Cocycles in categories of fibrant objects

(arXiv:1502.03925)

Zhen Lin Low

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

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- Karol Szumiło. 'Two models for the homotopy theory of cocomplete homotopy theories'. PhD thesis. University of Bonn, 2014
- Chris Kapulkin and Karol Szumiło. *Quasicategories of frames of cofibration categories*. 29th June 2015. arXiv: 1506.08681

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- The category of small categories of fibrant objects is itself a category of fibrant objects. This is a result of Szumiło.

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- The degeneracy operators insert a row of identity arrows.

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  This terminology is due to Jardine.

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Can we do better?

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In that situation, an old result of Dwyer and Kan says that the mapping spaces are homotopy equivalent to the nerves of the categories of cocycles.

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- Moreover, the data (p, v) and j are homotopically unique, i.e. the space of such choices is contractible.

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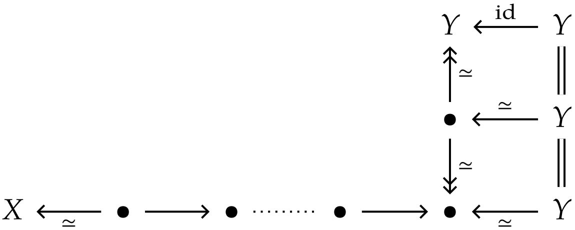

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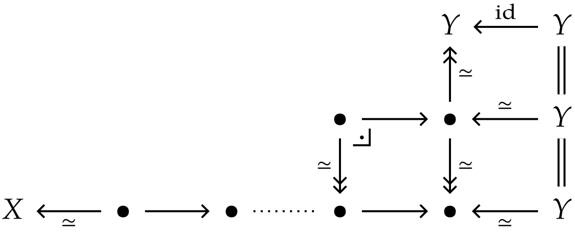

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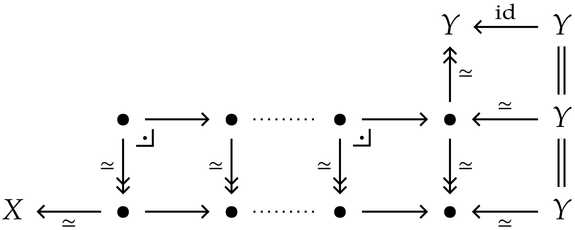

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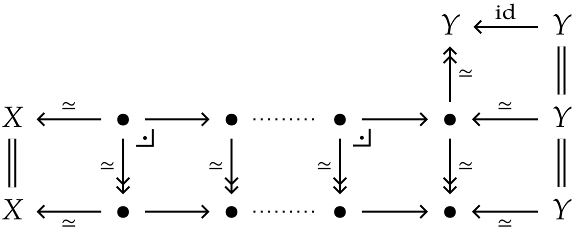

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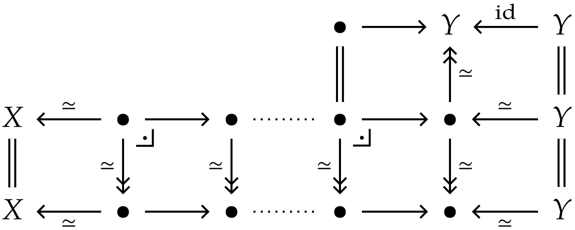

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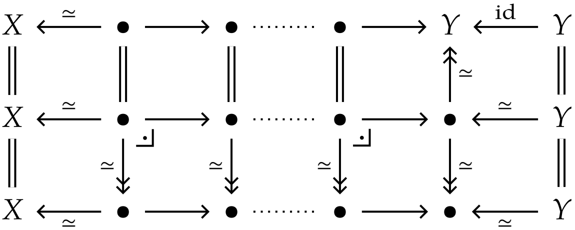

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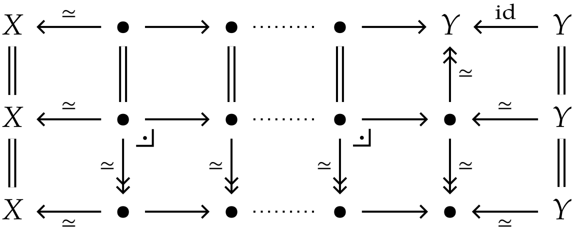

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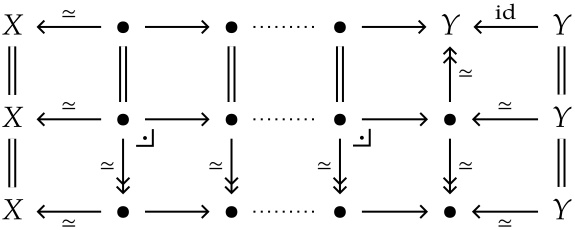

The first step is well-defined up to a contractible space of choices. All the other steps are functorial.

