A Theory of Elementary Higher Toposes

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9/14/2017

The roots of Topos theory: Elementary Toposes

Topos theory was first developed by Grothendieck and the Bourbaki school to be able to expand the notion of a sheaves [GJ72]. Using this new approach they were able to study algebro geometric objects from a new angle and prove new results such as Weil conjecture by using etale cohomology.

Definition 1.1. A *Grothendieck topos* $\mathcal{G}$ is a left-exact localization of a category of set valued presheaves on a small category. That means we have an adjunction

$$Fun(\mathcal{C}^{op}, \mathcal{S}\text{et}) \underset{i}{\overset{a}{\rightleftarrows}} \mathcal{G}$$

where i is an embedding. a is often called the *sheafification functor*.

Example 1.2. Let X be a topological space. Then we can define the category $Open_X$.

- *Objects*: Open subsets of X
- *Morphisms*: Inclusions of open sets

Using this we can define $Fun(Open_X^{op}, \mathcal{S}\text{et})$. Now let $Shv(X)$ be the subcategory whose objects are all functors $F : Open_X^{op} \to \mathcal{S}\text{et}$ such that if $U = \cup_i U_i$ then $F(U)$ is the equalizer of the diagram

$$F(U) \dashrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

This gives us a Grothendieck topos and can be used to study the space X.

At the same time, from a more logical perspective Lavwere was working on a axiomatization of the category of sets [La65]. It was at that time he worked with Tierney and saw the connections between toposes as developed by Grothendieck and axiomatized categories of sets. This lead to a development of a category that was able to combine both concepts, an *elementary topos* [Ti73].

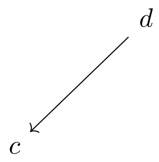
Definition 1.3. An *elementary topos* is a locally cartesian closed category with subobject classifier

Definition 1.4. A category $\mathcal{C}$ is Cartesian closed if it has finite products and for every two object $x, y \in \mathcal{C}$ there is an object y^x and map $x \times y^x \to y$ such that we following equivalence holds

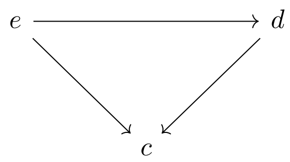
$$Hom_{\mathcal{C}}(z \times x, y) \cong Hom_{\mathcal{C}}(z, y^x)$$

Definition 1.5. A category is locally Cartesian closed, if for every object $c \in \mathcal{C}$ the slice category $\mathcal{C}_{/c}$ is Cartesian closed. $\mathcal{C}_{/c}$ is a category with

1. Objects:



2. Morphisms:



Definition 1.6. A subobject classifier Ω is any object that represents the functor $Sub(-)$, the functor that takes each object to the set of subobjects.

Concretely,

$$Sub(A) = \{\text{mono maps with target } A\} = Hom(A, \Omega)$$

Remark 1.7. We did not assume the existence of finite colimits because this follows from the other axioms.

Example 1.8. Ignoring some details a set theory in the sense of ZFC can be defined as an elementary topos which satisfies following conditions

1. It is generated by the final object 1.
2. It has a natural number object.
3. It satisfies the axiom of choice.

For more details see [MM12]

Using this definition Lavwere and Tierney were able to construct non-standard categories of sets. In particular, they managed to show that ZFC is independent of the continuum hypothesis using categorical models [Ti72].

First Steps in Higher Categories: Higher Toposes

The roots of higher topos theory can be traced back to similar events that lead to the rise of Grothendieck toposes in the first place. In particular, the strong interest in derived algebraic geometry lead Jacob Lurie to do a thorough study of higher toposes [Lu09]. After tremendous theoretical work we get following very analogous definition for a higher topos.

Definition 2.1. A higher topos $\mathcal{X}$ is a left exact accessible localization of a higher category of presheaves of spaces on a small higher category.

$$Fun(\mathcal{C}^{op}, Spaces) \underset{i}{\overset{a}{\rightleftarrows}} \mathcal{X}$$

Basically up until now we have following picture

$$\begin{array}{ccc} \textit{Grothendieck Topos} & \longrightarrow & \textit{Elementary Topos} \\ \downarrow & & \downarrow \\ \textit{Higher Topos} & \longrightarrow & ? \end{array}$$

This question remains what we want to put in that last spot. Intuitively it should behave like some sort of push out of the given diagram above. This leads to seriously consider a notion of a higher elementary topos.

First Step towards EHTs

The first thing we can do is to consider an analogous definition to the one given for elementary toposes above.

