A dihedral version of the Jones isomorphism

Notes for a talk given at the Young Topologists Meeting, EPFL, Lausanne, Switzerland

Massimiliano Ungheretti

July 7, 2015

The Jones isomorphism relates Hochschild homology $HH_{-*}(S^*X)$ and cohomology of the free loop space $H^*(\mathcal{L}X)$, for any simply connected space X. This and its S^1-equivariant version, have provided algebraic models for string topology. In work in progress, we use similar simplicial methods to explore the $O(2)$-equivariant case and give an isomorphism $DH_{-*}(S^*X) \cong H^*_{O(2)}(\mathcal{L}X)$, involving a flavour of dihedral homology.

Warning: as this abstract is too ambitious I will mainly talk about the Jones isomorphism and only make some remarks about its generalizations.

Free loop space

The object of study in this talk will be *the space of free loops* $\mathcal{L}X := \mathrm{Map}(S^1, X)$, where X is a topological space. This space occurs in several talks this week and plays a role in for example string theory and string topology. Its Betti numbers can be used for counting geodesics by the Gromoll-Meyer theorem.

History

In a way, it all started with the bar construction (Eilenberg, MacLane, Cartan), which led to the Adams' paper from '56 'On the cobar construction'. In that paper, a game is played for based loop spaces that is very similar to what we will do today. Then in the 80's work by Goodwillie[Goo85] and Burghelea-Fiedorowicz[BF86] established an isomorphism:

$$HH_*(S_*(\Omega X)) \cong H_*(\mathcal{L}X)$$

That is, Hochschild homology of singular chains on based loop space with concatenation product is a way of calculation homology of free loop spaces. Today will be about an analogous statement in *co*homology.

Cosimplicial model for $\mathcal{L}X$

We will need the concept of a cosimplicial space. This is an object similar to a simplicial set/space, but dual.

Definition. The category of *cosimplicial spaces* is defined as $\mathsf{cTop} := Fun(\Delta, \mathsf{Top})$ (compare $\mathsf{sSet} = Fun(\Delta^{op}, \mathsf{Set})$, $\mathsf{sTop} = Fun(\Delta^{op}, \mathsf{Top})$). Just as every simplicial set/space has a geometric realization, to each cosimplicial space $Y^\bullet$ we associate (functorially) a space called the *totalization*:

$$\operatorname{tot} Y^\bullet := \operatorname{Nat}_\Delta(\Delta^\bullet, Y^\bullet) \subset \Pi_n \operatorname{Map}(\Delta^n, Y^n)$$

Here, $\Delta^\bullet \in \mathsf{cTop}$ is the geometric $n-$simplex in degree n. I.e. it is the realization of the Yoneda embedding of Δ.

Example. If $K_\bullet$ is a simplicial set and X a space, then we can form a cosimplicial mapping space $(X^K)^\bullet$.

$$(X^K)^n = \operatorname{Map}(K_n, X) \cong \Pi_{K_n} X$$

Lemma. *This cosimplicial space is a model for a mapping space in the sense that we have a homeomorphism* $\operatorname{tot}(X^K)^\bullet \cong \operatorname{Map}(|K|, X)$.

Proof. To see this, analyse which subset we get after rewriting:

$$\operatorname{tot}(X^K)^\bullet \subset \Pi_n \operatorname{Map}(\Delta^n, \operatorname{Map}(K_n, X)) \cong \operatorname{Map}(\Pi_n \Delta^n \times K_n, X)$$

□

So, any simplicial model of a circle will give us a cosimplicial model of free loop space. In particular, we will use the simplicial set $S^1_\bullet = \Delta^1/\partial\Delta^1$. This simplicial set has two non-degenerate simplices: a basepoint and the fundamental simplex. In total it has $n+1-$simplices in degree n. We will now describe the associated cosimplicial mapping space $(X^{S^1})^\bullet$: the space of $n-$simplices is the product $X^{\times n+1}$, the cofaces δ^i copy coordinates and the codegeneracies σ^i forget coordinates.

The comparison map

Let $Y^\bullet \in \mathsf{cTop}$, then by applying the *singular cochain* functor we obtain a simplicial object in chain complexes $S^*Y \in \mathsf{sCh}$. The simplicial structure maps are by definition chain maps so we can associate to it a bicomplex with differentials $(\Sigma(-1)^i d_i, \delta_{\text{Singular}})$. We write $\operatorname{Tot}_\oplus S^*Y \in \mathsf{Ch}$ for the totalization of this bicomplex. We will now describe a chain map $\psi : \operatorname{Tot}_\oplus S^*Y \to S^* \operatorname{tot} Y$.

From the definition of $\operatorname{tot} Y$, we see that there are evaluation maps $\alpha_n : \Delta^n \times \operatorname{tot} Y \to Y^n$. We could try to pull back classes along this map, but we'd end up with $\alpha_n^*(x) \in S^*(\Delta^n \times \operatorname{tot} Y)$. So we can try to 'integrate out' one of the coordinates by using the slant product $-/- : S^q(A \times B) \otimes S_p(A) \to S^{q-p}(B)$. Now for a $x \in S^*(Y^n)$, define $\psi(x) = \alpha_n^*(x)/[\Delta^n]$, where $[\Delta^n]$ is the fundamental simplex/chain.

