A chain rule for Goodwillie calculus

sarah yeakel

ytm 2015

1 Background

Let F be an endofunctor of based spaces which

- preserves weak equivalences
- preserves filtered hocolimits
- is continuous, i.e., the induced map $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X), F(Y))$ is a continuous homomorphism of spaces

Then F has a Taylor tower (analogous to a Taylor series expanded at $*$)

$$F(X) \longrightarrow \qquad \cdots \longrightarrow P_n F(X) \longrightarrow P_{n-1}F(X) \longrightarrow \cdots \qquad P_1F(X) \longrightarrow P_0F(X) \simeq F(*)$$

The P_nF are called "n-excisive" and are analogous to polynomial of degree $\leq n$.

One wishes to study the functor F by studying its Taylor tower, and this is quite a good approximation when F is *analytic*, which implies $F(X) \simeq \mathrm{holim}_n\, P_nF(X)$ for sufficiently connected X. Many functors are analytic, for example, the identity functor of spaces is analytic.

Alas, the n-excisive approximations are hard to compute, so we consider the *layers* of the Taylor tower, defined as

$$D_nF = \mathrm{hofib}(P_nF \to P_{n-1}F)$$

The layers are called *n-homogeneous* and you can consider the fiber as a difference of the nth polynomial approximation from the $n-1$st. Indeed, this is a good analogy as we can see from squinting at Goodwillie's classification of the layers.

Theorem 1.1 (Goodwillie)**.**

$$D_nF(X) \simeq \Omega^\infty \left(\partial_n F \wedge X^{\wedge n}\right)_{h\Sigma_n} \sim \frac{f^{(n)}(*)\cdot x^n}{n!}$$

where ∂_nF is a spectrum with Σ_n-action called the n-th derivative of F*.*

Goodwillie went further to identify the homotopy type of these derivatives.

Theorem 1.2 (Goodwillie)**.** *The n-th derivative of F is equivalent to the multilinearization of the nth cross effect.*

$$(\Omega^\infty)\partial_n^G F \simeq \operatorname*{hocolim}_{k_1,\ldots,k_n\to\infty} \Omega^{k_1}\cdots\Omega^{k_n} cr_nF(\Sigma^{k_1}S^0,\ldots,\Sigma^{k_n}S^0)$$

The Σ_n-action is induced by permuting the variables of cr_nF. The nth cross effect is a functor of n variables which can be thought of as a measurement of the failure of F to be degree $n-1$ (in an additive sense). For example, $cr_1F(X) = \mathrm{hofib}(F(X) \to F(*))$, so if F were degree 0 (or constant), cr_1F would be trivial.

If we consider all the derivatives of a functor together, we have a symmetric sequence in spectra. Thus we may think of the derivatives as a functor

$$\partial_* : [Top_*, Top_*] \to [\Sigma, \mathcal{S}\mathrm{p}]$$

Question 1. *Is ∂_* (lax) monoidal?*

That is, we are asking for a natural transformation $\partial_* F \circ \partial_* G \to \partial_*(F \circ G)$ and a map $S^0 \to \partial_1 Id$, where the first $\circ$ is the composition product of symmetric sequences and the second is composition of functors.

It is easy to construct a composition map which is associative and unital up to homotopy, but strict associativity requires a different model for the derivatives.

If ∂_* is monoidal, some immediate consequences would be

- $\partial_* Id$ is an operad
- $\partial_* F$ is $\partial_* Id$-bimodule

These consequences have been proven. The first is due to work of Johnson, Arone-Mahowald, and Ching, and the second is work of Arone-Ching. Arone and Ching went on to show that the derivatives have an even nicer property, a chain rule.

Theorem 1.3 (Arone-Ching)**.** *For reduced, finitary, cofibrant functors F, G, $\partial_* F \circ_{\partial_* Id} \partial_* G \simeq \partial_*(F \circ G)$.*

The functors are cofibrant in the projective model structure on the category of functors, restricting domains to finite things to make sense of this (check "Operads and chain rules for the calculus of functors" for details). Taking the composition-product over $\partial_* Id$ is a derived product, i.e., the left hand side is a two-sided bar construction. The equivalence proven by Arone and Ching comes as a zigzag of equivalences, and there is no direct map for the chain rule.