Definition 3.1. (First try) An elementary higher topos is a locally Cartesian category with subobject classifier.

Turns out this definition is terrible. The first reason that might come to mind is that this category does not satisfy *descent*, which is definitely a condition we would expect from any definition of a EHT (and is in particular satisfied by any higher topos in the sense of Lurie). We need a far stronger conditions. The trick is to not just accept subobject classifiers but rather to supercharge everything as much possible and preserve as much information as we can.

Object Classifiers

This brings us to following reasonable question: Why did restrict our classifying object to subobjects and didn't go further. The answer is non-trivial automorphisms! As soon as we allow maps that are not mono we get non trivial automorphisms. As our category only has hom sets that information cannot be captured anymore.

However, in a higher category no such restriction exists as every higher category gives us *mapping spaces*. Therefore we are able to actually classify all maps over a certain object and not just subobjects.

Notation 4.1. Note that there are set theoretical issues with this that we will amend later on. For now we will ignore such issues to focus on the main problem.

This gives us following definition

Definition 4.2. Let $\mathcal{C}$ be a higher category with finite limits. An *object classifier* in the higher category $\mathcal{C}$ is an object $\mathcal{U}$ that classifies the space valued presheaf

$$(\mathcal{C}_{/-})^{core} : \mathcal{C}^{op} \to \mathcal{S}$$

which takes an object c to the core of the over category $(\mathcal{C}_{/c})^{core}$.

So for every object we have an equivalence of spaces

$$(\mathcal{C}_{/c})^{core} \simeq map_{\mathcal{C}}(c, \mathcal{U})$$

Remark 4.3. There is a way to make this definition more precise using the language of right fibrations. That, however, would require a more thorough explanation of the machinery.

Using the language of higher categories we were able to significantly strengthen our classifying object, however we are still not done yet. In the definition above the classifying object only recovers the core whereas we would prefer to be able to preserve all of the categorical data. However, that is literally impossible with one object as we are basically asking whether a single space can recover all of the categorical data. In order to be able to do that we need simplicial objects which satisfy appropriate conditions, i.e. *complete Segal objects*.

Complete Segal Objects

Before we delve into the world of complete Segal objects, let us step back for a second. Where did that word suddenly come from? *Complete Segal Spaces* are a model for higher categories developed by Charles Rezk [Re01]. Compared to quasicategories it has the amazing property that it does not rely on any intrinsic properties of spaces except for the fact that spaces have finite limits. Thus it has the capacity to be generalized to any category that has finite limits. Concretely this gives us following definition:

Definition 5.1. Let $\mathcal{C}$ be a higher category with finite limits. A *complete Segal object* is a simplicial object $X_\bullet : \Delta^{op} \to \mathcal{C}$ that satisfies following conditions:

1. *Segal Condition*: The maps

$$X_n \to X_1 \times_{X_0} ... \times_{X_0} X_1$$

are weak equivalences in $\mathcal{C}$.

2. *Completeness Conditions*: The map

$$X_0 \to X_3 \times_{(X_1 \times X_1)} (X_0 \times X_0)$$

is a weak equivalence in $\mathcal{C}$.

Remark 5.2. The first condition gives us a notion of *composition*. The second tells us that *homotopies* are the same as *weak equivalences*.

We already know that each object $y \in \mathcal{C}$ gives rise to a representable functor valued in spaces

$$\mathcal{Y}_y : \mathcal{C}^{op} \to Spaces$$
$$z \mapsto map_{\mathcal{C}}(z, y)$$

that takes an object z to $map(z, y)$. Intuively we want to say something similar holds in the case of those complete Segal objects. In other words, we want a functor valued in higher categories that takes each z to the simplicial space $map(z, x_\bullet)$, which itself should be a complete Segal space and thus just a higher category.

$$\mathcal{Y}_{X_\bullet} : \mathcal{C}^{op} \to \mathcal{C}\mathrm{at}_\infty$$
$$z \mapsto map_{\mathcal{C}}(z, X_\bullet)$$

where

$$map_{\mathcal{C}}(z, X_\bullet) =$$
$$map_{\mathcal{C}}(z, X_0) \leftleftarrows\!\!\rightrightarrows map_{\mathcal{C}}(z, X_1) \cdots map_{\mathcal{C}}(z, X_2) \cdots \cdots$$

In practice this is quite tricky. As far as we can tell there is no obvious direct way to do this. The proper way is by employing the language of fibrations. For the purposes of this talk I will skip the details here, but to summarize we study presheaves valued in spaces by using *right fibrations*, as already mentioned, and we study presheaves valued in higher categories by using *Cartesian fibrations*.

