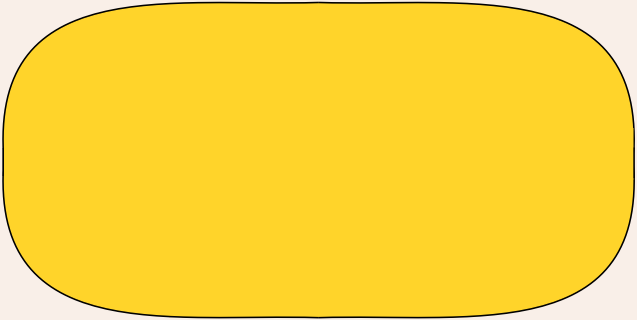
Parametrized Homology & Parametrized Alexander Duality Theorem

Sara Kališnik Verovšek

Young Topologists Meeting, EPF Lausanne

Motivation: Sensor Networks

The setting



$D \subset \mathbb{R}^d$ domain homeomorphic to a d-ball, ∂D domain boundary.

Motivation: Sensor Networks

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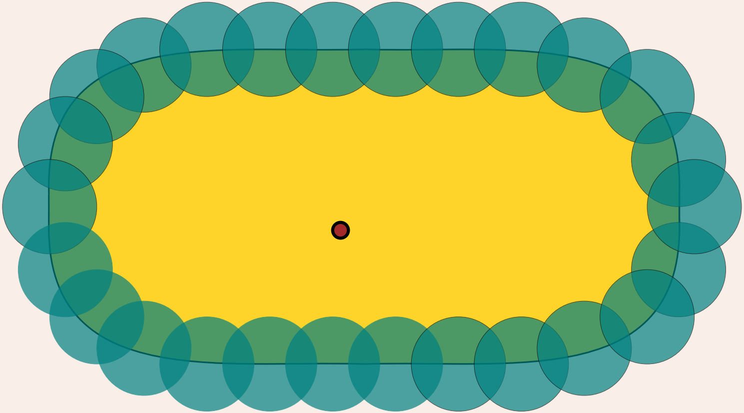
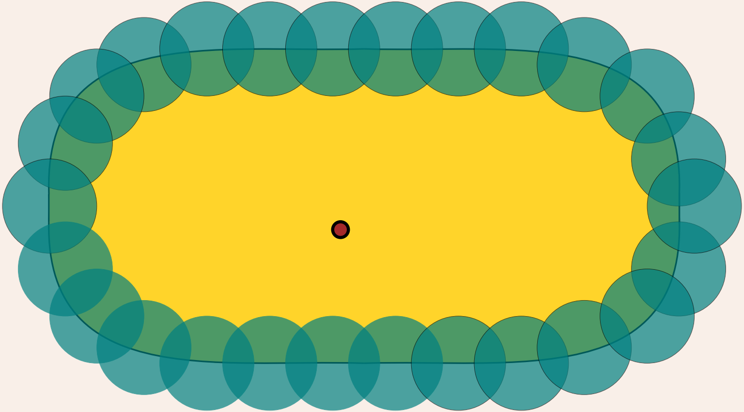


Let K be the region covered by sensors, U the uncovered region. We suppose throughout that the boundary is covered.

Motivation: Sensor Networks

Question: Are there any gaps in the sensor network?

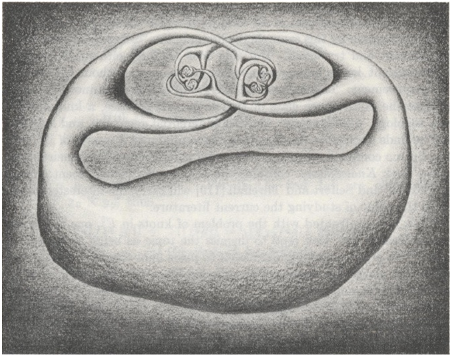


Motivation: Sensor Networks

Alexander Duality

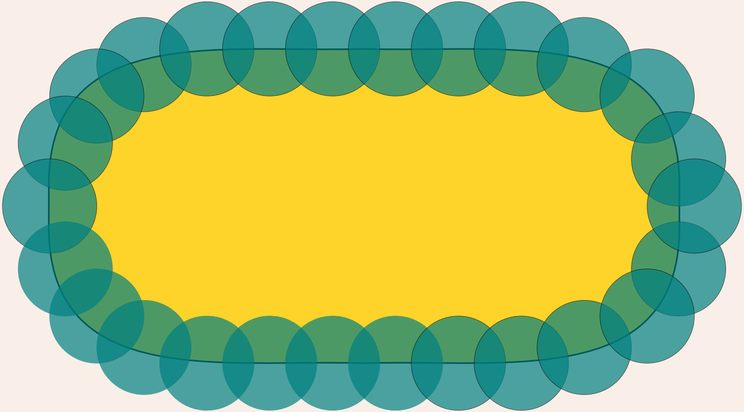
Let X be a compact, locally contractible subspace of $\mathbb{R}^d$. Then

