Homology of the real projective plane

April 14, 2019

This is essentially a solution for exercise 2 of set 6 in which one was supposed to find a(n abstract) simplicial complex K whose realization is homeomorphic to the real projective plane $\mathbb{R}P^2$ and compute its simplicial homology using a Mayer–Vietoris sequence. For the sake of clarity, this document is more detailed than what would be expected from a submission.

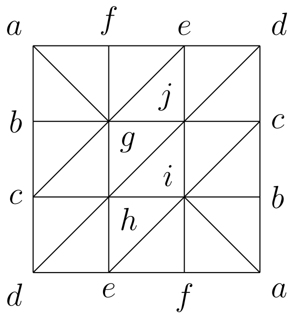
In the following, we freely use some usual conventions and abuses of notation such as representing abstract simplicial complexes and their labelings with pictures, using the alphabetical ordering of the vertices for homology calculations, writing v instead of $\{v\}$ for a 0-simplex etc.

Fixing possible sign issues is left as an exercise to the reader.

The complex and the decomposition

Here is a possible way to realize $\mathbb{R}P^2$ as a simplicial complex:[1]

K :



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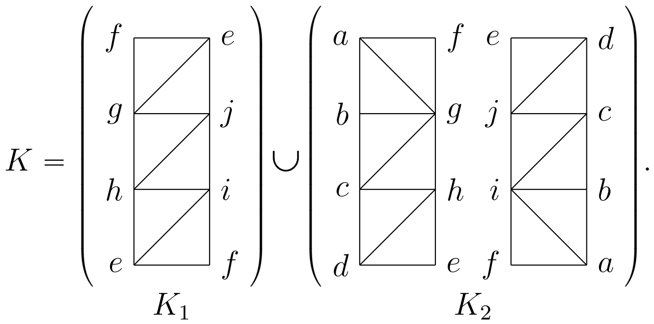
To use the Mayer–Vietoris sequence, we want to write K as the union of two subcomplexes whose homology (and that of their intersection) we know well (or can compute easily). One of the many possibilities to do this is as follows:

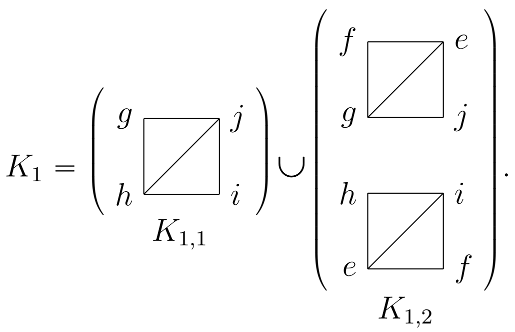
[1] There is actually a simplicial complex with only ten 2-simplicies that realizes $\mathbb{R}P^2$, but we'll stick to the one above because it's somewhat more straightforward to come up with: One can obtain $\mathbb{R}P^2$ from a square by identifying "antipodal" points of its boundary, and inspired by how one realizes the cylinder as a simplicial complex, one can subdivide that square into three "layers" vertically and horizontally to obtain the simplicial complex above.



First subcomplex

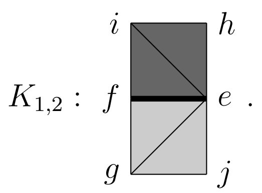
Note that K_1 is a representation of the Möbius strip. You may have seen that its homology is isomorphic to that of $\partial\Delta^2$, but we will compute it to have another demonstration of how to use the Mayer–Vietoris sequence and because it will be important to know an explicit generator of $H_1(K_1) \cong H_1(\partial\Delta^2) \cong \mathbb{Z}$.

Here is the decomposition of K_1 that we will use to compute its homology:



Note that $K_{1,1}$ is the union of two copies of Δ^2 whose intersection is isomorphic to Δ^1. Thus it is acyclic as the union of two acyclic subcomplexes whose intersection is also acyclic.

Doing the identifications given by the labeling, we see that $K_{1,2}$ can be written as the union of two copies of $K_{1,1}$ whose intersection is isomorphic to Δ^1:



Hence $K_{1,2}$ is also acyclic.

The intersection $K_{1,1} \cap K_{1,2}$ consists of two disjoint copies of Δ^1 (namely those spanned by $\{g, j\}$ and $\{h, i\}$). Thus $H_n(K_{1,1} \cap K_{1,2}) \cong 0$ for $n > 0$ and $H_0(K_{1,1} \cap K_{1,2}) \cong \mathbb{Z}^2$ is a free abelian group on the generators $[g]$ and $[h]$.

