A Step towards Non-Presentable Models of Homotopy Type Theory

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Syntax vs. Semantics

Mathematics

Syntax
(Type Theories)

Semantics
(Category Theories)

Internal Language

Models

Extensional Type Theories

Intensional Type Theories

1 – Categories

(∞, 1) – Categories
Syntax vs. Semantics

Mathematics

Syntax
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Extensional Type Theories

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(∞, 1) − Categories
### Extensional Type Theories vs. 1-Categories

- Type Theory
  - Extensional Martin-Löf Type Theory
  - Higher order type theory

- Category Theory
  - Cartesian closed 1-categories
  - Locally Cartesian closed 1-categories
  - Elementary 1-topoi

#### Lambek

#### Lambek, Scott
- Introduction to higher order categorical logic (1988)
Intensional Type Theories vs. \((\infty, 1)\)-Categories

Type Theory

- Type Theory with dependent sums and intensional identity types
  
  \[
  \mathsf{UI} \\
  \mathsf{Pi}, \mathsf{Sigma}, \mathsf{id} - \text{types} \\
  \mathsf{UI} \\
  \mathsf{Homotopy type theories}
  \]

Category Theory

- Intensional Martin-Löf Type Theory with \(\mathsf{Pi}, \mathsf{Sigma}, \mathsf{id} - \text{types}\)

- \(\mathsf{UI}\)

- Homotopy type theories

Internal Language

Kapulkin-Szumilo

- \(\mathsf{Kapulkin, Szumilo}\)

- Finely complete \((\infty, 1) - \text{categories}\)

Kapulkin

- \(\mathsf{Kapulkin}\)

- Locally Cartesian closed \((\infty, 1) - \text{categories}\)

- \(\mathsf{UI}\)

- ???


Is there anything we can say about models of intensional type theory?

If we add one non-elementary condition to the \((\infty, 1)\)-category side, namely \textit{presentability}, we do get interesting models:

1. \textit{Presentable locally Cartesian closed} \((\infty, 1)\)-\textit{categories} are models of Intensional Martin-Löf Type Theory with \(\Pi\)-, \(\Sigma\)-, and \textit{id}-types. [Gepner-Kock, 2017], [Lumsdaine-Warren, 2015], [Shulman, 2015].

2. \textit{Grothendieck} \((\infty, 1)\)-\textit{topoi} (presentable locally Cartesian closed \((\infty, 1)\)-categories that satisfy \textit{descent}) are a model for homotopy type theory [Shulman, 2019].
Intensional Type Theories vs. $(\infty, 1)$-Categories

Type Theory

- Type Theory with dependent sums and intensional identity types
- Intensional Martin-Löf Type Theory with $\Pi, \Sigma, \text{id} - \text{types}$
- Homotopy type theories

Category Theory

- finitely complete $(\infty, 1) - \text{categories}$
- Locally Cartesian closed $(\infty, 1) - \text{categories}$

Internal Language

Models

Kapulkin, Szumilo

Kapulkin

if Presentable

if Grothendieck

???
Intensional Type Theories vs. $(\infty, 1)$-Categories

Type Theory

Type Theory with dependent sums and intensional identity types

UI

Intensional Martin-Löf Type Theory with $\Pi, \Sigma, \text{id} -$ types

UI

Homotopy type theories

Category Theory

Internal Language

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Kapulkin, Szumilo

finitely complete $(\infty, 1)$ – categories

UI

Locally Cartesian closed $(\infty, 1)$ – categories

UI

Elementary $(\infty, 1)$– topos ???

Kapulkin

if Presentable

if Grothendieck
Elementary $(\infty, 1)$-topoi are the Answer ...

This suggests that we should develop

**Elementary $(\infty, 1)$-Topos Theory**

and prove a result analogous to the relation between extensional type theories and elementary 1-topoi.
... but they are difficult to study

We know some things about elementary \((\infty, 1)\)-topoi, but not yet enough to relate it to homotopy type theory. Here is a more realistic step:

**Goal**

1. *Construct a specific elementary \((\infty, 1)\)-topos.*
2. *Prove it is a model for homotopy type theory.*

This talk focuses on Step 1.
Can we even define *elementary $(\infty, 1)$-topoi*?

