Koszul duality and rational homotopy theory

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Plan of the presentation

- Motivation and some facts from rational homotopy theory

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- Introduction to Koszul duality

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- Introduction to Koszul duality
- Applications to rational homotopy theory

Rational homotopy theory

Convention

We assume that all spaces we consider are simply connected CW complexes.

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Definition

Let $f : X \to Y$ be a map of spaces, then we call f a rational homotopy equivalence if one of the following equivalent conditions holds

1. $f_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism,
2. $f_* : H^*(X; \mathbb{Q}) \to H^*(Y; \mathbb{Q})$ is an isomorphism.

Rational homotopy theory

Theorem (Quillen)

There is an equivalence between the homotopy category of rational spaces and the homotopy category of differential graded Lie algebras over $\mathbb{Q}$.

Rational homotopy theory

Theorem (Sullivan)

There is an equivalence between the homotopy category of rational spaces and the homotopy category of commutative differential graded algebras over $\mathbb{Q}$.

Commutative differential graded algebras

Definition

A commutative differential graded algebra is a differential graded vector space A together with a product

$$\cdot : A \otimes A \to A.$$

Such that $\cdot$ is

- associative
- graded commutative $a \cdot b = (-1)^{deg(a)deg(b)} b \cdot a$,
- and satisfies the Leibniz rule $d(a \cdot b) = d(a) \cdot b + (-1)^{deg(a)} a \cdot d(b)$.

Quasi-free CDGA's

Definition

A quasi-free CDGA $(\Lambda V, d)$, is a free commutative differential graded algebra ΛV together with a differential d.

Sullivan models

Definition

A Sullivan model for a space X is a quasi-free commutative differential graded algebra

$$(\Lambda V, d) \xrightarrow{\sim} A_{PL}(X),$$

where $A_{PL}(X)$ is the algebra of polynomial de Rham forms. We also require that the differential satisfies some properties.

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Definition

A Sullivan model $(\Lambda(V), d)$ is called minimal if $d(V) \subseteq \Lambda^{\geq 2} V$.

Minimal Sullivan models

Theorem

The minimal Sullivan model is unique up to isomorphism and two spaces X and Y are rational homotopy equivalent if and only if their minimal Sullivan models are isomorphic.

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Theorem

Let X be a space and $(\Lambda V, d)$ its minimal Sullivan model, then there is an isomorphism

$$\pi_k(X) \otimes \mathbb{Q} \cong Hom_{\mathbb{Q}}(V^k, \mathbb{Q}).$$

Examples of Sullivan models

Example (Odd dimensional spheres)

The minimal Sullivan model for S^{2n+1} is given by Λa the free CDGA on one generator a of degree $2n+1$ and with a zero differential.

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Example (Even dimensional spheres)

The minimal Sullivan model for S^{2n} is given by $\Lambda a, b$ such that $deg(a) = 2n$ and $deg(b) = 4n-1$ and the differential is given by $d(b) = a^2$.

Minimal Sullivan models

Question

How do we construct a minimal Sullivan model for an algebra A?

Minimal Sullivan models

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Answer

Some complicated inductive algorithm.

Koszul duality

But in special cases we have a technique which makes computing the minimal Sullivan model extremely easy.

Koszul duality

Convention

For simplicity we will now work with associative algebras. But all these ideas work in much greater generality.

The bar and cobar constructions

In general if we want a free resolution of an algebra A we have the following standard resolution which is given by

$$\Omega BA \xrightarrow{\sim} A.$$

Where Ω is the cobar construction and B the bar construction.

The bar construction

Definition

The bar construction is a functor

$$B : \text{Associative algebras} \to \text{Coassociative coalgebras}$$

which is given by $BA = (T^c(sA), d_B)$.

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The cobar construction is a functor

$$\Omega : \text{Coassociative coalgebras} \to \text{Associative algebras}$$

which is given by $\Omega C = (T(s^{-1}C), d_\Omega)$.