Remark. The slant product is a chain map, that is, $d(\gamma/[\Delta^n]) = d\gamma/[\Delta^n] \pm \gamma/d[\Delta^n]$. The first term is the internal/singular differential in $\operatorname{Tot}_\oplus S^*Y$ and the second is compensated for by the simplicial differential. The result is that ψ is a chain map from the totalization.

Example. In the special case of the cosimplicial mapping space, the evaluation maps $\alpha_n : \Delta^n \times \mathcal{L}X \to X^{\times n+1}$ are simply $(s_1, \ldots, s_n, \gamma) \mapsto (\gamma(0), \gamma(s_1), \ldots, \gamma(s_n))$.

In the case we are interested in, namely $Y^\bullet = (X^{S^1})^\bullet$, we can compare the $\operatorname{Tot}_\oplus$ to another chain complex called the *cyclic bar construction* or *Hochschild complex* (see also Cary Malkewich's talk from this morning).

Definition. Given a differential graded algebra A, the Hochschild complex $C_*(A)$ is the totalization of a simplicial chain complex. This simplicial object has $n-$simplices $A^{\otimes n+1}$ and structure maps analogous to those of X^{S^1}. See for example [Wei95, Ch 9].

As singular cochains S^*X form a differential graded algebra using the cup product, we can form $C_*(S^*X)$. The Alexander-Whitney maps $S^*X \otimes S^*X \to S^*(X \times X)$ now define a map of simplicial chain complexes

$$AW : (S^*X)^{\otimes \bullet+1} \to S^*((X^{S^1})^\bullet)$$

In fact, this map is a quasi isomorphism level-wise and thus also on the totalization.

Remark. It is worth rememberig that the cup product is defined using the diagonal map followed by the Alexander-Whitney map. This is the reason the two construction can be compared.

Theorem (Jones isomorphism[Jon87])**.** *Assume X is connected and simply connected. Assume furthermore that we are working over a field. Then we have a quasi isomorphism:*

$$C_*(S^*X) \xrightarrow{AW} \operatorname{Tot}_\otimes(S^*(X^{S^1})^\bullet) \xrightarrow{\psi} S^*(\mathcal{L}X)$$

In other words, $HH_*(S^*X) \cong H^*(\mathcal{L}X)$.

Proof. As the Alexander-Whitney map is always a quasi-isomorphism, it remains to show that ψ is. This can be done using spectral sequences. See for example 'A generalization of the Eilenberg-Moore spectral sequence' by D.W. Anderson. □

Application

The Jones isomorphism can help us calculate the cohomology these very big spaces by reducing the problem to a more algebraic one. For example, take the following corollary a variation of which appeared earlier in Kaj Börjeson's talk.

Corollary. *Let X be a connected, simply connected* formal manifold. *Assume furthermore that the ground-field is $\mathbb{R}$ or $\mathbb{C}$. Then*

$$HH_*(H^*X) \cong H^*(\mathcal{L}X)$$

Proof. The definition of a manifold being formal is that $\Omega^*_{dR}(X) \simeq H^*(X)$ as cdga's. We also have the dga version of de Rham's isomorphism $S^*(X) \simeq \Omega^*_{dR}(X)$ as dga's. Hochschild homology sends quasi isomorphic dga's to isomorphic graded vector spaces. □

Exercise. Play this game for spheres S^n, $n \geq 2$. This example was used to detect non-trivial string operations in [Wah12]. The cohomology of this free loop space can also be obtained using a Serre spectral sequence.

Extensions

Note that the circle group acts on free loop spaces by rotating loops $U(1) = S^1 \curvearrowright \mathcal{L}X$. The thing that Jones proved is actually the following statement:

$$HC^-_*(S^*X) \cong H^*_{S^1}(\mathcal{L}X) := H^*(\mathcal{L}X_{hS^1})$$

The left hand side is called *negative cyclic homology.* It is also built using the cyclic bar complex but takes into account the analogy between the Connes B operator on Hochschild chains and the circle action in cohomology.

In fact, we have $O(2)$ acting by rotating and flipping loops so we could ask for more. My current project is to formulate the right analogue of negative cyclic homology in order to establish an isomorphism $HD^-_*(S^*X) \cong H^*_{O(2)}(\mathcal{L}X)$. Here the D stands for *dihedral* in analogy to how cyclic groups appear for S^1.

References

[BF86] Dan Burghelea and Z Fiedorowicz. Cyclic homology and algebraic k-theory of spaces—ii. *Topology*, 25(3):303–317, 1986.

[Goo85] Thomas G Goodwillie. Cyclic homology, derivations, and the free loopspace. *Topology*, 24(2):187–215, 1985.

[Jon87] John DS Jones. Cyclic homology and equivariant homology. *Invent. Math.*, 87(2):403–423, 1987.

[Wah12] Nathalie Wahl. Universal operations in Hochschild homology. *arXiv preprint arXiv:1212.6498*, 2012.

[Wei95] Charles A Weibel. *An introduction to homological algebra.* Number 38. Cambridge university press, 1995.