$$\widetilde{\mathrm{H}}_{d-j-1}(\mathbb{R}^d - X) \cong \mathrm{H}^j(X).$$



Motivation: Sensor Networks

Example



$$\dim \mathrm{H}^1(K) = \dim \mathrm{H}_0(U) = 1$$

Motivation: Sensor Networks

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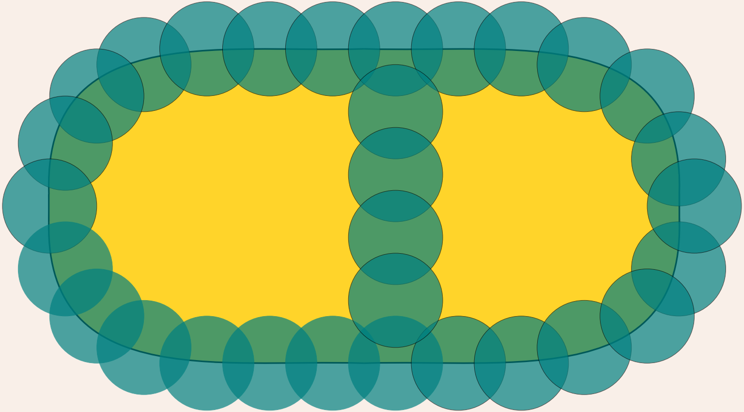

$$\dim \mathrm{H}^1(K) = \dim \mathrm{H}_0(U) = 2$$

Motivation: Sensor Networks

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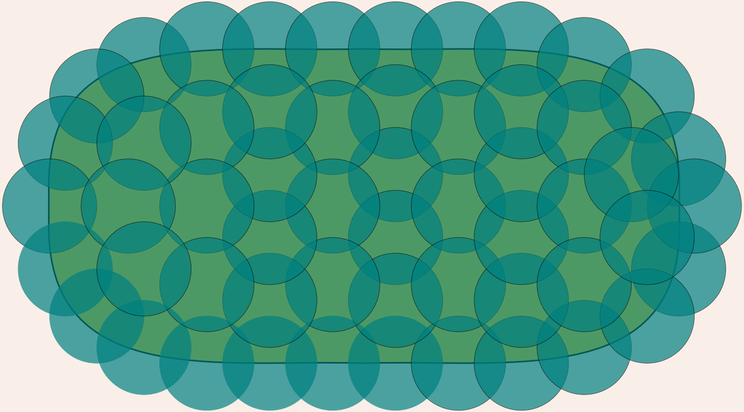
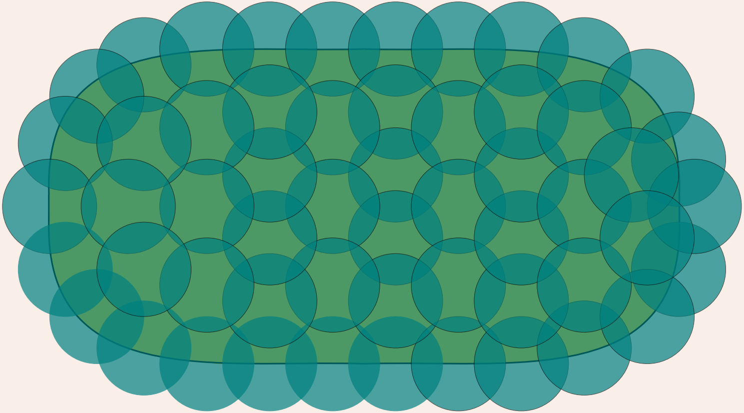


