An Axiomatic Approach to Algebraic Topology

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The goal of this talk is to introduce an abstract axiomatic framework for the homotopy theory of spaces. After giving a quick motivation and definition we move on to show how we can reproduce various results from classical algebraic topology in this abstract setting.

1. Classical Story
2. Elementary Higher Topos
3. Natural Number Objects
4. Truncations
5. Blakers-Massey Theorem
6. Where to go from here?

Set Theory and Elementary Toposes

A lot of classical mathematics is based on set theory. That necessitates a good axiomatic framework for set theory. We can achieve this in several ways.

1. Logic
2. Category Theory

Here are examples of some conditions from both perspectives

Axiom	**Logical**	**Categorical**
Extensionality	$f = g \Leftrightarrow f(x) = g(x)$	Generated by the final object
Union	$X \cup Y$ exists	Finite Coproducts Exist
$\vdots$	$\vdots$	$\vdots$

More concretely the precise way to characterize set theories via categories is by defining an *elementary topos*, which is a category with certain representability conditions [Jo03].

I am a homotopy theorist and like homotopical structures. For that reason I like to find a similar foundational approach to spaces, which are the basic building blocks of homotopy theory. As in the case of sets there are various ways we can approach this question. My goal is to focus on the categorical approach.

Elementary Higher Topos

In this section I give a definition for a category that can serve as a basis for the homotopy theory of spaces. As we have entered the realm of homotopy theory everything I say henceforth will automatically be "higher", $(\infty, 1)$ or ∞. As disclaimer, when I say higher category I am thinking of a complete Segal space but everything also works for quasi-categories. However, I we will not require any particular aspects of those models in this talk.

Definition 2.1. [Ra18a] An elementary higher topos $\mathcal{E}$ is a category $\mathcal{E}$ which satisfies following conditions:

1. It has finite limits and colimits.
2. It is locally Cartesian closed.
3. It has a subobject classifier Ω.
4. It has sufficient universes $\mathcal{U}$.

Definition 2.2. $\mathcal{E}$ is locally Cartesian closed if for every map $f : x \to y$, the pullback functor

$$\begin{tikzcd} \mathcal{E}_{/y} \arrow[r, shift left, "f^*"] & \mathcal{E}_{/x} \arrow[l, shift left, "f_*"] \end{tikzcd}$$

has a right adjoint f_*.

Definition 2.3. There is a functor

$$Sub(-) : \mathcal{E}^{op} \to \mathcal{S}\mathrm{et}$$

that takes every object x to the isomorphism classes of subobjects $Sub(x)$. A *subobject classifier* Ω is an object that classifies the functor $Sub(-)$.

Definition 2.4. $\mathcal{E}$ has sufficient universes if there exists a chain of objects $\{\mathcal{U}^\kappa\}_\kappa$ such that for every map $f : y \to x$ there exists a pullback square

$$\begin{tikzcd} y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}^\kappa_* \arrow[d] \\ x \arrow[r] & \mathcal{U}^\kappa \end{tikzcd}$$

Example 2.5. The category of spaces is an elementary higher topos.

1. Spaces have finite limits and colimits.
2. Spaces are locally Cartesian closed.
3. It has a subobject classifier $\Omega = \{0,1\}$ [Notice a mono map in spaces is a map that is an equivalence on path-components].
4. Before we show spaces have sufficient universes, we need following conventions. Fix a large cardinal κ:
   (a) A space X is κ-small if $|\pi_*(X)| < \kappa$
   (b) A map is κ-small if the fiber is κ-small.
   (c) We denote the subcategory of κ-small spaces by $\mathcal{S}\mathrm{paces}^\kappa$
   (d) Forgetful map $U : \mathcal{S}\mathrm{paces}_* \to \mathcal{S}\mathrm{paces}$ from pointed spaces.
   (e) $(\mathcal{S}\mathrm{paces}^\kappa)^{core}$ is the maximal subgroupoid.

   Now for any map of spaces $f : Y \to X$ there exists a pullback square:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\ulcorner"] & (\mathcal{S}\mathrm{paces}_*^\kappa)^{core} \arrow[d, "U^{core}"] \\ X \arrow[r] & (\mathcal{S}\mathrm{paces}^\kappa)^{core} \end{tikzcd}$$

   As any given map of spaces has κ-small fibers for a large enough cardinal κ, this means spaces have sufficient universes.

Example 2.6. Let us give one explicit example of this. For a κ-small space X we have a pullback square:

$$\begin{tikzcd} X \simeq \{(X,x) : x \in X\} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & (\mathcal{S}\mathrm{paces}_*^\kappa)^{core} \arrow[d] \\ * \arrow[r, "X"'] & (\mathcal{S}\mathrm{paces}^\kappa)^{core} \end{tikzcd}$$

The category of spaces is fascinating and has been the object of extensive studies in algebraic topology. For the remainder of this talk we want to show how various classical results from algebraic topology can be reproduced in this very abstract setting.