We can now compute the homology of K_1. First we note that K_1 is connected, so $H_0(K_1) \cong \mathbb{Z}$ (which can also be seen from the Mayer–Vietoris sequence).

To calculate $H_1(K_1)$, we have a look at the corresponding segment of the Mayer–Vietoris sequence:

$$0 \cong H_1(K_{1,1}) \oplus H_1(K_{1,2}) \to H_1(K_1) \xrightarrow{\partial_1} H_0(K_{1,1} \cap K_{1,2}) \xrightarrow{\phi_0} H_0(K_{1,1}) \oplus H_0(K_{1,2}).$$

This means that ∂_1 is injective and thus an isomorphism onto its image $\operatorname{im} \partial_1 = \ker \phi_0$.

To determine $\ker \phi_0$, we note that $[h] = [g]$ in $H_0(K_{1,1})$ and $H_0(K_{1,2})$, so

$$\phi_0(m[g] + n[h]) = ((m+n)[g], -(m+n)[g]) \in H_0(K_{1,1}) \oplus H_0(K_{1,2})$$

which is zero if and only if $m = -n$. Hence we have

$$\ker \phi_0 = \{-k[g] + k[h] \mid k \in \mathbb{Z}\} = \mathbb{Z} \cdot ([h] - [g]) \subseteq H_0(K_{1,1} \cap K_{1,2}) = \mathbb{Z} \cdot [g] \oplus \mathbb{Z} \cdot [h].$$

Thus $H_1(K_1) \cong \mathbb{Z}$ and the preimage of $[h] - [g]$ under ∂_1 is a generator.

Intuitively speaking, this preimage is represented by two sequences of edges connecting g and h in $K_{1,1}$ resp. $K_{1,2}$ such that their union is a cycle in K_1. An example of this would be taking $(\{g,h\})$ in $K_{1,1}$ and $(\{f,g\},\{e,f\},\{e,h\})$ in $K_{1,2}$, which would yield the generator $[\{f,g\} + \{g,h\} - \{e,h\} + \{e,f\}] \in H_1(K_1)$ after choosing appropriate signs.

In order to be more precise about this, we have to recall how ∂_1 is defined using the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1(K_{1,1} \cap K_{1,2}) & \xrightarrow{\varphi_1} & C_1(K_{1,1}) \oplus C_1(K_{1,2}) & \xrightarrow{\varrho_1} & C_1(K_1) & \longrightarrow & 0 \\
 & & \downarrow d_1^{K_{1,1} \cap K_{1,2}} & & \downarrow d_1^{K_{1,1}} \oplus d_1^{K_{1,2}} & & \downarrow d_1^{K_1} & & \\
0 & \longrightarrow & C_0(K_{1,1} \cap K_{1,2}) & \xrightarrow[\varphi_0]{} & C_0(K_{1,1}) \oplus C_0(K_{1,2}) & \xrightarrow[\varrho_0]{} & C_0(K_1) & \longrightarrow & 0
\end{array}.$$

Namely, given a 1-cycle η in K_1, one lifts η along ϱ_1, checks that the image of the lift under $d_1^{K_{1,1}} \oplus d_1^{K_{1,2}}$ comes from a 0-cycle η' in $C_0(K_{1,1} \cap K_{1,2})$ and sets $\partial_1([\eta])$ to be $[\eta'] \in H_0(K_{1,1} \cap K_{1,2})$.

Hence, if we can find $\alpha \in C_1(K_{1,2})$ and $\beta \in C_1(K_{1,2})$ such that $(d_1^{K_{1,1}}(\alpha), d_1^{K_{1,2}}(\beta)) = (h-g, g-h) = \varphi_0(h-g)$ and $\gamma := \varrho_1(\alpha,\beta) = \alpha + \beta \in \ker d_1^{K_1}$, we will have $\partial_1([\gamma]) = [h] - [g]$, which means that $[\gamma]$ is a generator of $H_1(K_1)$.

To realize the example from above, we set $\alpha = \{g,h\}$ and $\beta = -\{e,h\} + \{e,f\} + \{f,g\}$. Then we indeed have $d_1^{K_{1,1}}(\alpha) = h - g$ and $d_1^{K_{1,2}}(\beta) = e - h + f - e + g - f = g - h$. Moreover, a straightforward calculation shows that $\gamma := \varrho_1(\alpha,\beta) = \{f,g\} + \{g,h\} - \{e,h\} + \{e,f\}$ is a cycle, so $[\gamma] = [\{f,g\} + \{g,h\} - \{e,h\} + \{e,f\}]$ is indeed a generator of $H_1(K_1)$.