2 A monoidal model

Definition 2.1. Let $\mathbb{I}$ be the category of finite sets and injective maps.

Remark 2.2. Bökstedt used this category to define THH in the 80s before spectra had a good smash product. It has since been used in many settings, including homological stability, units of K-theory, and models for commutative $H\mathbb{Z}$-algebras.

Definition 2.3. Let

$$\partial_n F = \operatorname*{hocolim}_{U_1,\dots,U_n \in \mathbb{I}} \Omega^{U_1} \cdots \Omega^{U_n} cr_n F(S^{U_1}, \dots, S^{U_n})$$

Lemma 2.4. *If F is analytic, then $\partial_*^G F \simeq \partial_* F$, by Bökstedt's approximation lemma.*

Theorem 2.5. *(Y.) The model for ∂_* given in Definition 2.3 is monoidal.*

Proof. (sketch) We are looking for a natural transformation $\partial_* F \circ \partial_* G \to \partial_*(F \circ G)$, which should be a levelwise map of symmetric sequences. On level $j = j_1 + \cdots + j_k$, this boils down to defining maps

$$\partial_k F \wedge \partial_{j_1} G \wedge \cdots \wedge \partial_{j_k} G \longrightarrow \partial_j(F \circ G).$$

The crucial maps are the assembly maps $\alpha_F : F(X) \wedge Y \to F(X \wedge Y)$ (which any continuous functor F possesses), $\star : cr_1 F \circ cr_1 G \to cr_1(F \circ G)$ (choose models carefully for this), and $\coprod : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ which induces an associative map on homotopy colimits. Let's give an idea of how this goes on the first level $\partial_1 F \wedge \partial_1 G \to \partial_1(F \circ G)$:

$$\begin{array}{c}
\operatorname{hocolim}_{U \in \mathbb{I}} \Omega^U cr_1 F(S^U) \wedge \operatorname{hocolim}_{V \in \mathbb{I}} \Omega^V cr_1 G(S^V) \\
\Big\downarrow {\scriptstyle \alpha_{\mathrm{hocolim}},\, \alpha_\Omega} \\
\operatorname{hocolim}_{U \in \mathbb{I}} \operatorname{hocolim}_{V \in \mathbb{I}} \Omega^U \Omega^V cr_1 F(S^U) \wedge cr_1 G(S^V) \\
\Big\downarrow {\scriptstyle \alpha_{cr_1}} \\
\operatorname{hocolim}_{U \in \mathbb{I}} \operatorname{hocolim}_{V \in \mathbb{I}} \Omega^U \Omega^V cr_1 F(cr_1 G(S^U \wedge S^V) \\
\Big\downarrow {\scriptstyle \star} \\
\operatorname{hocolim}_{U,V \in \mathbb{I}} \Omega^{U \amalg V} cr_1(F \circ G)(S^{U \amalg V}) \\
\Big\downarrow {\scriptstyle \amalg_*} \\
\operatorname{hocolim}_{W \in \mathbb{I}} \Omega^W cr_1(F \circ G)(S^W)
\end{array}$$

This generalizes to higher level maps. Notice the last step was the key reason for using $\mathbb{I}$; if the homotopy colimit is defined over $\mathbb{N}$, the map can be defined, but it will not be associative on homotopy colimits. This is similar to the reason naive spectra do not have a good smash product, but symmetric spectra have enough extra structure to encode the smash product in an associative way. □

Immediate consequences of the monoidal structure is that the derivatives of a monad (for example the identity functor) have the structure of an operad and derivatives of other functors inherit a module structure over the derivatives of the identity. We hope to generalize this to functors of other categories to get natural operad and module structures in new settings.

Finally, a chain rule was promised, so one shall be given:

Claim 2.6. For F, G cofibrant (in projective model structure) endofunctors of spaces, we have

$$\partial_* F \circ_{\partial_* Id} \partial_* G \xrightarrow{\simeq} \partial_*(F \circ G)$$

Remark 2.7. There is a direct map in this chain rule. There is hope to extend more of Arone and Ching's work; for example, they develop a classification of n-excisive functors using information from the derivatives.