Natural Number Objects

The first thing we can notice about the definition is that it lacks any infinite axiom. Many classical construction requires us to be able to do some infinite constructions. In the classical setting of sets, we get a notion of infinity via natural number objects.

Definition 3.1. Let $\mathcal{E}$ be a topos. An object $\mathbb{N}$ along with two maps $s : \mathbb{N} \to \mathbb{N}$ and $o : 1 \to \mathbb{N}$ is called a natural number object if for any other object X with maps $b : 1 \to X$, $u : X \to X$ the diagram below has a unique filling.

$$\begin{tikzcd} & \mathbb{N} \arrow[r, "s"] \arrow[d, dashed, "\exists!"] & \mathbb{N} \arrow[d, dashed, "\exists!"] \\ 1 \arrow[ur, "o"] \arrow[dr, "b"'] & & \\ & X \arrow[r, "u"] & X \end{tikzcd}$$

Example 3.2. The actual natural numbers $\mathbb{N}$ with $0 \in \mathbb{N}$ and $succ : \mathbb{N} \to \mathbb{N}$ is a natural number object in spaces.

We can use natural number objects in the classical setting of an elementary topos to do various constructions which are inherently infinite without actually assuming infinite limits or colimits exists. For example we can construct free monoids.

However, not every elementary topos has a natural number object and so we have to assume its existence (a counterexample is the category of finite sets). In the axiomatic context this corresponds to *the axiom of infinity.* In an elementary higher topos we can use the inherent homotopy theory to actually prove the following.

Theorem 3.3. *[Ra18c] Every elementary higher topos $\mathcal{E}$ has a natural number object.*

The idea of the proof is the following. We have finite limits and colimits. Thus we can do the following. First we can take the coequalizer

$$\begin{tikzcd} 1 \arrow[r, shift left, "id"] \arrow[r, shift right, "id"'] & 1 \arrow[r] & S^1 \end{tikzcd}$$

Then we can take the pullback square

$$\begin{tikzcd} \Omega S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & 1 \arrow[d] \\ 1 \arrow[r] & S^1 \end{tikzcd}$$

In the case of spaces the object ΩS^1 is exactly the loop space of the circle, which are just the integers $\mathbb{Z}$. We can use this intuition to prove various important results about ΩS^1 such that

1. $\Omega S^1 \cong \Omega S^1 \coprod \Omega S^1$.
2. ΩS^1 is a group object.
3. $\cdots$

Truncations

One classical construction in the category of spaces are truncation functors.

Definition 4.1. A space is X is n-truncated if $\pi_k(X)$ is trivial for $k > n$. We denote the subcategory of n-truncated spaces by $\tau_n\mathcal{S}\text{paces}$.

There is an adjunction

$$\mathcal{S}\text{paces} \underset{i}{\overset{\tau_n}{\rightleftarrows}} \tau_n\mathcal{S}\text{paces}$$

The adjunction implies that $\tau_n X$ is a universal truncation. Traditionally, we construct this truncation by a small object argument. We basically fill all the higher sphere with balls until there is nothing left to fill. This clearly needs lots of infinite colimits. However, we want to show that we can deduce a similar result without any infinite colimits.

The trick is to use an idea from Cech covers. For a given space X we can form following simplicial diagram

$$\begin{tikzcd} X & X \times X \arrow[l, shift left, "\pi_1"'] \arrow[l, shift right, "\pi_2"] & X \times X \times X \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & \cdots \arrow[l, shift left=3] \arrow[l, shift left] \arrow[l, shift right] \arrow[l, shift right=3] \end{tikzcd}$$

The colimit of this simplicial diagram is $\tau_{-1}X$ [Lu09] [Re05]. The problem is that we cannot take colimits of simplicial diagrams (as we don't have infinite colimits).

Fortunately, there is a workaround. Egbert Rijke has shown that if we replace the product by joins then we can simplify the diagram from a simplicial diagram to a sequential one [Ri17]. We can repeat that idea in context of an elementary higher topos.

Definition 4.2. An object X in $\mathcal{E}$ is n-truncated if for every other object Y the mapping space $map(Y, X)$ is n-truncated.

Definition 4.3. Let A and B be two objects in $\mathcal{E}$. We define the join $A * B$ as the pushout

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & B \arrow[d, "inr"] \\ A \arrow[r, "inl"] & A * B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Theorem 4.4. *[Ra18d] For a given object A the sequential colimit of the sequential diagram*

$$A \xrightarrow{inl} A * A \xrightarrow{inl} (A * A) * A \xrightarrow{inl} \cdots$$

is the -1-truncation of A, $\tau_{-1}A$.