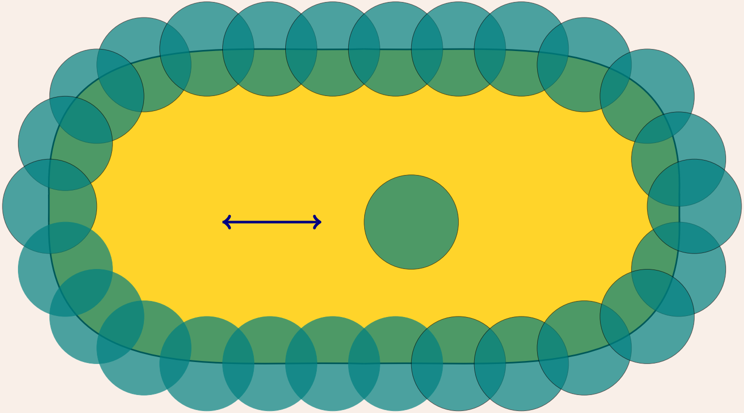
$$\dim \mathrm{H}^1(K) = \dim \mathrm{H}_0(U) = 0$$

Motivation: Sensor Networks

Test for STATIC coverage: dim $\mathrm{H}^{d-1}(K) = 0$.



Time-varying Sensor Networks



Motivation: Sensor Networks

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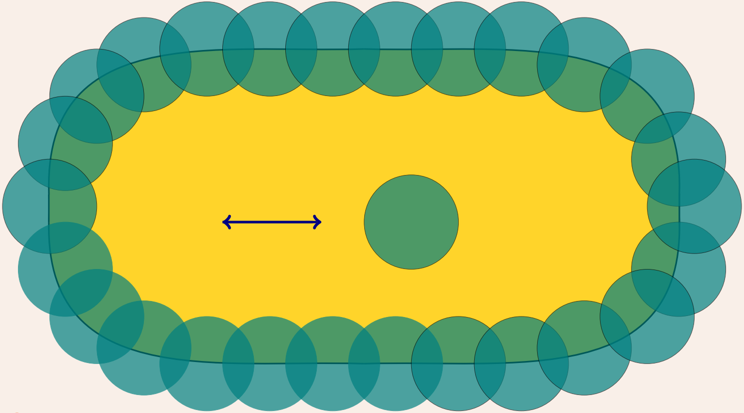
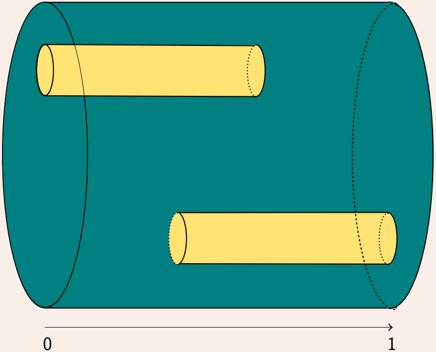


Question: Can an intruder escape detection by moving continuously within the domain of a sensor network?

Time-varying Sensor Networks



D	$=$	domain	K	$=$	covered region $\subset D \times I$
∂D	$=$	domain boundary	K_t	$=$	$K \cap (D \times \{t\})$
I	$=$	time interval	U	$=$	uncovered region $\subset D \times I$
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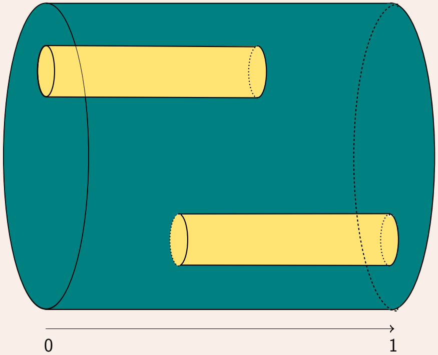

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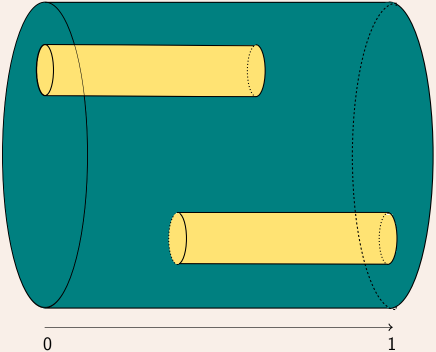

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Evasion Problem: Can we determine the existence of an evasion path? ($p \colon I \to U$ such that $\mathrm{proj} \circ p = \mathrm{Id}_I$)

Motivation: Sensor Networks

Test for coverage in a dynamic sensor network

Vin De Silva and Robert Ghrist give a partial answer to the evasion problem:

If $c \in \mathrm{H}_d(K, \partial D \times I)$ exists such that $0 \neq \partial c \in \mathrm{H}_{d-1}(\partial D \times I)$, then no evasion path exists.