Using the language of Cartesian fibrations, we can generalize things to the following result:

Theorem 5.3. *For each complete Segal object $x_\bullet$ there is a Cartesian fibration $\mathcal{Y}_{x_\bullet}$ that models the functor that takes a point y to the higher category $map(y, x_\bullet)$.*

As is appropriate for any object worthy of being called representable we also have a *Yoneda Lemma*:

Lemma 5.4. *For any two complete Segal objects $x_\bullet$ and $y_\bullet$, we have an equivalence*

$$map(x_\bullet, y_\bullet) \simeq Map(\mathcal{Y}_{x_\bullet}, \mathcal{Y}_{y_\bullet})$$

Having constructed such a Cartesian fibration, we can give following definition.

Definition 5.5. We say a presheaf valued in higher categories is *representable* if it is equivalent to one of the form $\mathcal{Y}_{x_\bullet}$. In that case we say it is represented by $x_\bullet$.

Having gone on this abstract detour we know have all the tools to expand our definition. Before we give our definition, we recall that we have to adjust for set theoretical issues.

Definition 5.6. We say $\mathcal{U}_\bullet$ is a *object classifying complete Segal object* closed under finite colimits and limits if it represents a sub functor of the presheaf $\mathcal{C}_{/-}$ closed under limits and colimits. By that we mean there is an embedding

$$\mathcal{Y}_{\mathcal{U}_\bullet} \hookrightarrow \mathcal{C}_{/-}$$

such that the image is closed under finite limits and colimits. Henceforth we call such an object just an object classifier. Such objects are also called *universes.*

Definition of an Elementary Higher Topos

We finally have assembled all the tools to define an elementary higher topos.

Definition 6.1. We say a higher category $\mathcal{C}$ is an elementary higher topos if it satisfies following conditions:

1. It has finite limits and colimits
2. It has a subobject classifier Ω
3. For every map f in $\mathcal{C}$ there exists an object classifier $\mathcal{U}_\bullet$ such that f is in the image of $\mathcal{Y}_{\mathcal{U}_\bullet}$.

We have discussed before why the previous definition was not reasonable. Why would this definition be reasonable? Let's look at some of the evidence.

Theorem 6.2. *If $\mathcal{C}$ is an EHT then $\mathcal{C}_{/c}$ is also an EHT.*

Remark 6.3. For reasons unknown the speaker this theorem is called the "fundamental theorem of topos theory" in the case of elementary toposes. Regardless of the precise reason this shows the importance of this theorem.

Proof. The case for finite limits and colimits are clear. The existence of a subobject classifier follows from the same argument for elementary toposes ($\Omega \times c$ is a the subobject classifier). Finally the existence of object classifiers is analogous to the argument for subobject classifiers ($\mathcal{U}_\bullet \times c$ is an object classifier). □

Theorem 6.4. *Every elementary higher topos is locally Cartesian closed.*

Remark 6.5. This is very powerful as in the case of classical category theory we had it as part of the conditions. Here it just follows from a stronger notion of an object classifier.

Proof. It suffices to show it is Cartesian closed and the local part will follow from the previous theorem. For that it suffices to show that the map $- \times y : \mathcal{E} \to \mathcal{E}$ has a right adjoint. We will avoid the details but just to point out the idea.

The idea is that complete Segal objects are internal higher category objects and so behave like higher categories. Thus they have a notion of "objects" and "mapping objects". In particular, for every object x in $\mathcal{C}$. I get a mapping object $map_{\mathcal{U}}(y,x)$, which is just the pullback

$$\begin{array}{ccc} map_{\mathcal{U}}(y,x) & \longrightarrow & \mathcal{U}_1 \\ \big\downarrow & \lrcorner & \big\downarrow \\ * & \xrightarrow[(x,y)]{} & \mathcal{U}_0 \times \mathcal{U}_0 \end{array}$$

The claim is that this is the internal mapping object. Indeed, what we need is a unit map $x \to map_{\mathcal{U}}(y, x \times y)$ However, by the properties of object classifiers such map is just a map $x \times y \to x \times x \times y$ over x. However, there is one obvious choice, namely $(x_0, y_0) \mapsto (x_0, x_0, y_0)$, which gives us our unit map. □

Proposition 6.6. *Every elementary higher topos satisfies descent.*

Proof. For a careful definition of descent take a look at work by Rezk [Re05]. One equivalent condition is that the functor $(\mathcal{C}_{/-})^{core}$ takes colimits to limits. That follows immediately from the fact that the fuctor is representable. □

Examples

Having done all of this let us see some examples

Example 7.1. Let $\mathcal{X}$ be a higher topos in the sense of Lurie. Clearly, it has finite limits and colimits. Also we can show that the sub object functor takes colimits to limits, which by presentability implies it has to be representable.