**Warning**

There are definitions of *elementary $(\infty, 1)$-topoi* that have been proposed, but the “correct” definition depends on its relation to *homotopy type theory*.

Nonetheless we will work with a definition throughout this talk!
Definition (Shulman, R.)

An *elementary $(\infty, 1)$-topos* is an $(\infty, 1)$-category $\mathcal{E}$ satisfying the following conditions:

1. $\mathcal{E}$ has finite limits and colimits.
2. $\mathcal{E}$ is locally Cartesian closed.
3. $\mathcal{E}$ has a subobject classifier $\Omega$.
4. There exists a class of object $\mathcal{U}^S$ (*universes*) and embeddings of functors

$$\mathcal{I}^S : \text{Map}(\_, \mathcal{U}^S) \hookrightarrow (\mathcal{E}/\_)^\sim$$

such that the family of embeddings $\{\mathcal{I}^S\}_S$ is jointly surjective.
What does any of this mean?

1. $\text{Map}( -, \mathcal{U}_S ) \to (\mathcal{E}/-) \simeq \Rightarrow$ universal fibration $\tilde{\mathcal{U}}_S \to \mathcal{U}_S$.
2. $\text{Map}( -, \mathcal{U}_S ) \hookrightarrow (\mathcal{E}/-) \simeq$ an embedding $\Rightarrow \tilde{\mathcal{U}}_S \to \mathcal{U}_S$ univalent
3. $\mathcal{I}_S$ jointly surjective $\Rightarrow$ every map classified by some $\mathcal{U}_S$.
4. Disagreement on how to characterize universes.
5. We often want the image of $\mathcal{I}_S$ to be closed under operations (limits, colimits, ... ).
How does it relate to other definitions?

Here is a basic result relating various notions of topoi.

**Lemma (R.)**

Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos.

1. The subcategory of 0-truncated objects is an elementary 1-topos.

2. $\mathcal{E}$ satisfies descent. In particular $\mathcal{E}$ is presentable if and only if it is a Grothendieck $(\infty, 1)$-topos.

So, it is a common generalization of elementary 1-topoi and Grothendieck $(\infty, 1)$-topoi.
Only Non-Presentability Counts

The result by Shulman implies that presentable elementary $(\infty, 1)$-topoi are already models and we should focus on non-presentable ones.

Question
How can we construct non-presentable elementary $(\infty, 1)$-topoi?
Filter Construction: Introduction

Let $\mathcal{E}$ be a finitely complete 1-category. Let $\mathcal{F} \subset \text{Sub}(1)$ be a filter of subterminal objects, meaning:

1. **Non-Empty**: $1 \in \mathcal{F}$.
2. **Intersections**: $U, V \in \mathcal{F} \Rightarrow U \times V \in \mathcal{F}$.
3. **Upwards closed**: $U \in \mathcal{F}, U \leq V \Rightarrow V \in \mathcal{F}$

Then we will define a new category $\mathcal{E}_\mathcal{F}$. 
Filter Construction: Construction

- Objects of $\mathcal{E}_F$ are objects of $\mathcal{E}$.
- For two objects $X, Y$ we have

$$\text{Hom}_{\mathcal{E}_F}(X, Y) = \{f : X \times U \to Y : U \in \mathcal{F}\} / \sim$$

where for $f : X \times U \to Y$, $g : X \times V \to Y$

$$f \sim g \iff \exists W \in \mathcal{F}(f \times \text{id}_W = g \times \text{id}_W)$$
Filter Construction: Results

Lemma (Johnstone: Sketches of an Elephant)

The quotient map

\[ \mathcal{P}_\mathcal{F} : \mathcal{E} \to \mathcal{E}_\mathcal{F} \]

preserves

1. finite limits and colimits,
2. locally Cartesian structure,
3. subobject classifier.