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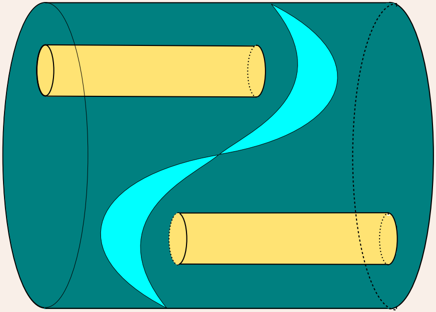

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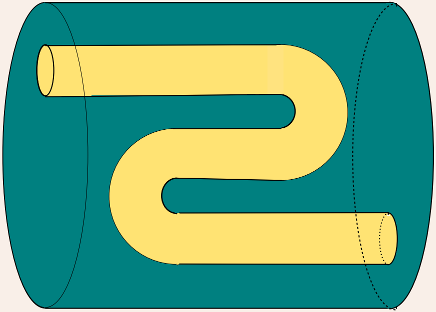

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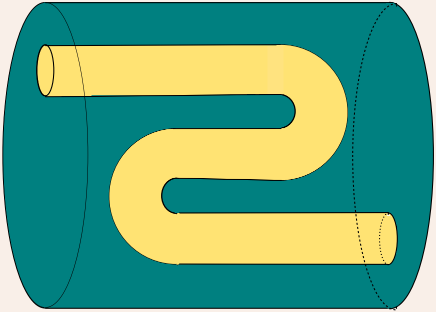

Henry Adams et. al began studying the evasion problem with the goal of finding an if-and-only-if criterion for the existence of an evasion path using zigzag persistence.

Parametrized Spaces

Parametrized Space

A parametrized space is a pair $\mathbb{X} = (X, p)$ where X is a topological space and $p \colon X \to \mathbb{R}$ is a continuous function. We can view

$$(\mathbb{X}_a^a = p^{-1}(a) \mid a \in \mathbb{R})$$

as a 1-parameter family of topological spaces (the topology on the total space X gives it the structure of a 'family').

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We call

- preimages of points, denoted by $\mathbb{X}_a^a$, levelsets of $\mathbb{X}$;
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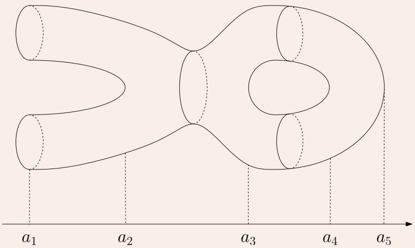
'Levelset zigzag persistence'
appeared in Zigzag Persistent Homology and Real-valued Functions (G. Carlsson, V. de Silva, D. Morozov).

(X, p) is of Morse-type if..

- the homology of levelsets changes only at finitely many critical values a_i, $i = 1, \ldots, n$;
- homology groups of slices and levelsets are finite dimensional
- Let $I \in \{(-\infty, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n), (a_n, \infty)\}$. Then $p^{-1}(I)$ is homeomorphic to products of the form $Y \times I$, and each homeomorphism $Y \times I \to p^{-1}(I)$ extends to a continuous function $Y \times \bar{I} \to p^{-1}(\bar{I})$, where $\bar{I}$ is the closure of $I \subset \mathbb{R}$.



Levelset Zigzag Persistence

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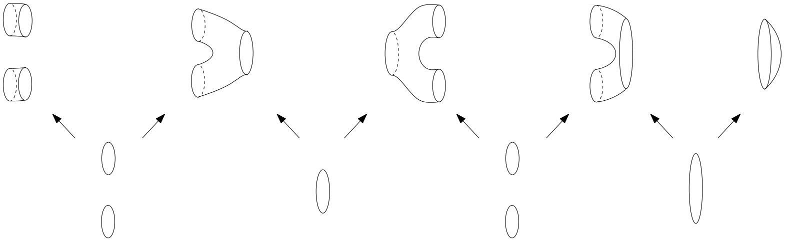