Next, we see that $H_2(K_1)$ is "squeezed between trivial groups" in the MV sequence:

$$0 \cong H_2(K_{1,1}) \oplus H_2(K_{1,2}) \to H_2(K_1) \to H_1(K_{1,1} \cap K_{1,2}) \cong 0,$$

so it also is trivial. Moreover, $H_n(K_1) \cong 0$ for $n > 2$ as K_1 is a 2-dimensional complex.

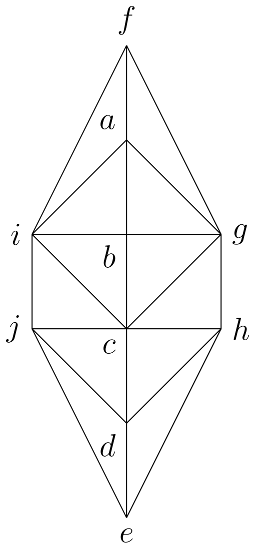
All in all, we have calculated that

$$H_n(K_1) \cong \begin{cases} \mathbb{Z} & n \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

where $H_1(K_1) = \mathbb{Z} \cdot [\{f,g\} + \{g,h\} - \{e,h\} + \{e,f\}]$.

Second subcomplex

After doing the identifications given by the labeling, K_2 looks as follows:

K_2 : 

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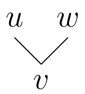
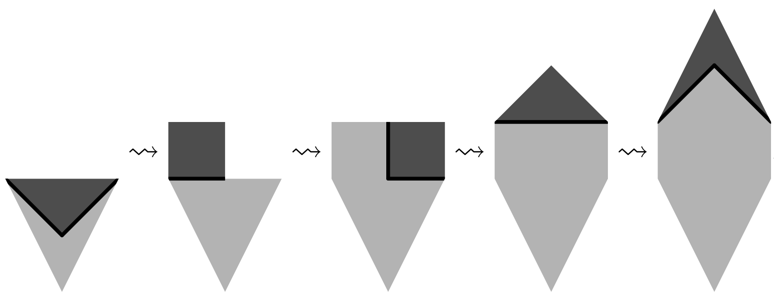
The picture makes it evident that $|K_2|$ is homeomorphic to a disk and we will show that K_2 is indeed acyclic by decomposing it into two acyclic subcomplexes whose intersection is acyclic.

First we note that the simplicial complex

L : 

is acyclic because it is the union of two copies of Δ^1 whose intersection is isomorphic to Δ^0.

Now we can iteratively build K_2 by starting with a complex isomorphic to the acyclic complex $K_{1,1}$ from above and in each step adding a copy of $K_{1,1}$ in a way that the intersection is isomorphic to Δ^1 or L (thus also acyclic), which means that each complex in the sequence is acyclic:



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The intersection and the final MV sequence

The intersection $K_1 \cap K_2$ is given by

$$K_1 \cap K_2 : \quad \begin{array}{c|c|c} f & \qquad & e \\ g & & j \\ h & & i \\ e & & f \end{array}$$

which represents a hexagon with vertices f, g, h, e, j, i after doing the identifications indicated by the labeling.[2]

We refrain from computing its homology here which can be done directly or using a Mayer–Vietoris sequence. The result is

$$H_n(K_1 \cap K_2) \cong \begin{cases} \mathbb{Z} & n \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

where $H_1(K_1 \cap K_2)$ is generated by the class of $\theta := \{f, g\} + \{g, h\} - \{e, h\} + \{e, j\} - \{i, j\} - \{f, i\}$, i. e. a generating cycle is given by "going around the circle once".

Now we start computing the homology of K. Since K is connected, $H_0(K) \cong \mathbb{Z}$.

To compute $H_1(K)$, we will analyze the following segment of the Mayer–Vietoris sequence:

$$H_1(K_1 \cap K_2) \xrightarrow{\phi_1} H_1(K_1) \oplus H_1(K_2) \xrightarrow{\rho_1} H_1(K) \xrightarrow{\partial_1} H_0(K_1 \cap K_2) \xrightarrow{\phi_0} H_0(K_1) \oplus H_0(K_2).$$

Using the homology class of g as a generator of 0-th homology groups of K_1, K_2 and $K_1 \cap K_2$, we see that

$$\begin{aligned} \mathbb{Z} \cong H_0(K_1 \cap K_2) &\xrightarrow{\phi_0} H_0(K_1) \oplus H_0(K_2) \cong \mathbb{Z}^2 \\ m[g] &\mapsto (m[g], -m[g]) \end{aligned}$$

is injective, i. e. $\ker \phi_0 = 0$.