So, if \( \mathcal{E} \) is an elementary 1-topos then \( \mathcal{E}_\mathcal{F} \) is one as well.
Filter Construction: Generalization

We want to generalize this construction to $(\infty, 1)$-categories. Here we need to care about which model of $(\infty, 1)$-categories we are using:

1. Kan enriched categories
2. Quasi-Categories
3. Complete Segal spaces
Filter Construction: Kan enriched categories

1. **Input**: A finitely complete Kan enriched category $\mathcal{C}$ and a filter of subterminal objects $\mathcal{F}$.
2. We can take $\mathcal{C}$ to be a simplicial object in categories:
   \[
   \mathcal{C}_\bullet : \Delta^{op} \to \mathbf{Cat}.
   \]
3. Construct $(\mathcal{C}_\bullet)_\mathcal{F} : \Delta^{op} \to \mathbf{Cat}$.
4. **Output**: The simplicial category $\mathcal{C}_\mathcal{F}$, which is a Kan enriched category.
Let $\mathcal{C}$ be a finitely complete quasi-category or CSS and $\mathcal{F}$ a filter of subterminal objects. Then define the functor

$$\mathcal{C}/_- : \mathcal{F}^{op} \longrightarrow \mathcal{Cat}_\infty$$

Then we define the *filter construction* as the colimit

$$\mathcal{C}_\mathcal{F} = \text{colim}(\mathcal{C}/_- : \mathcal{F}^{op} \rightarrow \mathcal{Cat}_\infty).$$
The Filter Construction and Topos Theory

**Theorem (R.)**

Let \( \mathcal{C} \) be finitely complete \((\infty,1)\)-category and \( \mathcal{F} \) a filter of subterminal objects. Then we have a quotient functor

\[
P_\mathcal{F} : \mathcal{C} \to \mathcal{C}_\mathcal{F}
\]

which preserves

1. **finite limits and colimits**
2. *locally Cartesian closed structure*
3. subobject classifiers
4. universes

So, in particular if \( \mathcal{E} \) is an **elementary \((\infty,1)\)-topos** then \( \mathcal{E}_\mathcal{F} \) is one as well.
How do we get non-Presentable Examples?

**Theorem (Adelman-Johnstone 82)**

Let $\mathcal{I}$ be a set and $\mathcal{F}$ a non-principal filter on $\text{Set}^\mathcal{I}$ (which is just a filter on $\mathcal{P}(\mathcal{I})$). Then the filter construction $(\text{Set}^\mathcal{I})_\mathcal{F}$ is non-presentable elementary $1$-topos and so a non-presentable model of higher order type theory.

This result generalizes appropriately.
How do we get non-Presentable Examples?

**Theorem (R.)**

Let $I$ be a set and $\mathcal{F}$ a non-principal filter on $\text{Kan}^I$. Then the filter construction $(\text{Kan}^I)_\mathcal{F}$ is a non-presentable elementary $(\infty, 1)$-topos.

**Example (R.)**

Let $\mathcal{F}$ be the filter of co-finite subsets on $\mathbb{N}$ (the Fréchet filter). Then $(\text{Kan}^\mathbb{N})_\mathcal{F}$ is an elementary $(\infty, 1)$-topos, such that:

1. It is not presentable.
2. It has no infinite coproducts (except for initial object).
3. The natural number object is non-standard.
Let’s Summarize!

1. We want models of homotopy type theory.
2. We defined elementary \((\infty, 1)\)-topoi and hope to prove they give us the desired models.
3. Shulman’s result covers the presentable case so the focus should be on non-presentable ones.
4. Using the filter construction, we get a method for construction non-presentable elementary \((\infty, 1)\)-topoi.
5. Can we prove these are models?
How does this tie to Type Theory?

The filter construction is a (filtered) colimit.

**Question**

Are models of *homotopy type theory* closed under (filtered) colimits?

The results by Shulman only prove closure under presheaf and localization constructions.
References. Thank you! Questions?

For more details see:

Filter Quotients and Non-Presentable $(\infty, 1)$-Toposes,

Thank You!

Questions?