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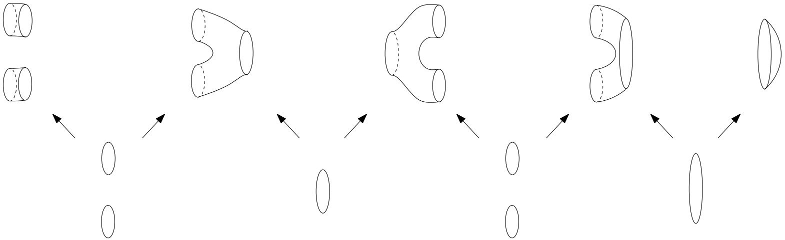

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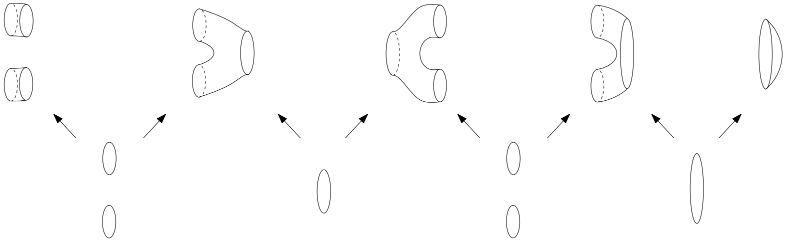
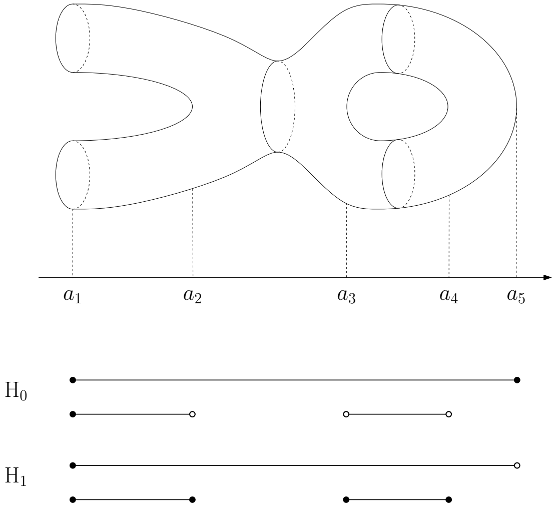
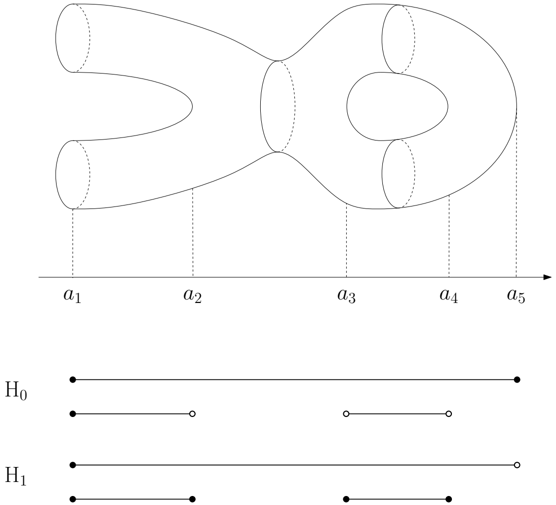


Apply homology functor. The resulting quiver representation is decomposable by Gabriel's theorem.
The collection of intervals over all dimensions is called Levelset Zigzag Persistence.

Levelset Zigzag Persistence



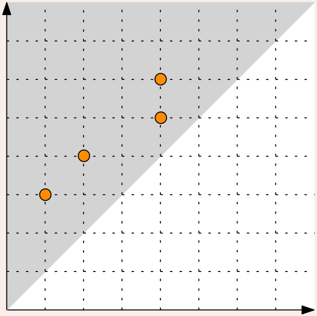


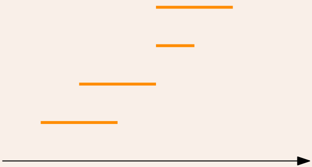
Can we define ‘parametrized homology’ for a larger class of parametrized spaces?

Persistence

Persistence is commonly described in two formats:

- The barcode is a collection of intervals;
- The (undecorated) persistence diagram is a multiset of points lying above the main diagonal in $\mathbb{R}^2$.





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Decorated Diagrams

Frederic Chazal et al. introduced decorated real numbers. We translate between decorated points and intervals as follows:

(p^-, q^-)	is written	$[p, q)$	and drawn	
(p^-, q^+)	is written	$[p, q]$	and drawn	
(p^+, q^-)	is written	(p, q)	and drawn	
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A multiset of decorated points lying above the diagonal is called a decorated persistence diagram.

