The Baez-Dolan Stabilization Hypothesis

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A Confession

The Baez-Dolan Stabilization Hypothesis and Left Proper Model Structures on Operads
We will begin with an overview of an old problem, due to Baez and Dolan, and discuss where in the solution of this problem model category theoretic considerations arise. This discussion will relate TQFTs, higher category theory, Rezk's Theta n spaces, model categories, and operads. The solution requires placing left proper model structures on algebras over certain colored operads, most notably on the category of non-reduced symmetric operads. We will discuss how this can be accomplished and why left properness is so essential.

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The Baez-Dolan Stabilization Hypothesis

A statement about higher category theory which arose in *Higher Dimensional Algebra and Topological Quantum Field Theory* (1995) by John Baez and James Dolan.

This paper studied n-dimensional TQFTs via n-category representations, generalizing group representations.

An n-category has objects, morphisms, 2-morphisms, ..., n-morphisms, and composition laws encoded by diagrams. These are NOT (∞, n)-categories

Weak n-categories: diagrams commute up to coherent equivalence. We use Rezk's model.
$nCat := \tau_{\leq n}(\Theta_n\text{-spaces})$

Stabilization and Eckmann-Hilton

Consider the reindexing functor $U : n\text{-cat} \to (n-1)\text{-cat}$:

$$\begin{tikzcd} obj & mor \arrow[dl] & 2-mor \dots \arrow[dl] & n-mor \arrow[dl] \\ obj & mor \dots & (n-1)-mor & \end{tikzcd}$$

In the image, objects have a composition law, morphisms have vertical and horizontal composition, etc. That's extra structure!

Example: start with a 2-category $C = (x, 1_x, hom(1_x, 1_x))$ with 1 object and 1 morphism, reindex twice: 2-cat$\to$ 0-cat. Eckmann-Hilton argument implies $hom(1_x, 1_x)$ is a commutative monoid.

3-cat$\to$ 0-cat gives no further structure; reindexing

k-tuply monoidal weak n-categories

Given an $(n+k)$-category $\mathcal{C}$ with only one cell in each dimension $< n$, reindex to an n-category $\mathcal{D}$ with extra structure. Call any such $\mathcal{D}$ a **k-tuply monoidal n-category**. Let $nCat_k$ be the category of such $\mathcal{D}$. Forgetful $U : nCat_k \to nCat_{k-1}$ has left adjoint *suspension* S.

Hypothesis (Baez-Dolan Stabilization Hypothesis)

If $k \geq n+2$ then $S : nCat_k \to nCat_{k+1}$ is an equivalence of categories.

Batanin: the extra structure on $\mathcal{D}$ is that of an algebra in (weak) *nCat* over a k-operad (e.g. a certain $G_{k,n}$ he constructed). We need to work "up to homotopy" because action of cells in dim $< n$ can be non-trivial on higher cells in general.

Conjecture (Baez-Dolan Stabilization Hypothesis)

Batanin's Proof of Baez-Dolan

Theorem (Batanin, 2015)

Let $(\mathcal{M}, I, \otimes)$ be an n-truncated monoidal model category I cofibrant ($\forall X, Y$, $map(X, Y)$ is n-truncated). Let 1_k is the terminal k-operad and $G_{k,n}$ is the n-truncation of its cofibrant replacement in Op_k. If $k \geq n+2$ then $Alg_{G_{k,n}}(\mathcal{M})$ is Quillen equivalent to $Alg_{G_{k+1,n}}(\mathcal{M})$.

Proof uses $Alg_{G_{k,n}}(\mathcal{M}) \leftrightarrows Alg_{E_{\infty,n}}(\mathcal{M}) \leftrightarrows Alg_{G_{k+1,n}}(\mathcal{M})$

Theorem is provable by passing through symmetric operads **if** $SOp(nCat)$ is a left proper model category.

Unfortunately, that's not true in general, or even for $\mathcal{M} = sSet$!

See Hackney-Robertson-Yau arxiv 1411.4668

Left Properness

$\mathcal{M}$ is *left proper* if in any pushout diagram with cofibrations $A \to C$ and $B \to D$:

$$\begin{tikzcd} A \arrow[r, "\simeq"] \arrow[d, hook] & B \arrow[d, hook] \\ C \arrow[r] & D \end{tikzcd}$$

where $A \to B$ is a weak equivalence, then $C \to D$ is a weak equivalence. It's like the gluing lemma. Used to control homotopy pushouts and for left Bousfield localization.

Even without it, the language of semi-model categories can be used to prove the theorem. Just need simplicial mapping spaces, hocolims, and homotopy invariance of certain operad-algebras.

