1. Introduction to homological algebra

Goal: Introduce a language that is useful in various different fields of mathematics and develop some tools based on that which we will need throughout the course.

a. Elements of category theory

Here are some examples we would like to generalize:

- k-vector spaces and k-linear maps (for a field k)
- groups and group homomorphisms
- topological spaces and continuous maps

The axiomatization is as follows:

Definition 1.1. A *category* $\mathscr{C}$ consists of

- a class $\operatorname{Ob}\mathscr{C}$ whose elements are called the *objects* of $\mathscr{C}$,
- a class $\mathscr{C}(A, B)$ for all $A, B \in \operatorname{Ob}\mathscr{C}$ whose elements are called *morphisms* from A to B,
  Notation: $\operatorname{Mor}\mathscr{C} := \coprod_{A,B\in\operatorname{Ob}\mathscr{C}} \mathscr{C}(A, B)$, $f\colon A \to B$ means $f \in \mathscr{C}(A, B)$.
- a *composition operation*

$$\begin{aligned}\circ\colon \mathscr{C}(A, B) \times \mathscr{C}(B, C) &\to \mathscr{C}(A, C)\\ (f, g) &\mapsto g \circ f\end{aligned}$$

  for all $A, B, C \in \operatorname{Ob}\mathscr{C}$ which satisfies
  - (associativity)

$$\forall A, B, C, D \in \operatorname{Ob}\mathscr{C}\ \forall f \in \mathscr{C}(A, B)\ \forall g \in \mathscr{C}(B, C)\ \forall h \in \mathscr{C}(C, D):\\ h \circ (g \circ f) = (h \circ g) \circ f \in \mathscr{C}(A, D),$$

  - (unitality)

$$\forall A \in \operatorname{Ob}\mathscr{C}\ \exists \mathrm{Id}_A \in \mathscr{C}(A, A)\ \forall B \in \operatorname{Ob}\mathscr{C}:\\ (\forall f \in \mathscr{C}(A, B)\ f \circ \mathrm{Id}_A = f) \wedge (\forall g \in \mathscr{C}(B, A)\ \mathrm{Id}_A \circ g = g).$$

A category $\mathscr{C}$ is called *locally small* if for all $A, B \in \operatorname{Ob}\mathscr{C}$, $\mathscr{C}(A, B)$ is a set.
A locally small category $\mathscr{C}$ is called *small* if in addition $\operatorname{Ob}\mathscr{C}$ is a set.

Example 1.2.

1. $\underline{\mathrm{Set}}$
2. $\underline{\mathrm{Mon}}$, $\underline{\mathrm{Gr}}$, $\underline{\mathrm{Ab}}$ (first two contain the next one as a "subcategory")
3. $\underline{\mathrm{Ring}}$, $\underline{\mathrm{CRing}}$ (the former contains the latter as a "subcategory")
4. $\underline{\mathrm{Vect}}_k$ for a field k, $\underline{\mathrm{Mod}}_R$ for a ring R
5. $\underline{\mathrm{Top}}$, $\underline{\mathrm{Top}}_*$
6. "$\tilde{P}$" for $(P, \leqslant)$ (small)
7. BM for a monoid M (small)
8. $\mathscr{C}^{\mathrm{op}}$ $((\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{C})$

9. $\mathscr{C} \times \mathscr{D}$ (composition left as an exercise)
10. $\mathscr{C}^{\to}$

Definition 1.3. Let $\mathscr{C}$ be a category and $A, B \in \operatorname{Ob}\mathscr{C}$. A morphism $f\colon A \to B$ is called an *isomorphism* if there exists a $g\colon B \to A$ such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$.

Notation: We write $A \cong B$ if there exists an isomorphism $A \to B$ (or equivalently $B \to A$).

