An Example of an Elementary $(\infty, 1)$-Topos that is not a Grothendieck $(\infty, 1)$-Topos

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In this talk we introduce a new construction, *the filter quotient*, and show how the filter quotient of every elementary $(\infty, 1)$-topos is again an elementary $(\infty, 1)$-topos. Then we show how in specific instances this gives us ways to contruct elementary $(\infty, 1)$-toposes that are not Grothendieck $(\infty, 1)$-toposes.

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A Theory of Elementary $(\infty, 1)$-Toposes

Elementary topos theory studies logical aspects in category. Roughly speaking an elementary topos is a category that behaves "like the category of sets". It has been developed quite extensively by people such as Lawvere, Tierney, Johnstone[Jo03], ...

My (and some other peoples) project is to develop an $(\infty, 1)$-categorical analogue of elementary topos theory, which can be captured under the term *elementary* $(\infty, 1)$*-topos*. Following the same logic an elementary $(\infty, 1)$-topos should be an $(\infty, 1)$-category that behaves "like the category of spaces". The current plan is to develop elementary $(\infty, 1)$-topos theory in theory and application.

Remark 1.1. The word $(\infty, 1)$-category is a term generally used when talking about categories with a notion of equivalence. There are various ways to make this notion precise, which are called "models of $(\infty, 1)$-categories". Whenever there is any need to be precise we will refer to the precise model we are using.

Here is the definition we will use throughout.

Definition 1.2. An elementary $(\infty, 1)$-topos $\mathcal{E}$ is an $(\infty, 1)$-category such that

1. It has finite limits and colimits.
2. It is locally Cartesian closed.
3. It has a subobject classifier.
4. It has sufficient universes.

Definition 1.3. A subobject classifier is an object Ω and natural isomorphisms

$$\mathcal{S}\mathrm{ub}(X) \cong Hom(X, \Omega).$$

Definition 1.4. A universe is the data of a class of maps S, an object $\mathcal{U}$ and natural equivalences of spaces

$$((\mathcal{E}_{/X})^S)^{core} \simeq Map_{\mathcal{E}}(X, \mathcal{U})$$

Definition 1.5. $\mathcal{E}$ has sufficient universes if the collection $((\mathcal{E}_{/X})^S)^{core}$ is jointly surjective on $(\mathcal{E}_{/X})^{core}$.

Remark 1.6. We often make assumptions on $\mathcal{U}$ such that the category $(\mathcal{E}_{/X})$ is closed under certain constructions such as limits, colimits, Π-types, ...

Let us the give the most important example of such an elementary $(\infty, 1)$-topos.

Example 1.7. Let $\mathcal{K}\mathrm{an}$ be the category of Kan complexes. We want to confirm it satisfies the four axioms:

1. It has finite limits and colimits.
2. It is locally Cartesian closed.
3. The subobject classifier is $\Omega = \{0, 1\}$. The key observation is that a map in $\mathcal{K}\mathrm{an}$ is mono if and only if it is an inclusion of path components i.e. a local weak equivalence.
4. We now need to show we have enough universes. Let κ be a large enough cardinal. Let $\mathcal{K}\mathrm{an}^\kappa$ be the category of κ-small Kan complexes. Let $(\mathcal{K}\mathrm{an}^\kappa)^{core}$ be the subcategory with morphisms weak equivalences. Then we can form a Kan complex $N((\mathcal{K}\mathrm{an}^\kappa)^{core})$ in which the n-cells roughly correspond to a choice of n-composable weak equivalences. Note this Kan complex is certainly not κ-small. Moreover, there is an equivalence of Kan complexes

   $$N(((\mathcal{K}\mathrm{an}_{/X})^\kappa)^{core}) \simeq Map(X, N((\mathcal{K}\mathrm{an}^\kappa)^{core}))$$

   where the left hand side is κ-small maps over X. For example, let $X = \Delta^0$ then we have

   $$N(\mathcal{K}\mathrm{an}^\kappa)^{core} \simeq Map(\Delta^0, N((\mathcal{K}\mathrm{an}^\kappa)^{core}))$$

We can think of an elementary $(\infty,1)$-topos as a common generalization of a Grothendieck $(\infty,1)$-topos and an elementary topos.

Theorem 1.8. *Every Grothendieck $(\infty,1)$-topos (in the sense of Lurie) is an elementary $(\infty,1)$-topos.*

Theorem 1.9. *The subcategory of 0-truncated objects $\tau_0\mathcal{E}$ is an elementary topos. We call $\tau_0\mathcal{E}$ the underlying elementary topos.*

There are some things we can already prove about such an elementary $(\infty,1)$-topos.