Frederic Chazal et al. introduced a third format for expressing persistence, which works particularly well with a continuous parameter.

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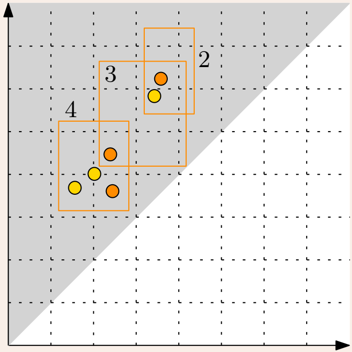
r-measures

The intuition is that if we know how many points of the diagram are contained in each rectangle in the half plane, then we know the diagram itself.



r-measures

Let $\mathcal{H} = \{(p,q) \in \mathbb{R}^2 \mid p < q\}$. The set of rectangles in $\mathcal{H}$ is

$$\mathrm{Rect}(\mathcal{H}) = \{[a,b] \times [c,d] \subset \mathcal{H} \,|\, a < b < c < d\}.$$

A rectangle measure or r-measure on $\mathcal{H}$ is a function

$$\mu\colon\ \mathrm{Rect}(\mathcal{H}) \to \{0,1,2,3,\ldots\} \cup \{\infty\}$$

which is additive, meaning that $\mu(R) = \mu(R_1) + \mu(R_2)$ whenever

$$\boxed{R} = \boxed{R_1}\boxed{R_2} \quad \text{or} \quad \boxed{R} = \begin{array}{|c|}\hline R_1 \\ \hline R_2 \\ \hline \end{array}$$

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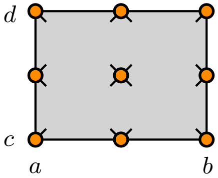
To describe the correspondence between r-measures and decorated persistence diagrams, we need to relate decorated points to rectangles.

Let $R = [a,b] \times [c,d] \in \mathrm{Rect}(\mathcal{H})$ and let (p^*, q^*) be a decorated point. Then

$$\begin{aligned} (p^*, q^*) \in R \quad &\Leftrightarrow \quad [b,c] \subset (p^*, q^*) \subset (a,d) \\ &\Leftrightarrow \quad (p,q) \text{ and its decoration tick are contained in } R. \end{aligned}$$



The Equivalence Theorem

There is a bijective correspondence between:

- Finite r-measures μ on $\mathcal{H}$. Here 'finite' means that $\mu(R) < \infty$ for every $R \in \mathrm{Rect}(\mathcal{H})$;
- Locally finite multisets A in $\mathcal{H}$. Here 'locally finite' means that $\mathrm{card}(A|_R) < \infty$ for every $R \in \mathrm{Rect}(\mathcal{H})$.

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We now have three ways to describe a persistence diagram:

- barcode of finite real intervals;
- diagram of decorated points;
- rectangle measure.

Measure theory approach is more flexible and works under a larger range of circumstances.

We combine this approach with levelset zigzag persistence to define parametrized homology.

(joint work with Carlsson, de Silva, Morozov)

Four Measures

Given a rectangle $R = [a, b] \times [c, d] \in \mathrm{Rect}(\mathcal{H})$, the aim is to count the homological features of $\mathbb{X}$ that persist over the closed interval $[b, c]$, but not over the open interval (a, d).

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Consider the following diagram of spaces and inclusion maps.

$$\begin{array}{ccccccc} & \mathbb{X}_a^b & & \mathbb{X}_b^c & & \mathbb{X}_c^d & \\ \nearrow & & \nwarrow \; \nearrow & & \nwarrow \; \nearrow & & \nwarrow \\ \mathbb{X}_a & & \mathbb{X}_b & & \mathbb{X}_c & & \mathbb{X}_d \end{array}$$

Apply the j-dimensional homology functor H to obtain:

$$\begin{array}{ccccccc} & \mathrm{H}(\mathbb{X}_a^b) & & \mathrm{H}(\mathbb{X}_b^c) & & \mathrm{H}(\mathbb{X}_c^d) & \\ \nearrow & & \nwarrow \; \nearrow & & \nwarrow \; \nearrow & & \nwarrow \\ \mathrm{H}(\mathbb{X}_a) & & \mathrm{H}(\mathbb{X}_b) & & \mathrm{H}(\mathbb{X}_c) & & \mathrm{H}(\mathbb{X}_d). \end{array}$$

We denote this quiver by $\mathrm{H}\mathbb{X}_{\{a,b,c,d\}}$. It is decomposable by Gabriel's theorem.