The bar and cobar construction

Proposition

The bar and cobar construction both preserve quasi-isomorphisms.

Koszul duality

In the case that the coalgebra BA is formal, i.e. BA is quasi isomorphic to its homology HBA, then ΩHBA is also a resolution for A and it is minimal.

Koszul duality

Definition

An algebra A is called Koszul if the bar construction is formal.

Koszul duality

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Theorem

If A is a Koszul algebra then ΩHBA is a minimal model for A.

Quadratic algebras

Definition

An algebra A is called quadratic if it has a presentation of the form

$$A = T(V)/(R),$$

where $R \subseteq V \otimes V$.

Homology of the bar construction

Definition

Let A be a quadratic algebra, then we define an extra grading, called the weight grading, on A by giving each generator degree 1.

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Definition

On the bar construction of A we define an extra grading called the bar length, which is defined by assigning every element in A degree 1.

The bar complex

Bar length $\rightarrow$ / Weight $\downarrow$	1		2		3
1	V	$\xleftarrow{d}$	0		0
2	$\frac{V^2}{R}$	$\xleftarrow{d}$	$V^{\otimes 2}$	$\xleftarrow{d}$	0
3	$\frac{V^3}{(VR \oplus RV)}$	$\xleftarrow{d}$	$(\frac{V^2}{R} \otimes V) \oplus (V \otimes \frac{V^2}{R})$	$\xleftarrow{d}$	$V^{\otimes 3}$

The diagonal

Definition

The diagonal $\mathcal{D}$ of the bar construction is the sub complex of the bar construction of all elements such that the bar length is equal to the weight.

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Note that the homology of the diagonal is very easy to compute.

Koszul duality

Definition (Alternative definition of Koszul duality)

An algebra A is Koszul if $HBA \subseteq \mathcal{D}$

The Koszul dual

Definition

Let A be a quadratic algebra then we define the Koszul dual algebra $A^!$ as

$$A^! = T(sV^*)/(R^\perp),$$

In this case $R^\perp$ is the annihilator of R under the pairing $V^* \otimes V^* \otimes V \otimes V \to \mathbb{Q}$.

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If A is Koszul then $A^!$ is isomorphic to the $(HBA)^*$.

Examples

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The algebra $\mathbb{Q}[x]/x^2$ is Koszul.

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Generalizations of Koszul duality

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Theorem

The operads $\mathcal{COM}$ and $\mathcal{LIE}$ are Koszul dual to each other.

Koszul spaces

Definition

A space X is called a Koszul space if it is rationally equivalent to the derived spatial realization of a Koszul algebra.

Examples of Koszul spaces

- Spheres

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- Loop spaces

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- Products and wedges of Koszul spaces
- Ordered configuration spaces of points in $\mathbb{R}^n$
- Highly connected manifolds

Applications

Theorem (Berglund)

Let X be a Koszul space then the homotopy and cohomology groups are Koszul dual, i.e. we have an isomorphism

$$\pi_*(\Omega X) \otimes \mathbb{Q} = H^*(X;\mathbb{Q})^{!_{Lie}}.$$

Applications

Theorem (Berglund)

Let X be an n-connected Koszul space then there is the following isomorphism

$$H_*(\Omega^n X;\mathbb{Q}) \cong H^*(X;\mathbb{Q})^{!_{G_n}}.$$

Example: The wedge of two spheres

- The space $S^2 \vee S^3$ is a Koszul space.

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Example: The wedge of two spheres

- The space $S^2 \vee S^3$ is a Koszul space.
- The cohomology is given by $\mathbb{Q}[x, y]/(x^2, xy, y^2)$ with $deg(x) = 2$ and $deg(y) = 3$.
- The rational homotopy Lie algebra of the loop space is then given by $\pi_*(\Omega(S^2 \vee S^3)) \otimes \mathbb{Q} = \mathcal{LIE}(a, b)$, with $deg(a) = 1$ and $deg(b) = 2$.

Thank you for your attention.