GSTGC 2016 - Indiana University
A new Approach to Straightening

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Notes

You can find these slides at:

http://math.illinois.edu/∼rasekh2/GSTGC2016.pdf

The Case of Sets

Theorem

Let X be a set. There is an equivalence of categories:

$$Set_{/X} \underset{\amalg}{\overset{p^{-1}()}{\rightleftarrows}} Fun(X, Set)$$

between sets over X and set-valued maps from X.

The Case of Sets

One side:

$$\begin{array}{c} Y \\ \downarrow p \\ X \end{array} \Longrightarrow \begin{array}{ccc} X & \longrightarrow & Set \\ x & \longmapsto & p^{-1}(x) \end{array}$$

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Other side:

$$X \xrightarrow{F} Set \quad \Longrightarrow \quad \begin{array}{c} \amalg_{x \in X} F(x) \\ \big\downarrow \\ X \end{array}$$

Grothendieck Construction

Can be generalized to categories:

Theorem (Grothendieck)

Let $\mathcal{C}$ be a category. There is the following adjunction:

$$\mathcal{C}at_{/\mathcal{C}} \underset{\int}{\overset{colim}{\rightleftarrows}} Fun_{\mathcal{C}at}(\mathcal{C}^{op}, Set)$$

between categories over $\mathcal{C}$ and set-valued functors from $\mathcal{C}$, which becomes the following equivalence:

$$Fib(\mathcal{C}) \underset{\int}{\overset{colim}{\rightleftarrows}} Fun_{\mathcal{C}at}(\mathcal{C}^{op}, Set)$$

if we restrict to categories fibered in sets over $\mathcal{C}$.

The Idea of Straightening

Idea

For $\mathcal{C}$ a "higher category", there is an "equivalence":

$\{\text{certain objects over } \mathcal{C}\} \approx \{\text{functors from } \mathcal{C}^{op} \text{ into spaces}\}$

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2. What is an equivalence?

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1. What is a higher category?
2. What is an equivalence?
3. What are certain objects?

What is a higher Category?

Idea (Idea of a Higher Category)

A category which has "higher morphisms".

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- Different ways to concretely encode this idea.
- We will focus on quasi-categories

Basic Idea of Quasi-Categories

Example (2-Cell = 2-Simplex = Triangle = Δ^2)

$$\begin{array}{ccc} 0 & \xrightarrow{\quad h \quad} & 2 \\ & {\scriptstyle f}\searrow \quad \Uparrow\alpha \quad \nearrow{\scriptstyle g} & \\ & 1 & \end{array}$$

It has ...

Basic Idea of Quasi-Categories

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It has ...

1. 3 Objects represented by the numbers
2. 3 (non-dg) 1-Morphisms represented by the lines
3. 1 (non-dg) 2-Morphism represented by the arrow

Think of 2-morphism α as "homotopy" between h and $g \circ f$

Basic Idea of Quasi-Categories

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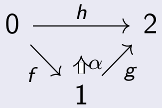

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Way to encode it:

$$\{0,1,2\} \underset{s}{\overset{t}{\leftleftarrows}} \{f,g,h\} \quad \{\alpha\} \quad \cdots$$

Example of Quasi-Categories

Example (Quasi-Category of Spaces)

$\mathcal{S}$: Quasi-Category of spaces.

$$Spaces \underset{s}{\overset{t}{\leftleftarrows\!\!\rightarrow}} Cont.Maps \;\substack{\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow} \; Homotopies \;\substack{\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow}\; \cdots$$

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Other Examples:

1. $\mathcal{QC}at$ the (large) quasi-category of (small) quasi-categories
2. If X is a quasi-category then X^{op}
3. If X and Y are two quasi-categories then $Fun(X, Y)$
4. Other particular $Fun(X^{op}, \mathcal{S})$

Back to Straightening (More Precise Version)

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For $\mathcal{C}$ a "higher category", there is an "equivalence":

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1. Higher category: quasi-category
2. Equivalence: adjunctions & equivalences
3. Certain objects: Right fibrations over $\mathcal{C}$

The Straightening Construction

Theorem (Lurie)

Let X be a quasi-category. There is the following adjunction:

$$\mathcal{QC}at_{/X} \underset{Un_X}{\overset{St_X}{\rightleftarrows}} Fun(X^{op}, \mathcal{S})$$

between quasi-categories over X and space-valued functors from X^{op}, which becomes the following equivalence:

$$RFib(X) \underset{Un_X}{\overset{St_X}{\rightleftarrows}} Fun(\mathcal{C}^{op}, \mathcal{S})$$

if we restrict to right fibrations over X.

Unstraightening Functor (Definition)

Construct functor $Un_X : Fun(X^{op}, \mathcal{S}) \to \mathcal{S}_{/X}$.