Proposition 1.4. *If $f\colon A \to B$ is an isomorphism, then there exists a* unique *$g\colon B \to A$ such that $f \circ g = \mathrm{Id}_B$ and $g \circ f = \mathrm{Id}_A$.*

Notation: *Denote such a g by f^{-1}.*

Proof. If $f \circ g' = \mathrm{Id}_B$, then we have

$$g = g \circ \mathrm{Id}_B = g \circ (f \circ g') = (g \circ f) \circ g' = \mathrm{Id}_A \circ g' = g'.$$

A similar argument works for another right inverse of f. □

Now we want to be able to compare categories with each other.

Definition 1.5. Let $\mathscr{C}$, $\mathscr{D}$ be categories. A *functor* consists of
- a function $F_0\colon \operatorname{Ob}\mathscr{C} \to \operatorname{Ob}\mathscr{D}$,
- functions $(F_1)_{A,B}\colon \mathscr{C}(A, B) \to \mathscr{C}(F_0(A), F_0(B))$

such that
- (preservation of composition)

$$\forall A, B, C \in \operatorname{Ob}\mathscr{C}\ \forall f \in \mathscr{C}(A, B)\ \forall g \in \mathscr{C}(B, C)\colon$$
$$(F_1)_{B,C}(g) \circ_{\mathscr{D}} (F_1)_{A,B}(f) = (F_1)_{A,C}(g \circ_{\mathscr{C}} f),$$

- (preservation of identities)

$$\forall A \in \operatorname{Ob}\mathscr{C}\ (F_1)_{A,A}(\mathrm{Id}_A) = \mathrm{Id}_{F_0(A)}.$$

Notation: We usually just write F instead of (F_0, F_1).

Example 1.6.
1. Forgetful functors: $\underline{\mathrm{Vect}}_k \to \underline{\mathrm{Ab}} \to \underline{\mathrm{Gr}} \to \underline{\mathrm{Mon}} \to \underline{\mathrm{Set}}$, $\underline{\mathrm{Top}}_* \to \underline{\mathrm{Top}} \to \underline{\mathrm{Set}}$, ... (usually implicit)
2. $\langle - \rangle\colon \underline{\mathrm{Set}} \to \underline{\mathrm{Vect}}_k$, $F_{\underline{\mathrm{Ab}}}\colon \underline{\mathrm{Set}} \to \underline{\mathrm{Ab}}$
3. $(-)^{\mathrm{op}}\colon \underline{\mathrm{Gr}} \to \underline{\mathrm{Gr}}$
4. $(-)^*\colon \underline{\mathrm{Vect}}_k \to \underline{\mathrm{Vect}}_k^{\mathrm{op}}$
5. $M_n\colon \underline{\mathrm{Ring}} \to \underline{\mathrm{Ring}}$, $(-)^\bullet\colon \underline{\mathrm{Ring}} \to \underline{\mathrm{Mon}}$,
6. $\pi_1\colon \underline{\mathrm{Top}}_* \to \underline{\mathrm{Gr}}$
7. $f\colon P \to Q$ order preserving $\rightsquigarrow \tilde{f}\colon \tilde{P} \to \tilde{Q}$
8. $\varphi\colon M \to N$ monoid homomorphism $\rightsquigarrow B\varphi\colon BM \to BN$
9. $\mathrm{Id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$
10. $\mathscr{C}(X, -)\colon \mathscr{C} \to \underline{\mathrm{Set}}$ for $\mathscr{C}$ locally small and $X \in \operatorname{Ob}\mathscr{C}$ – can also be "refined" in some cases, e. g. for $X \in \operatorname{Ob}\underline{\mathrm{Ab}}$ we have $\operatorname{Hom}(X, -)\colon \underline{\mathrm{Ab}} \to \underline{\mathrm{Ab}}$.