Theorem 1.10. *Every elementary $(\infty,1)$-topos has a natural number object.*

Remark 1.11. There are three different ways to define a natural number object, Lawvere, Freyd and Peano, and they all coincide.

Theorem 1.12. *For every natural number $n : \mathbb{N}$ there is a truncation functor*

$$\tau_n : \mathcal{E} \to \tau_n\mathcal{E}$$

The key about those two previous results is that they are pretty straightforward if $\mathcal{E}$ has infinite colimits, but are highly non-trivial if we only have finite limits and colimits. Moreover, the previous holds for every natural number, which gains its full strength when we have a non-standard natural number object, which is impossible if we have infinite colimits. So, in particular, the results are not very interesting when we are working with Grothendieck $(\infty,1)$-toposes, as the natural number object is just $\coprod_{\mathbb{N}} 1$ and truncation follows from localizing at certain spheres.

This means we should care about finding examples of elementary $(\infty,1)$-toposes that aren't Grothendieck $(\infty,1)$-toposes. This has proven to be quite difficult.

Example 1.13 ([Lo19])**.** The category of finite sets is an elementary 1-topos, which does not have have infinite colimits. Thus we would hope that some analogous notion of finite spaces gives us an elementary $(\infty,1)$-topos. However, this is not correct because any notion of finite spaces lacks universes.

The way to fix things is to take a suitable notion of large cardinal, a 1-*inaccessible cardinal.* And show small spaces form an elementary $(\infty,1)$-topos that is not Grothendieck. This has been done by Lo Monaco in recent work. Thus this method does not give us an elementary $(\infty,1)$-topos without infinite colimits.

The goal of this talk is to construct an elementary $(\infty,1)$-topos that isn't a Grothendieck $(\infty,1)$-topos.

The key observation is following proposition:

Proposition 1.14. *Let $\mathcal{E}$ be an elementary $(\infty,1)$-topos such that its underlying topos $\tau_0\mathcal{E}$ is not a Grothendieck topos. Then $\mathcal{E}$ is not a Grothendieck $(\infty,1)$-topos.*

The proof is just the fact that the underlying elementary topos of a Grothendieck $(\infty, 1)$-topos is a Grothendieck 1-topos.

It is not known if the opposite statement holds.

The Filter Construction of an Elementary Topos

The goal is to give a construction that helps us build new toposes. The construction involves a notion of filter.

Definition 2.1. Let $(P, \leq)$ be a partially ordered set. A *filter* F is a subset of P that satisfies following conditions.

1. $F \neq \emptyset$.
2. F is downward directed, meaning that for any two object $x, y \in F$ there exists $z \in F$ such that $z \leq x$ and $z \leq y$.
3. F is upward closed, meaning that if $x \leq y$ and $x \in F$, then $y \in F$.

Example 2.2. For any poset P and object x the subset $\{y \in P : x \leq y\}$ is a filter. We call such filter a *principal filter*.

Definition 2.3. A maximal filter $F \subsetneq P$ is called an *ultrafilter*.

Remark 2.4. We will only care about filters on the Heyting algebra $\mathcal{S}\mathrm{ub}(1)$, which has maximal object and meets and thus for us a filter is any subset that contains the maximal elements, is closed under meets and is upward closed.

Let $\mathcal{E}$ be an elementary 1-topos. Then $\mathcal{S}\mathrm{ub}(1)$ is a poset (and in fact a Heyting algebra). Let Φ be a filter on the poset $\mathcal{S}\mathrm{ub}(1)$. We want to define a new topos $\mathcal{E}_\Phi$ as follows.

1. It has objects the same objects as $\mathcal{E}$.
2. For two objects A, B, the hom set $Hom_{\mathcal{E}_\Phi}(A, B)$, is the defined as the set
$$Hom_{\mathcal{E}_\Phi}(A, B) = \{f : A \times U \to B : U \in \Phi\}/\sim$$
where the equivalence relation is defined as follows: $f \sim g$ if there is a commutative square
$$\begin{tikzcd} A \times W \arrow[r] \arrow[d] & A \times U \arrow[d, "f"] \\ A \times V \arrow[r, "g"] & B \end{tikzcd}$$

So, we have the same set of objects, but morphisms are partially defined maps, up to morphisms that are equal "from some point on". We have following basic observation about this construction.

Example 2.5 ([Jo03, Example A2.1.13])**.** The filter quotient construction $\mathcal{E}_\Phi$ is an elementary topos.

Remark 2.6. The main reason we care about this construction is that Grothendieck 1-toposes are not closed under filter quotients.

We want an analogous construction for elementary $(\infty, 1)$-topos. The problem is that morphisms of $(\infty, 1)$-categories are defined up to homotopy and thus special care is required.