Thus every elementary higher topos has a (-1)-truncation. However, this process cannot be used for n-truncations. For that we need a different argument and the key is to use induction. Again this idea is originally due to Rijke in homotopy type theory [Ri17]. Instead of giving the general argument we will focus on the case of spaces to make the argument more understandable.

Let us assume we know how to construct n-truncation $\tau_n X$ of a space X. How can we use it to construct $\tau_{n+1}X$? The space $\tau_{n+1}X$ should be thought of as the space which has the same points as X but where we n-truncate each loop space $\tau_n(\Omega_x X)$. How can make this into an actual mathematical argument?

So, assume we have a truncation functor

$$\tau_n : \mathcal{S}\text{paces} \to \tau_n\mathcal{S}\text{paces}$$

For every space X there is a map of spaces

$$\mathcal{Y} : X \to Fun(X, \mathcal{S}\text{paces}^{core})$$

that takes a point x to the map of spaces $Path(-,x) : X \to \mathcal{S}\text{paces}$ that maps a point y to the path space $Path(y,x)$. This map is in fact an embedding of spaces. Now using our truncation functor τ_n we get a composition

$$(\tau_n)_* \circ \mathcal{Y} = \mathcal{Y}_n : X \to Fun(X, \tau_n\mathcal{S}\text{paces}^{core})$$

Clearly, this map is usually NOT an embedding anymore. So what is the image of X under the map $\mathcal{Y}_n$? As $\mathcal{Y}$ is an embedding it suffices to determine which objects are identified via $(\tau_n)_*$. For a functor $F : X \to \mathcal{S}\text{paces}^{core}$, we have $(\tau_n)_*(F)(x) = \tau_n(F(x))$. Thus $\mathcal{Y}_n(X)$ is the subspace of $Fun(X, \tau_n\mathcal{S}\text{paces}^{core})$ generated by the points $\tau_n(Path(-,x))$.

We can capture the argument in the following diagram

$$\begin{tikzcd}
X \arrow[r, hook, "\mathcal{Y}"] \arrow[d, two heads] & Fun(X, \mathcal{S}\text{paces}^{core}) \arrow[r, "(\tau_n)_*"] & Fun(X, \tau_n(\mathcal{S}\text{paces}^{core})) \\
\tau_{n+1}(X) \arrow[urr, hook] & &
\end{tikzcd}$$

Thus, we can define the $n+1$-truncation by the image of the map $\mathcal{Y}_n(X)$ in $Fun(X, \tau_n(\mathcal{S}\text{paces}^{core}))$.

The beauty of the argument is that it can be directly generalized to an elementary higher topos. In that direction we have following results.

Theorem 4.5. *[Ra18c] (Yoneda Lemma in an Elementary Higher Topos) Let $\mathcal{U}$ be a universe that classifier the diagonal map $\Delta : X \to X \times X$, meaning we have a pullback square*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}_* \arrow[d] \\ X \times X \arrow[r] & \mathcal{U} \end{tikzcd}$$

The adjunction of the bottom map $\mathcal{Y}_X : X \to \mathcal{U}^X$ is a mono map in $\mathcal{E}$.

Theorem 4.6. *[Ra18d] For every $n \geq -2$, there is an adjunction*

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift left, "\tau_n"] & \tau_n \mathcal{E} \arrow[l, shift left, "i"] \end{tikzcd}$$

Blakers-Massey Theorem

Now that we have been able to construct truncations we can ask ourselves other classical questions from algebraic topology related to truncated and connected maps. Here we look at one classical result, namely Blakers-Massey theorem.

Definition 5.1. An object $f : Y \to X$ is n-connected if for all n-truncated maps $Z \to X$ the map

$$f^* : map(X, Z) \to map_(Y, Z)$$

Theorem 5.2. *(Classical Blakers-Massey Theorem) Let us assume we have a pushout square such that f is m-connected and g is n-connected.*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] \arrow[ur, phantom, "\ulcorner", very near end] & W \end{tikzcd}$$

Then, the gap map $(f, g) : Z \to X \times_W Y$ is $(m + n)$-connected.

In a paper by Anel, Biedermann, Finster and Joyal show that every presentable elementary higher topos (there called higher topos) satisfies a very general version of the Blakers-Massey theorem [ABFJ17]. I have shown that the same proof (with minor modifications) holds in the setting of an elementary higher topos.

Where do we go from here?

There are so many places we can go from here:

1. We have truncations and we have spheres, which means we have homotopy groups. Thus we can now ask how homotopy groups of spheres in $\mathcal{E}$ compare to homotopy groups of spheres in spaces.

2. In particular, Freudenthal suspension theorem and homotopy groups implies that we have stabilizations and stable homotopy groups. Thus we can now ask for comparison between the stabilization of an elementary higher topos and spectra.

3. Moreover, given that in the classical setting we were able to use natural number objects to construct free algebras, the next step would be to show that we can construct free algebras in every elementary higher topos.

References

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