Four measures

There are four types of indecomposable summands, which meet b and c, but not a and d. By counting each of these summands, we get four quantities:

$\mu_{\mathbb{X}}^{\backslash\backslash}(R) = \langle$  $\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}}\rangle$

$\mu_{\mathbb{X}}^{\vee}(R) = \langle$  $\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}}\rangle$

$\mu_{\mathbb{X}}^{\wedge}(R) = \langle$  $\mid \mathrm{H}\mathbb{X}_{\{a,b,c,d\}}\rangle$

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If these four quantities are finite r-measures, by the equivalence theorem each determines a decorated persistence diagram. Let $\mathrm{Dgm}^*(\mathrm{H}\mathbb{X})$ be the diagram determined by $\mu_{\mathbb{X}}^*$.

We say that $\mathbb{X} = (X, f)$ has a well-defined parametrized (co)homology when the four quantities defined above are finite r-measures. This happens for $\mathbb{X}$, where:

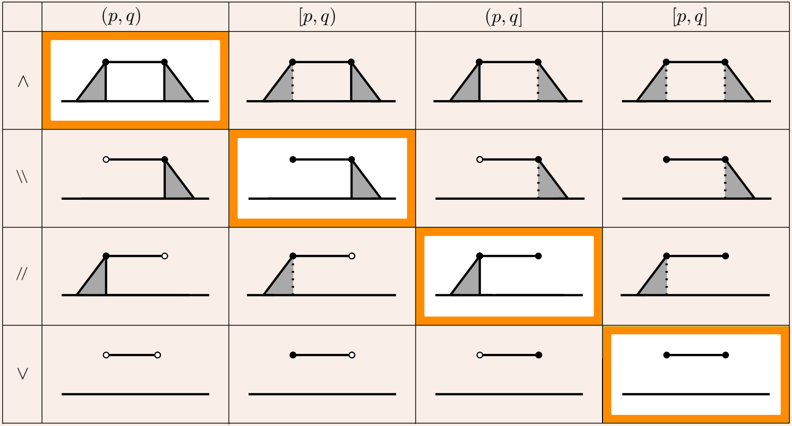
(i) X is a locally compact polyhedron, f is proper, and H is Steenrod–Sitnikov homology.

(ii) X is a smooth manifold and f is a proper Morse function.

(iii) X is a locally compact polyhedron and f is a proper piecewise-linear map.

Four measures

These four diagrams demonstrate how homological features perish - whether cycles are killed in homology by higher dimensional chains or whether they cease to exist:



Parametrized Homology & Parametrized Alexander Duality Theorem

Parametrized homology

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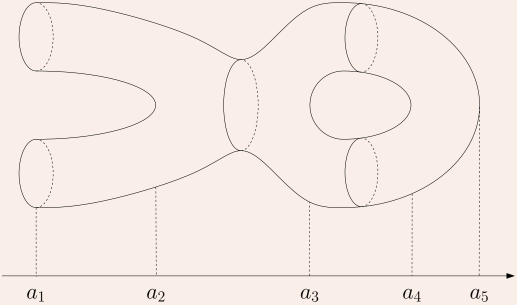
Other (co)homology theories

(co)homology theory	Diagrams	parametrized version
reduced singular homology	$\mathrm{Dgm}^{\backslash\backslash}(\widetilde{\mathrm{H}}\mathbb{X})$, $\mathrm{Dgm}^{\vee}(\widetilde{\mathrm{H}}\mathbb{X})$, $\mathrm{Dgm}^{\wedge}(\widetilde{\mathrm{H}}\mathbb{X})$, $\mathrm{Dgm}^{//}(\widetilde{\mathrm{H}}\mathbb{X})$	parametrized reduced homology
singular cohomology	$\mathrm{Dgm}^{\backslash\backslash}(\mathrm{H}\mathbb{X})$, $\mathrm{Dgm}^{\vee}(\mathrm{H}\mathbb{X})$, $\mathrm{Dgm}^{\wedge}(\mathrm{H}\mathbb{X})$, $\mathrm{Dgm}^{//}(\mathrm{H}\mathbb{X})$	parametrized cohomology
Čech cohomology	$\mathrm{Dgm}^{\backslash\backslash}(\check{\mathrm{H}}\mathbb{X})$, $\mathrm{Dgm}^{\vee}(\check{\mathrm{H}}\mathbb{X})$, $\mathrm{Dgm}^{\wedge}(\check{\mathrm{H}}\mathbb{X})$, $\mathrm{Dgm}^{//}(\check{\mathrm{H}}\mathbb{X})$	parametrized Čech cohomology