Filter Quotient of an Elementary $(\infty, 1)$-Topos

Up until now we have been very vague about what an $(\infty, 1)$-category is. However, in order to be able to give a correct construction of a filter quotient we need to be more precise.

Definition 3.1. A Kan enriched category is a category enriched over the category of Kan complexes.

Theorem 3.2 ([Be07])**.** *Kan enriched categories are a model of* $(\infty, 1)$*-categories.*

We can now state the main theorem.

Theorem 3.3. *Let $\mathcal{E}$ be elementary Kan enriched topos and Φ a filter in $\mathcal{S}ub(1)$. Then we can construct a new elementary Kan enriched topos $\mathcal{E}_\Phi$.*

Proof. The idea of the proof is as follows. A Kan enriched category $\mathcal{E}$ is a simplicial object

$$\mathcal{E}_\bullet : \Delta^{op} \to \mathcal{C}\mathrm{at}$$

thus we have level-wise 1-categories $\mathcal{E}_n$, which all have the same class of objects. We can thus apply the filter quotient on the category $\mathcal{E}_n$. It is not difficult to see that the construction is functorial and thus the we get a new Kan enriched category $\mathcal{E}_\Phi$ (here we are using the technical fact that the filtered colimit of Kan complexes is again a Kan complex).

We want to show that this Kan enriched category $\mathcal{E}_\Phi$ is an elementary Kan enriched topos. The key realization is that all the objects that satisfy the universal property in $\mathcal{E}$ still satisfy the same universal property in $\mathcal{E}_\Phi$.

In particular, the final object in $\mathcal{E}_\Phi$ is just the final object in $\mathcal{E}$. Moreover, if we take two maps $f : A \times U \to C$ and $g : B \times V \to C$, then they are equivalent to the maps $f\pi_1 : (A \times U) \times V \to C$ and $g\pi_1 : (B \times V) \times U \to C$ the pullback in $\mathcal{E}_\Phi$ is just the pullback of in $\mathcal{E}$, $A \times_C B \times U \times V$.

A careful analysis of subobjects in $\mathcal{E}_\Phi$ proves that Ω is stil the subobject classifier (here we use the fact that a map is mono in $\mathcal{E}_\Phi$ if and only if it is equivalent to any map that was mono in $\mathcal{E}$ i.e. mono maps are the "eventually mono maps"). Finally, the same universes $\mathcal{U}$ in $\mathcal{E}$ will be universes in $\mathcal{E}_\Phi$. $\square$

Remark 3.4. Notice, the argument about limits, colimits and subobject classifier is quite straightforward, but the argument about universes is a little bit involved and requires us to switch between various models of $(\infty, 1)$-categories.

Examples of Elementary $(\infty, 1)$-Toposes that is not Grothendieck $(\infty, 1)$-Toposes

We want to use the filter quotient to build some interesting elementary $(\infty, 1)$-toposes. For that we need an important observation.

Theorem 4.1. *Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos and Φ a filter. Then we have an equivalence of 1-categories*

$$\tau_0(\mathcal{E}_\Phi) \simeq (\tau_0(\mathcal{E}))_\Phi$$

Remark 4.2. This equivalence is not an isomorphisms. In particular objects in $\tau_0(\mathcal{E}_\Phi)$ are the "eventually 0-truncated objects", whereas objects in $\tau_0(\mathcal{E}))_\Phi$ are globally 0-truncated.

The upshot is: All we need to do is start with an elementary $(\infty, 1)$-topos $\mathcal{E}$ and a filter Φ such that $(\tau_0(\mathcal{E}))_\Phi$ is not a Grothendieck topos. Here we can use our vast knowledge about elementary 1-toposes.

Example 4.3. Let $\mathcal{K}\mathrm{an}$ be the Kan enriched category of Kan complexes. This is an example of an elementary $(\infty, 1)$-topos with underlying elementary topos $\mathcal{S}\mathrm{et}$. Let S be any set. Then the category $\mathcal{K}\mathrm{an}^S$ of S-indexed Kan complexes is also an elementary $(\infty, 1)$-topos with underlying elementary topos $\mathcal{S}\mathrm{et}^S$. Moreover, the final object in $\mathcal{K}\mathrm{an}^S$ is just $(1)_{s\in S}$. Thus, $\mathcal{S}\mathrm{ub}(1) = P(S)$, the set of subsets of S and so a filter on $\mathcal{K}\mathrm{an}^S$ is just a filter on the power set $P(S)$. For any such filter we get an elementary $(\infty, 1)$-topos $(\mathcal{K}\mathrm{an}^S)_\Phi$ such that the underlying elementary topos is $(\mathcal{S}\mathrm{et}^S)_\Phi$.