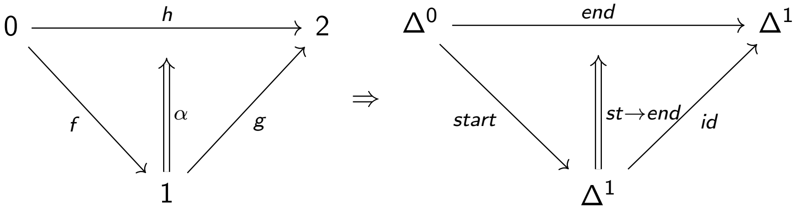
Definition

Fix $F : X^{op} \to \mathcal{S}$

$$(Un_X F)_n = \{ g \in X_n = Hom(\Delta^n, X) : \ (\Delta^n)^{op} \underset{G\circ(g)^{op}}{\overset{St_n}{\rightrightarrows}} \mathcal{S} \ \ \Downarrow \alpha \ \}$$

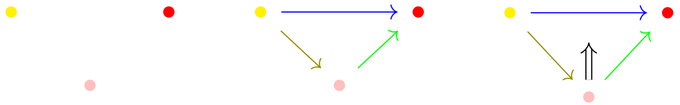
Unstraightening Functor (Concrete Example)

Let $F : \Delta^2 \to \mathcal{S}$ be the following:



Unstraightening Functor (Concrete Example)

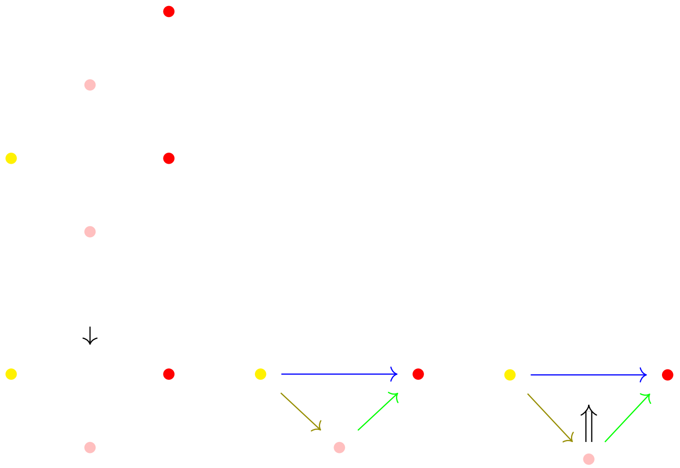
We will do construction level-wise.
Level-wise version of Δ^2:



$(\Delta^2)_0$ $(\Delta^2)_1$ $(\Delta^2)_2$

We build $Un_X F$ step by step over $X = \Delta^2$

Unstraightening Functor (Concrete Example)



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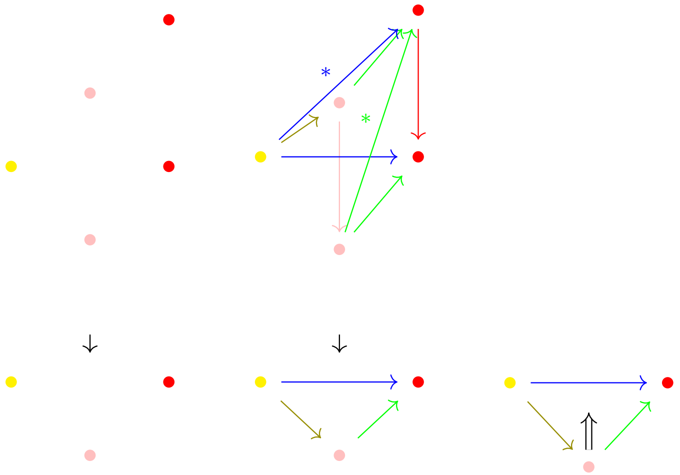

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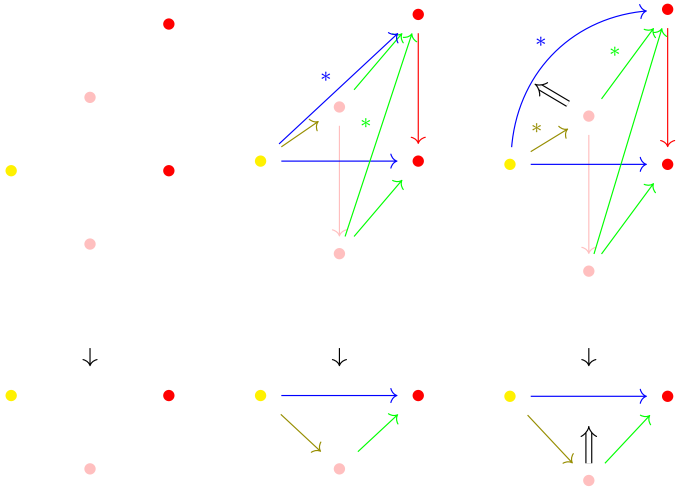