11. $F\colon \mathscr{C} \to \mathscr{D} \rightsquigarrow F^{\mathrm{op}}\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$
12. $F\colon \mathscr{C} \to \mathscr{D} \rightsquigarrow F^{\to}\colon \mathscr{C}^{\to} \to \mathscr{D}^{\to}$
13. $\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{G} \mathscr{E} \rightsquigarrow G \circ F\colon \mathscr{C} \to \mathscr{E}$

We can also relate functors with each other:

Definition 1.7. Let $F, G\colon \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation* τ from G to F, denoted by $\tau\colon F \Rightarrow G$, consists of a function $\tau\colon \operatorname{Ob}\mathscr{C} \to \operatorname{Mor}\mathscr{D}$ such that

- $$\forall A \in \operatorname{Ob}\mathscr{C}\colon \tau(A)\colon F(A) \to F(B),$$
- $$\forall (f\colon A \to B) \in \operatorname{Mor}\mathscr{C}\colon \tau(A) \circ G(f) = F(f) \circ \tau(B),$$

  in other words, the diagram

  $$\begin{tikzcd} F(A) \arrow[r, "\tau(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\tau(B)"'] & G(B) \end{tikzcd}$$

  commutes.

Example 1.8.
1. $F_{\underline{\mathrm{Ab}}} \circ U_{\underline{\mathrm{Ab}}} \Rightarrow \mathrm{Id}_{\underline{\mathrm{Ab}}}\colon \underline{\mathrm{Ab}} \to \underline{\mathrm{Ab}}$
2. $(-)^{\mathrm{op}} \overset{\mathrm{inv}}{\Longrightarrow} \mathrm{Id}_{\underline{\mathrm{Gr}}}\colon \underline{\mathrm{Gr}} \to \underline{\mathrm{Gr}}$
3. $M_n(-) \overset{\det}{\Longrightarrow} (-)^{\bullet}\colon \underline{\mathrm{CRing}} \to \underline{\mathrm{Mon}}$
4. $\mathrm{Id}_{\underline{\mathrm{Ab}}} \Rightarrow \operatorname{Hom}(\mathbb{Z}, -) : \underline{\mathrm{Ab}} \to \underline{\mathrm{Ab}}$
5. $\mathrm{Id}_{\underline{\mathrm{Vect}_k}} \Rightarrow ((-)^*)^*\colon \underline{\mathrm{Vect}_k} \to \underline{\mathrm{Vect}_k}$
6. $\pi_1(- \times -) \Rightarrow \pi_1(-) \times \pi_1(-)\colon \underline{\mathrm{Top}}_* \times \underline{\mathrm{Top}}_* \to \underline{\mathrm{Gr}}$
7. $\mathrm{Id}_F\colon F \Rightarrow F$
8. $(u\colon X \to Y) \in \operatorname{Mor}\mathscr{C} \rightsquigarrow \mathscr{C}(Y, -) \overset{u^*}{\Longrightarrow} \mathscr{C}(X, -)\colon \mathscr{C} \to \underline{\mathrm{Set}}$ (for locally small $\mathscr{C}$)
9. $\tau\colon F \Rightarrow G \rightsquigarrow \tau^{\mathrm{op}}\colon G^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$
10. $F \overset{\tau}{\Rightarrow} G \overset{\upsilon}{\Rightarrow} H \rightsquigarrow \upsilon \circ \tau\colon F \Rightarrow H$ ("vertical composition")
11. $F, F'\colon \mathscr{C} \to \mathscr{D}$, $\tau\colon F \Rightarrow F'$, $G, G'\colon \mathscr{D} \to \mathscr{E}$, $\upsilon\colon G \Rightarrow G' \rightsquigarrow \upsilon * \tau\colon G \circ F \Rightarrow G' \circ F'$ ("horizontal composition" / "whiskering")

Definition 1.9. Let $\tau\colon F \Rightarrow G$ be a natural transformation between functors $F, G\colon \mathscr{C} \to \mathscr{D}$. τ is called a *natural isomorphism* if for all $A \in \operatorname{Ob}\mathscr{C}$, $\tau(A)\colon F(A) \to G(A)$ is an isomorphism.

Notation: We write $F \cong G$ if there exists a natural isomorphism $F \Rightarrow G$ (or equivalently $G \Rightarrow F$).

