relation between extensional type theories and elementary 1-topoi.

... but they are difficult to study

We know some things about elementary $(\infty, 1)$-topoi, but not yet enough to relate it to homotopy type theory. Here is a more realistic step:

Goal

1. *Construct a specific elementary $(\infty, 1)$-topos.*
2. *Prove it is a model for homotopy type theory.*

This talk focuses on Step 1.

Can we even define elementary $(\infty, 1)$-topoi?

Warning

There are definitions of elementary $(\infty, 1)$-topoi that have been proposed, but the "correct" definition depends on its relation to homotopy type theory.

Nonetheless we will work with **a** definition throughout this talk!

Elementary $(\infty, 1)$-topos

Definition (Shulman, R.)

An *elementary $(\infty, 1)$-topos* is an $(\infty, 1)$-category $\mathcal{E}$ satisfying following conditions:

1. $\mathcal{E}$ has finite limits and colimits.
2. $\mathcal{E}$ is locally Cartesian closed.
3. $\mathcal{E}$ has a subobject classifier Ω.
4. There exists a class of object $\mathcal{U}^S$ (*universes*) and embeddings of functors
$$\mathcal{I}^S : \mathrm{Map}(-, \mathcal{U}^S) \hookrightarrow (\mathcal{E}_{/-})^{\simeq}$$
such that the family of embeddings $\{\mathcal{I}^S\}_S$ is jointly surjective.

What does any of this mean?

1. $\mathrm{Map}(-,\mathcal{U}^S) \to (\mathcal{E}_{/-})^{\simeq} \Rightarrow$ universal fibration $\tilde{\mathcal{U}}^S \twoheadrightarrow \mathcal{U}^S$.
2. $\mathrm{Map}(-,\mathcal{U}^S) \hookrightarrow (\mathcal{E}_{/-})^{\simeq}$ an embedding $\Rightarrow \tilde{\mathcal{U}}^S \twoheadrightarrow \mathcal{U}^S$ *univalent*
3. $\mathcal{I}^S$ jointly surjective $\Rightarrow$ every map *classified* by some $\mathcal{U}^S$.
4. Disagreement on how to characterize universes.
5. We often want the image of $\mathcal{I}^S$ to be closed under operations (limits, colimits, ...).

How does it relate to other definitions?

Here is a basic result relating various notions of topoi.

Lemma (R.)

Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos.

1. *The subcategory of 0-truncated objects is an elementary 1-topos.*
2. *$\mathcal{E}$ satisfies descent. In particular $\mathcal{E}$ is presentable if and only if it is a Grothendieck $(\infty, 1)$-topos.*

So, it is a common generalization of elementary 1-topoi and Grothendieck $(\infty, 1)$-topoi.

Only Non-Presentability Counts

The result by Shulman implies that presentable elementary $(\infty, 1)$-topoi are already models and we should focus on non-presentable ones.

Question

How can we construct non-presentable elementary $(\infty, 1)$-topoi?

Filter Construction: Introduction

Let $\mathcal{E}$ be a finitely complete 1-category. Let $\mathcal{F} \subset \mathrm{Sub}(1)$ be a *filter of subterminal objects*, meaning:

1. *Non-Empty:* $1 \in \mathcal{F}$.
2. *Intersections:* $U, V \in \mathcal{F} \Rightarrow U \times V \in \mathcal{F}$.
3. *Upwards closed:* $U \in \mathcal{F}, U \leq V \Rightarrow V \in \mathcal{F}$

Then we will define a new category $\mathcal{E}_{\mathcal{F}}$.

Filter Construction: Construction

- Objects of $\mathcal{E}_{\mathcal{F}}$ are objects of $\mathcal{E}$.
- For two object X, Y we have

$$\mathrm{Hom}_{\mathcal{E}_{\mathcal{F}}}(X, Y) = \{f : X \times U \to Y : U \in \mathcal{F}\}/ \sim$$

where for $f : X \times U \to Y$, $g : X \times V \to Y$

$$f \sim g \Leftrightarrow \exists W \in \mathcal{F}(f \times \mathrm{id}_W = g \times \mathrm{id}_W)$$

Filter Construction: Results

Lemma (Johnstone: Sketches of an Elephant)

The quotient map

$$\mathcal{P}_{\mathcal{F}} : \mathcal{E} \to \mathcal{E}_{\mathcal{F}}$$

preserves

1. **finite** *limits and colimits,*
2. *locally Cartesian structure,*
3. *subobject classifier.*

So, if $\mathcal{E}$ is an elementary 1-topos then $\mathcal{E}_{\mathcal{F}}$ is one as well.