Calculus of fractions

$$
\begin{array}{ccccccccccc}
X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots\cdots & \bullet & \longrightarrow & Y & \xleftarrow{\mathrm{id}} & Y \\
\| & & \| & & \| & & \| & & \big\uparrow{\simeq} & & \| \\
X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots\cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y \\
\| & & {\simeq}\big\downarrow & \lrcorner & {\simeq}\big\downarrow & & {\simeq}\big\downarrow & \lrcorner & \big\downarrow{\simeq} & & \| \\
X & \xleftarrow[\simeq]{} & \bullet & \longrightarrow & \bullet & \cdots\cdots & \bullet & \longrightarrow & \bullet & \xleftarrow[\simeq]{} & Y
\end{array}
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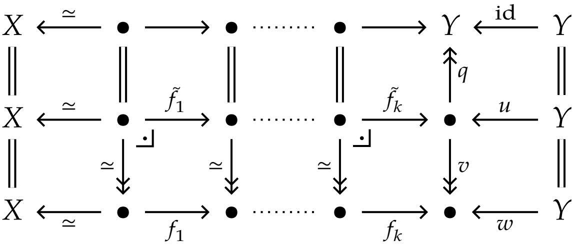
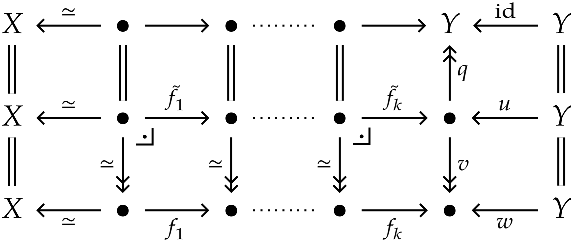


Calculus of fractions

Assume $w = \mathrm{id}_Y$.



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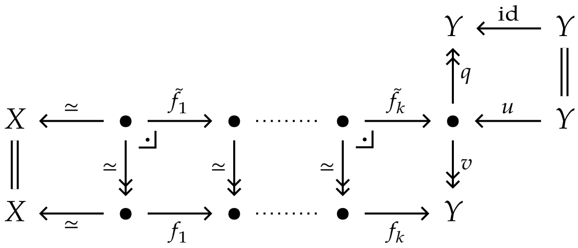
Assume $v \circ u = \mathrm{id}_Y$.



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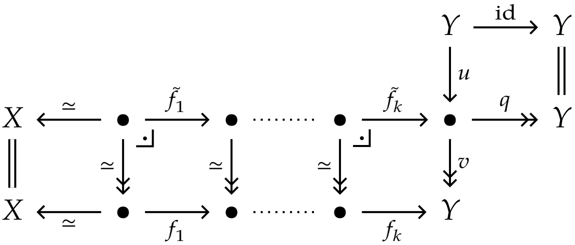

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 & & & & & & & & \big\downarrow u & & \big\| \\
X & \xleftarrow{\simeq} & \bullet & \xrightarrow{\tilde{f}_1} & \bullet & \cdots & \bullet & \xrightarrow{\tilde{f}_k} & \bullet & \xrightarrow{q} \!\!\!\!\rightarrow & Y \\
\big\| & & \simeq \big\downarrow\!\!\downarrow & & \simeq \big\downarrow\!\!\downarrow & & \simeq \big\downarrow\!\!\downarrow & & \big\downarrow\!\!\downarrow v & & \\
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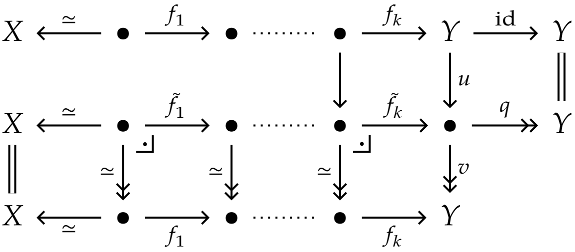

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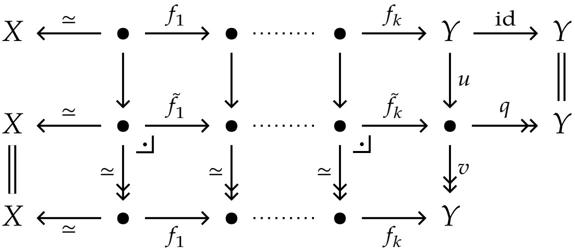

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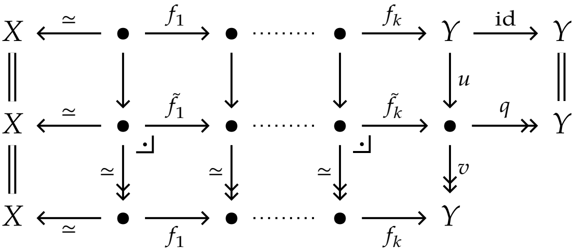

All of the above steps are functorial.
These two procedures lie at the heart of the proof that categories of fibrant objects admit a homotopical calculus of right fractions.

Revisiting cocycles

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Question. What is the connection between cocycles in the sense above and cocycles in cohomology?
Answer. The Verdier hypercovering theorem.

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In fact, we can replace $\mathcal{W}_{/X}$ with the full subcategory $\mathcal{Q}_X$ spanned by the trivial fibrations $\tilde{X} \to X$.

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Thus, hypercovers are generalisations of open covers.

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This is basically the Verdier hypercovering theorem.