According to Theorem 6.1.6.8 for each bounded local class S there exists an object classifier $\mathcal{U}_0^S$. Now we can repeat the same procedure with $\mathcal{X}^{\Delta[n]}$ as this is still a topos. Thus for every S and every n we get an object classifier $\mathcal{U}_n^S$. These give us simplicial objects $\mathcal{U}_\bullet^S$, which classify the full subcategory of $\mathcal{X}$ which has maps in S.

Finally, we notice that in a topos every map is in some bounded local class. Thus we have shown that for every map there is an object classifier.

Example 7.2. This example in particular implies that the category of spaces is an elementary higher topos. In this case we can actually understand the object classifier as a concrete object. Let $\mathcal{U}$ be an object classifier. As it is a space we can understand it by looking at it's points i.e. maps $X : * \to \mathcal{U}$. As it classifies spaces the data of the map X should just be some space. In order words we should have following pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{E} \\ \big\downarrow & \ulcorner & \big\downarrow \\ * & \longrightarrow & \mathcal{U}_0 \end{array}$$

where $\mathcal{E}$ is the universal bundle. This basically implies that $\mathcal{U}_0$ is the core of some subcategory of spaces that is closed under finite limits and colimits. And the higher simplicial levels give us the higher categorical data of that chosen subcategory.

So, in spaces each object classifier is just a subcategory thought of as a simplicial space.

Where do we go from here?

Having introduced this new notion of an elementary higher topos what can we do with it? There are two overall paths we can take from here. Study EHT for their own sake and try to use them in other branches of mathematics. We will suggest some possible questions in both directions.

For the sake of Elementary Higher Toposes:

1. *Comparison to Other Definitions:* In a recent blog post Mike Shulman has suggested a definition for a elementary higher topos. On face value his definition is different from mine as he assumes that is locally Cartesian closed, but that we only have object classifiers that classify the core and not the whole category. In light of the theorem above, my definition does imply the definition of Shulman. There is strong evidence that the opposite is also true but that needs some further study.

2. *Colimits:* In the case of elementary toposes we do not need to assume that we have colimits as it follows from the other axioms. Thus one interesting question is whether the same holds in the higher settings. In ongoing work with Jonas Frey we think we have a proof that the case for an initial object and coproducts directly generalizes. However, the case for pushouts seems unclear and requires further efforts.

3. *Univalence in EHTs:* In [GK17] Gepner and Kock introduced a notion of univalence in the context of locally Cartesian closed presentable higher categories. Moreover, they show there is a relationship between univalent

maps and object classifiers (which have to exist for presentability reasons). In our context we take the existence of object classifiers as part of the definition. Thus we can define univalent maps in EHT. The hope is that we might have a similar classification result for univalent maps in an EHT.

For the sake of other areas of mathematics:

1. *Non-Standard models of Spaces:* One of the initial questions that made people study elementary toposes was the desire to gain a better understanding of the category of sets and various ways it can be classifier. This is not just for fun but can actually help us gain better understanding of sets (recall [Ti72] where Tierney shows the independence of the continuum hypothesis). The plan is that something similar should happen in the higher categorical case. The hope is that by imposing the right condition we can give a categorical classification of the category of spaces. That way there is a chance we can realize what kind of properties hold for spaces.

   In a similar vain we can use this approach to study categories which can help us undestand the category of spaces.

2. *Homotopy Type Theory:* Homotopy type theory is very broadly speaking a foundational approach to mathematics that takes homotopy theory as one of it's building blocks. So, a notion of homotopy is built into the basic constructions, rather than in the context of set theory where we define and impose homotopies. We do have some understanding of what models for homotopy type theory are (see [KL12]), but definitely lack a complete list of models. But models are important as they explicitly show us the boundary of our theory and also give us an indicator on what kind of things we can prove. One possible solution to an exhaustive list of models is an elementary higher topos. In particular the object classifiers should play the role of our univalent universes.

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