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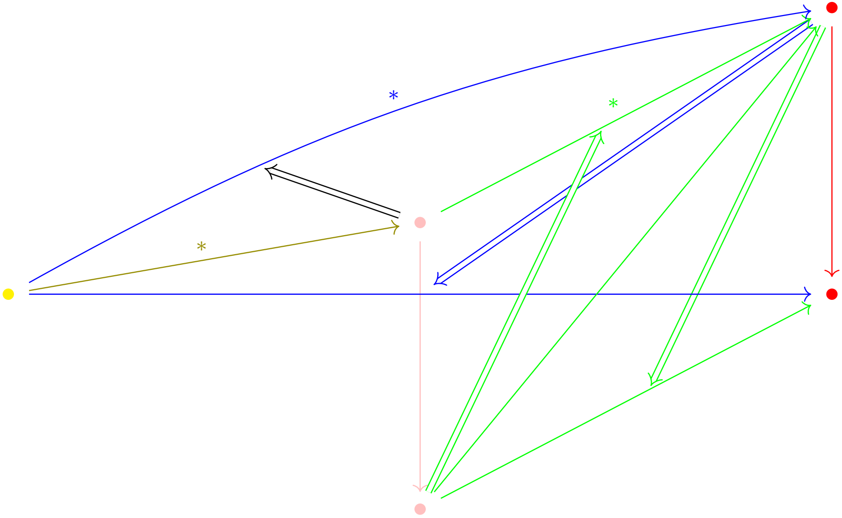

Unstraightening Functor (Moral of the story)

- The moral is that we build the unstraightening construction diagonally step by step.
- 0th step is exactly the case of sets we described in the beginning

Important Example

Let $\rho_x : X^{op} \to \mathcal{S}$ be the representable functor $(\rho_x(y) = Map_X(y, x))$

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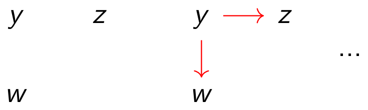

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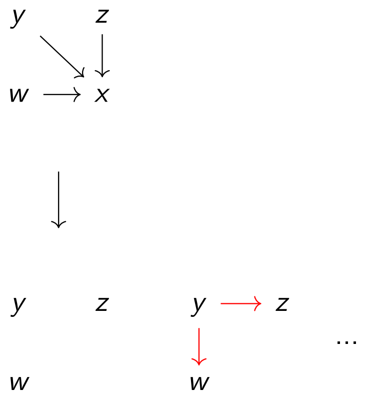

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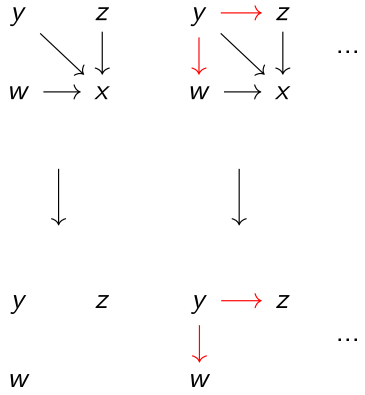

Over-Categories

This structure is familiar. It is an "over-category".

Lemma (Unstraightening of Representable)

For $x \in X$ an object we have $Un_X(\rho_x) = X_{/x}$

"Bundling up the functor with values *maps into* x gives us the category of *things over* x"

Representable Functors

Representable functors are special:

Lemma (Yoneda Lemma)

Let $F : X^{op} \to \mathcal{S}$ be a functor and x an object. Then we have following equivalence:

$$Map(\rho_x, F) \cong F(x)$$

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Corollary

Let $\alpha : F \to G : X^{op} \to \mathcal{S}$ be map of functors. Then α is an equivalence if and only if

$$Map(\rho_x, \alpha) : Map(\rho_x, F) \to Map(\rho_x, G)$$

is homotopy equivalence of spaces (for every object x in X).

Representable Maps for Right Fibrations

Over-Categories are similar for right fibration:

Lemma (Yoneda Lemma for Right Fibrations)

In the following diagram (Y,Z are right fibrations over X)

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

f is an equivalence if and only if

$$Map_{/X}(X_{/x}, f) : Map_{/X}(X_{/x}, Y) \to Map_{/X}(X_{/x}, Z)$$

is an equivalence of spaces (for every object x in X).

Representable Functors (Another Look)

There is a notion of "tensor" of categories. In particular:

Lemma

For $F : X^{op} \to \mathcal{S}$ and x an object we have an equivalence

$$F \otimes Map(x, -) \cong F(x)$$

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Representable Maps

The tensor construction allows for more general statements:

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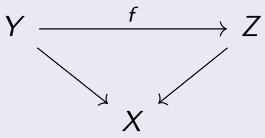
Lemma (Yoneda Lemma for Maps)

In the following diagram



f is an equivalence if and only if

$$X_{x/} \times_X f : X_{x/} \times_X Y \to X_{x/} \times_X Z$$

is an equivalence of spaces (for every object x in X). Here $X_{x/}$ is the category of object <u>under</u> x.

Fun Fact about Spaces

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- In the case of spaces a right fibration is a map of spaces.
- So, we get this:

Corollary

For S a space there is an equivalence of higher categories:

$$\mathcal{S}_{/S} \underset{Un_S}{\overset{St_S}{\rightleftarrows}} Fun(S, \mathcal{S})$$

- Note the similarity to the case of sets!

Cool Example

Let $S = BG$. Then we get

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BG as a higher category has one object

$$\mathcal{S}_{/BG} \underset{Un_{BG}}{\overset{St_{BG}}{\rightleftarrows}} G - Spaces$$

Unstraightening the one unique representable map gives us exactly $EG \to BG$.

Thank you!