Proposition 1.10. *Let $\tau\colon F \Rightarrow G$ be a natural isomorphism between functors $F, G\colon \mathscr{C} \to \mathscr{D}$. Then the assignment*

$$\begin{aligned} \operatorname{Ob}\mathscr{C} &\to \operatorname{Mor}\mathscr{D} \\ A &\mapsto \tau(A)^{-1} \end{aligned}$$

defines a natural transformation.

Proof. Let $(f\colon A \to B) \in \operatorname{Mor} \mathscr{C}$. Then, since τ is a natural transformation, we have

$$\tau(A) \circ G(f) = F(f) \circ \tau(B).$$

Composing both sides with $\tau(A)^{-1}$ from the right yields

$$G(f) = \tau(A)^{-1} \circ \tau(A) \circ G(f) = \tau(A)^{-1} \circ F(f) \circ \tau(B).$$

Composing both sides with $\tau(B)^{-1}$ from the left we obtain

$$G(f) \circ \tau(B)^{-1} = \tau(A)^{-1} \circ F(f) \circ \tau(B) \circ \tau(B)^{-1} = \tau(A)^{-1} \circ F(f),$$

i. e. the diagram

$$\begin{tikzcd} G(A) \arrow[r, "\tau(A)^{-1}"] \arrow[d, "G(f)"'] & F(A) \arrow[d, "F(f)"] \\ G(B) \arrow[r, "\tau(B)^{-1}"'] & F(B) \end{tikzcd}$$

commutes, which is what we needed to show. $\square$

Example 1.11. 1. $(-)^{\mathrm{op}} \cong \mathrm{Id}_{\underline{\mathrm{Gr}}}\colon \underline{\mathrm{Gr}} \to \underline{\mathrm{Gr}}$
2. $\mathrm{Id}_{\underline{\mathrm{Ab}}} \cong \mathrm{Hom}(\mathbb{Z}, -) : \underline{\mathrm{Ab}} \to \underline{\mathrm{Ab}}$
3. $\mathrm{Id}_{\underline{\mathrm{Vect}}_k^{\mathrm{fd}}} \cong ((-)^*)^*\colon \underline{\mathrm{Vect}}_k^{\mathrm{fd}} \to \underline{\mathrm{Vect}}_k^{\mathrm{fd}}$
4. $\pi_1(- \times -) \cong \pi_1(-) \times \pi_1(-)\colon \underline{\mathrm{Top}}_* \to \underline{\mathrm{Gr}}$
5. If $\mathscr{C}$ is a locally small category and $u\colon X \to Y$ is an isomorphism in $\mathscr{C}$, then $\mathscr{C}(Y, -) \overset{u^*}{\Longrightarrow} \mathscr{C}(X, -)\colon \mathscr{C} \to \underline{\mathrm{Set}}$ is a natural isomorphism.

Definition 1.12. A functor $F\colon \mathscr{C} \to \mathscr{D}$ is called
- an *isomorphism of categories* if there exists a functor $G\colon \mathscr{D} \to \mathscr{C}$ such that $F \circ G = \mathrm{Id}_{\mathscr{D}}$ and $G \circ F = \mathrm{Id}_{\mathscr{C}}$ (**Notation:** $\mathscr{C} \cong \mathscr{D}$),
- an *equivalence of categories* if there exists a functor $G\colon \mathscr{D} \to \mathscr{C}$ such that $F \circ G \cong \mathrm{Id}_{\mathscr{D}}$ and $G \circ F \cong \mathrm{Id}_{\mathscr{C}}$ (**Notation:** $\mathscr{C} \simeq \mathscr{D}$).

Example 1.13. 1. $M \cong N \in \underline{\mathrm{Mon}} \rightsquigarrow BM \cong BN$
2. $\underline{\mathrm{Vect}}_k^{\mathrm{fd}} \simeq \underline{\mathrm{Mat}}_k$ where $\operatorname{Ob} \underline{\mathrm{Mat}}_k = \mathbb{N}$, $\underline{\mathrm{Mat}}_k(n, m) = \mathrm{Mat}_{m \times n}(k)$ and composition in $\underline{\mathrm{Mat}}_k$ is given by matrix multiplication.