Semi-Model Categories

$\mathcal{M}$ is a *semi-model category* if it satisfies all the axioms of a model category except:

1. The factorization of $f : A \to B$ into a trivial cofibration followed by a fibration only holds when A is cofibrant.
2. A trivial cofibration $f : A \to B$ lifts against a fibration only when A is cofibrant.

Still have fibrant and cofibrant replacements, localization to homotopy category, Quillen functors & equivalences, framings, (co)simplicial resolutions, hammock localization, cube lemma, Ken Brown's lemma, and now left Bousfield localization.
If all objects cofibrant then same as model category.

Examples of Semi-Model Categories

Lemma (Schwede-Shipley; Kan's Transfer Principle; Crans)

Let P be a (colored) operad, $\mathcal{M}$ a combinatorial monoidal model category. The free-forgetful adjunction $F : \mathcal{M} \leftrightarrows Alg_P(\mathcal{M}) : U$ creates a model structure on $Alg_P(\mathcal{M})$ (where f is a fibration or weak equivalence if $U(f)$ is) if for all generating trivial cofibrations $j : K \to L$ in $\mathcal{M}$, the pushout in $Alg_P(\mathcal{M})$:

$$\begin{tikzcd} F(K) \arrow[r] \arrow[d] & F(L) \arrow[d] \arrow[d, phantom, shift right=3ex, "\searrow"] \\ A \arrow[r] & B \end{tikzcd}$$

has $A \to B$ a weak equivalence (and transfinite compositions too).

If it's only true for cofibrant A then $Alg_P(\mathcal{M})$ is a semi-model category.

Generalized Baez-Dolan Stabilization

Theorem (Batanin-W.)

Let $\mathcal{M}$ be n-truncated and combinatorial. Let Op_k^{loc} be the n-locally constant k-operads, obtained as a left Bousfield localization of Op_k. There is a suspension $S^* : Op_{k+1}^{loc}(\mathcal{M}) \to Op_k^{loc}(\mathcal{M})$ which, if $k \geq n+2$, is a right Quillen equivalence.

This implies the previous theorem about $Alg_{G_{k,n}}$ and the Baez-Dolan Stabilization Hypothesis. This version doesn't require a choice of k-operad to encode k-tuply monoidal n-categories.

The proof requires left Bousfield localization in non left proper settings, e.g. $Op_k(nCat)$.

Bousfield Localization Without Left Properness

Theorem (W.)

If $\mathcal{M}$ is a locally presentable, cofibrantly generated (semi-)model category in which the domains of the generating cofibrations are cofibrant then for any set of morphisms $\mathcal{C}$ in $\mathcal{M}$, there exists a cofibrantly generated semi-model structure $L_{\mathcal{C}}(\mathcal{M})$ on $\mathcal{M}$ with weak equivalences defined to be the $\mathcal{C}$-local equivalences, and cofibrations the cofibrations of $\mathcal{M}$.

It satisfies the universal property in the category of semi-model categories. It's Quillen equivalent as a semi to the model category $L_{\mathcal{C}}(\mathcal{M})$ should it exist. It allows us to focus on n-locally constant k-operads and solve the problem there via our earlier result.

Smith's Theorem for semi-model categories

Theorem (W.)

Given $\mathcal{M}, \mathscr{W}, I$ satisfying

1. $\mathscr{W}$ *is κ-accessible, closed under retracts, 2/3 property.*
2. *Any morphism in inj(I) is a weak equivalence.*
3. *The maps of I have cofibrant domain and the initial object in $\mathcal{M}$ is cofibrant.*
4. *Within the class of trivial cofibrations, maps with cofibrant domain are closed under pushouts to arbitrary cofibrant objects and under transfinite composition.*

Then $(\mathcal{M}, \mathscr{W}, I, J)$ is a combinatorial semi-model category.

Future: Version for cellular semi-model categories?

Model Structure on Operads

Theorem (Batanin, W.)

If $\mathcal{M}$ is cofibrantly generated with small domains, monoidal, and satisfies a generalized commutative monoid axiom then SOp$(\mathcal{M})$ inherits a model structure from $\mathcal{M}^{\mathbb{N}}$. Under a strong hypothesis regarding cofibrations and group actions, SOp$(\mathcal{M})$ is left proper.

Corollary (Batanin, W.)

For $\mathcal{M} = Ch(k)$, bounded or unbounded, with char$(k) = 0$, SOp$(Ch(k))$ is left proper.

Note: Only cofibrantly generated needed to make a semi-model category (necessarily relatively left proper)

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