Hence, by exactness, $\operatorname{im} \partial_1 = \ker \phi_0 = 0$. This yields, again by exactness, $H_1(K) = \ker \partial_1 = \operatorname{im} \rho_1$. Using that $\ker \rho_1 = \operatorname{im} \phi_1$, this means that $H_1(K) \cong \operatorname{coker} \phi_1$ by the first isomorphism theorem.

Now ϕ_1 is a homomorphism

$$\mathbb{Z} \cong H_1(K_1 \cap K_2) \to H_1(K_1) \oplus H_1(K_2) \cong H_1(K_1) \oplus 0 \cong \mathbb{Z},$$

so it maps the generator $[\theta] = [\{f, g\} + \{g, h\} - \{e, h\} + \{e, j\} - \{i, j\} - \{f, i\}]$ of $H_1(K_1 \cap K_2)$ to a multiple $k \cdot [\gamma]$ of the generator $[\gamma] = [\{f, g\} + \{g, h\} - \{e, h\} + \{e, f\}]$ of $H_1(K_1)$ and thus its image is $k \cdot \mathbb{Z} \cdot [\gamma]$, which means that its cokernel is isomorphic to $\mathbb{Z}/k\mathbb{Z}$.

[2] This complex may look familiar as one possible way to solve exercise 3a of set 5 was taking an $(n+2)$-gon for every $n \geqslant 1$, so some of you may have already computed its homology.

Intuitively speaking, one can say that "the cycle $\{f,g\}+\{g,h\}-\{e,h\}+\{e,j\}-\{i,j\}-\{f,i\}$ goes around the Möbius strip K_1 twice", so k must be 2. This can be made precise as follows:

Let $\gamma' := \{e,j\}-\{i,j\}-\{f,i\}-\{e,f\} \in C_1(K_1 \cap K_2) \subseteq C_1(K_1)$. Note that γ' is a cycle in K_1 and that $\theta = \gamma + \gamma'$, so $[\theta] = [\gamma] + [\gamma']$ in $H_1(K_1)$. Therefore it is enough to show that $[\gamma] = [\gamma']$, i. e. $[\gamma - \gamma'] = 0$, in $H_1(K_1)$. Also this has a geometric interpretation: In the representation of K_1 as a rectangle whose top and bottom edge are appropriately identified, $\gamma - \gamma'$ corresponds to the boundary of the rectangle, so it is the image of an appropriate sum of the 2-simplices in the rectangle under $d_2^{K_1}$:

$$\begin{aligned}
&=d_2^{K_1}(\{e,f,g\} + \{e,g,j\} + \{g,h,j\} + \{h,i,j\} - \{e,h,i\} + \{e,f,i\}) \\
&=(\{f,g\} - \{e,g\} + \{e,f\}) + (\{g,j\} - \{e,j\} + \{e,g\}) + (\{h,j\} - \{g,j\} + \{g,h\}) + \\
&\qquad (\{i,j\} - \{h,j\} + \{h,i\}) - (\{h,i\} - \{e,i\} + \{e,h\}) + (\{f,i\} - \{e,i\} + \{e,f\}) \\
&=(\{f,g\} + \{e,f\}) + (-\{e,j\}) + (\{g,h\}) + (\{i,j\}) - (\{e,h\}) + (\{f,i\} + \{e,f\}) \\
&=(\{f,g\} + \{g,h\} - \{e,h\} + \{e,f\}) + (-\{e,j\} + \{i,j\} + \{f,i\} + \{e,f\}) \\
&=\gamma - \gamma'.
\end{aligned}$$

Hence $\phi_1([\theta])$ indeed corresponds to $[\gamma'] + [\gamma] = [\gamma] + [\gamma] = 2[\gamma]$, so $H_1(K) \cong \mathbb{Z}/2\mathbb{Z}$.

Note that the calculation above also shows that $\ker \phi_1 = 0$ as ϕ_1 is essentially given by multiplication by 2. Hence, looking at the exact sequence

$$0 \cong H_2(K_1) \oplus H_1(K_2) \xrightarrow{\rho_2} H_2(K) \xrightarrow{\partial_2} H_1(K_1 \cap K_2) \xrightarrow{\phi_1} H_1(K_1) \oplus H_1(K_2),$$

we see that $0 = \ker \phi_1 = \operatorname{im} \partial_2$, so $H_2(K) = \ker \partial_2 = \operatorname{im} \rho_2 = 0$.

Moreover, as K is a 2-dimensional simplicial complex, we have $H_n(K) \cong 0$ for all $n > 2$.

All in all, we obtain

$$H_n(K) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$