Filter Construction: Generalization

We want to generalize this construction to $(\infty, 1)$-categories. Here we need to care about which model of $(\infty, 1)$-categories we are using:

1. Kan enriched categories
2. Quasi-Categories
3. Complete Segal spaces

Filter Construction: Kan enriched categories

1. **Input:** A finitely complete Kan enriched category $\mathcal{C}$ and a filter of subterminal objects $\mathcal{F}$.
2. We can take $\mathcal{C}$ to be a simplicial object in categories:

$$\mathcal{C}_\bullet : \Delta^{op} \to \mathcal{C}\mathrm{at}.$$

3. Construct $(\mathcal{C}_\bullet)_\mathcal{F} : \Delta^{op} \to \mathcal{C}\mathrm{at}$.
4. **Output:** The simplicial category $\mathcal{C}_\mathcal{F}$, which is a Kan enriched category.

Filter Construction: Quasi-categories and Complete Segal spaces

Let $\mathcal{C}$ be a finitely complete quasi-category or CSS and $\mathcal{F}$ a filter of subterminal objects. Then define the functor

$$\begin{array}{ccc} \mathcal{C}_{/-} : \mathcal{F}^{op} & \longrightarrow & \mathcal{C}\mathrm{at}_\infty \\ U & \longmapsto & \mathcal{C}_{/U} \\ \big\downarrow & & \big\uparrow {\scriptstyle - \times U} \\ V & \longmapsto & \mathcal{C}_{/V} \end{array}$$

Then we define the *filter construction* as the colimit

$$\mathcal{C}_{\mathcal{F}} = \mathrm{colim}(\mathcal{C}_{/-} : \mathcal{F}^{op} \to \mathcal{C}\mathrm{at}_\infty).$$

Filter Construction and Topos Theory

Theorem (R.)

Let $\mathcal{C}$ be finitely complete $(\infty, 1)$-category and $\mathcal{F}$ a filter of subterminal objects. Then we have a quotient functor

$$\mathcal{P}_{\mathcal{F}} : \mathcal{C} \to \mathcal{C}_{\mathcal{F}}$$

which preserves

1. **finite** *limits and colimits*
2. *locally Cartesian closed structure*
3. *subobject classifiers*
4. *universes*

So, in particular if $\mathcal{E}$ is an elementary $(\infty, 1)$-topos then $\mathcal{E}_{\mathcal{F}}$ is one as well.

How do we get non-Presentable Examples?

Theorem (Adelman-Johnstone 82)

Let $\mathcal{I}$ be a set and $\mathcal{F}$ a non-principal filter on $\mathcal{S}\mathrm{et}^{\mathcal{I}}$ (which is just a filter on $\mathrm{P}(\mathcal{I})$). Then the filter construction $(\mathcal{S}\mathrm{et}^{\mathcal{I}})_{\mathcal{F}}$ is non-presentable elementary 1-topos and so a non-presentable model of higher order type theory.

This result generalizes appropriately.

How do we get non-Presentable Examples?

Theorem (R.)

Let $\mathcal{I}$ be a set and $\mathcal{F}$ a non-principal filter on $\mathcal{K}\mathrm{an}^{\mathcal{I}}$. Then the filter construction $(\mathcal{K}\mathrm{an}^{\mathcal{I}})_{\mathcal{F}}$ is a non-presentable elementary $(\infty, 1)$-topos.

Example (R.)

Let $\mathcal{F}$ be the filter of co-finite subsets on $\mathbb{N}$ (the *Fréchet filter*). Then $(\mathcal{K}\mathrm{an}^{\mathbb{N}})_{\mathcal{F}}$ is an elementary $(\infty, 1)$-topos, such that:

1. It is not presentable.
2. It has no infinite coproducts (except for initial object).
3. The natural number object is non-standard.

Let's Summarize!

1. We want models of homotopy type theory.
2. We defined elementary $(\infty, 1)$-topoi and hope to prove they give us the desired models.
3. Shulman's result covers the presentable case so the focus should be on non-presentable ones.
4. Using the filter construction, we get a method for construction non-presentable elementary $(\infty, 1)$-topoi.
5. Can we prove these are models?

How does this tie to Type Theory?

The filter construction is a (filtered) colimit.

Question

Are models of homotopy type theory closed under (filtered) colimits?

The results by Shulman only prove closure under presheaf and localization constructions.

References. Thank you! Questions?

For more details see:

Filter Quotients and Non-Presentable $(\infty, 1)$-Toposes,
arXiv:2001.10088

Thank You!

Questions?