Pick a filter Φ on $P(S)$ such that the elementary topos $(\mathcal{S}\mathrm{et}^S)_\Phi$ is not a Grothendieck topos. Then $(\mathcal{K}\mathrm{an}^S)_\Phi$ is also an elementary $(\infty, 1)$-topos that is not a Grothendieck $(\infty, 1)$-topos.

Here is one example: Let S be infinite and Φ be a non-principal filter. Then $(\mathcal{S}\mathrm{et}^S)_\Phi$ is not a Grothendieck topos [AJ82]. Thus $(\mathcal{K}\mathrm{an}^S)_\Phi$ is also not a Grothendieck $(\infty, 1)$-topos.

The Filter Quotient and Truncations

Let us give one concrete example of the filter construction and its effects on truncation functors.

Example 5.1. Let $\mathcal{C}$ be an $(\infty, 1)$-category and n a natural number. An object X in $\mathcal{C}$ is n-truncated if for every other object Y, the mapping space $Map(Y, X)$ is an n-truncated Kan complex.

The goal is to show how we can expand this definition with non-standard natural number objects.

Example 5.2. This is motivated by [Jo03, D5.1.7]. Let $\mathbb{N}$ be the set of natural numbers, Φ be the filter of cofinite sets. This filter is clearly non-principal and so $(\mathcal{K}\mathrm{an}^{\mathbb{N}})_\Phi$ is not a Grothendieck $(\infty,1)$-topos. We can give a more explicit description of this category:

It's objects are sequences of Kan complexes $(X_n)_{n\in\mathbb{N}}$. Morphisms are maps of sequences

$$(f_n)_{n\geq N} : (X_n)_{n\geq N} \to (Y_n)_{n\geq N}$$

such that $f \sim g$ if and only if $f_n = g_n$ for n large enough.

Recall a natural number object $\mathbb{N}$ is the data

$$1 \xrightarrow{o} \mathbb{N} \xrightarrow{s} \mathbb{N}$$

that is initial among any such data (X, b, u).

The natural number object in $(\mathcal{K}\mathrm{an}^{\mathbb{N}})_\Phi$ is the level-wise Kan complex $(\mathbb{N})_{n\in\mathbb{N}}$. A natural number is a level-wise choice of natural number, which is exactly a sequence $[(a_n)_{n\in\mathbb{N}}]$, where two sequences are in the same class if they agree eventually.

The standard natural numbers n correspond to the class of sequences that converge to n. But then we also have non-standard natural numbers, such as the sequence

$$diag_n = n$$

that are not in the image of the standard natural numbers. To see this, notice that $diag_n$ intersects with each standard natural number only at level n, which means each intersection is eventually trivial, which means it is trivial in $(\mathcal{K}\mathrm{an}^{\mathbb{N}})_\Phi$. Thus the natural number object in $(\mathcal{K}\mathrm{an}^{\mathbb{N}})_\Phi$ has many non-standard natural numbers.

Whenever we have a natural number object and universes, we can give an internal definition of truncation, where the truncation levels correspond to the natural numbers in our natural number object. If the natural number object is standard, then we get nothing new. However, non-standard natural numbers give us truncations levels, which have no external realization.

We can see this in our example. If the sequence $[(a_n)_{n\in\mathbb{N}}]$ is eventually stable (and equal to m) then an object $(K_n)_{n\in\mathbb{N}}$ is $[(a_n)_{n\in\mathbb{N}}]$-truncated if and only if the mapping space

$$Map((L_n)_{n\in\mathbb{N}}, (K_n)_{n\in\mathbb{N}})$$

is an n-truncated space for any other object $(L_n)_{n\in\mathbb{N}}$. But if $[(a_n)_{n\in\mathbb{N}}]$ does not stabilize (for example if it is $[(diag_n)_{n\in\mathbb{N}}]$), then the mapping space does not seem to have any external manifestation.

Future Directions

Where do we go from here?

1. If Φ is a non-principal ultrafilter on $P(S)$ then the elementary topos $(\mathcal{S}\mathrm{et}^S)_\Phi$ has many similarities to the category of sets. In particular, it is generated by the final object and is Boolean. Yet it is not equivalent to the category of sets.

   The question now becomes how much this generalizes to $(\mathcal{K}\mathrm{an}^S)_\Phi$ and how it compares to Kan complexes.

2. We already know that every Grothendieck $(\infty, 1)$-topos is a model for homotopy type theory [Sh19]. But we don't have any non-Grothendieck topos examples yet and so one interesting question is whether this filter quotient construction gives us models for homotopy type theory.

References

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