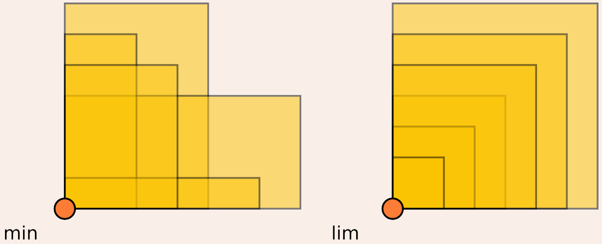
Example



Multiplicities

To formally determine the diagram, we must compute the multiplicities. Given a measure μ, the multiplicity of (p^*, q^*) is

$$\mathrm{m}_\mu(p^*, q^*) = \min\{\mu(R) \mid (p^*, q^*) \in R, R \in \mathrm{Rect}(\mathcal{H})\}$$



Alternatively. Let $R_1 \supset R_2 \supset R_3 \supset \dots$ be a sequence of closed rectangles which contain (p^*, q^*) and $\bigcap_n R_n = (p, q)$. Then

$$\mathrm{m}_\mu(p^*, q^*) = \lim_{n\to\infty} \mu(R_n).$$

Example

Let us calculate the multiplicities of (a_1^-, a_2^+). Pick ϵ small enough that $a_1 - \epsilon < a_1 < a_2 < a_2 + \epsilon$. We have

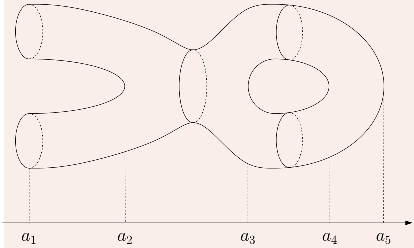
$$\mathrm{H}_1\mathbb{X}_{a_1-\epsilon, a_1, a_2, a_2+\epsilon} \cong$$

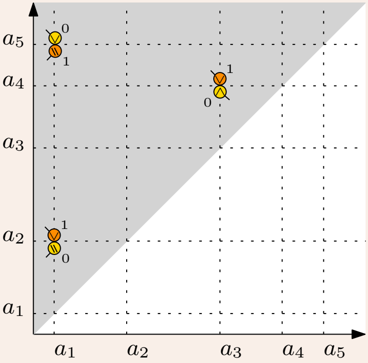
$$\oplus$$

The summand on the right is not registered by any of the measures, whereas the one on the left is detected by ${}_1\mu_{\mathbb{X}}^{\vee}$. Since these values are the same for all ϵ, we have

$$\begin{aligned}
m_{{}_1\mu_{\mathbb{X}}^{\backslash\backslash}}(a_1^-, a_2^+) &= \lim_{\epsilon\to 0} {}_1\mu_{\mathbb{X}}^{\backslash\backslash}([a_1-\epsilon, a_1]\times[a_2, a_2+\epsilon]) &= 0\\
m_{{}_1\mu_{\mathbb{X}}^{\vee}}(a_1^-, a_2^+) &= \lim_{\epsilon\to 0} {}_1\mu_{\mathbb{X}}^{\vee}([a_1-\epsilon, a_1]\times[a_2, a_2+\epsilon]) &= 1\\
m_{{}_1\mu_{\mathbb{X}}^{\wedge}}(a_1^-, a_2^+) &= \lim_{\epsilon\to 0} {}_1\mu_{\mathbb{X}}^{\wedge}([a_1-\epsilon, a_1]\times[a_2, a_2+\epsilon]) &= 0\\
m_{{}_1\mu_{\mathbb{X}}^{//}}(a_1^-, a_2^+) &= \lim_{\epsilon\to 0} {}_1\mu_{\mathbb{X}}^{//}([a_1-\epsilon, a_1]\times[a_2, a_2+\epsilon]) &= 0
\end{aligned}$$

Example





Alexander Duality for Parametrized Homology

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ with $n \geq 2$, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. We assume that the levelsets $\mathbb{X}_a$ for $a \in \mathbb{R}$, and slices $\mathbb{X}_a^b$ for $a < b$ are compact and locally contractible. If $\mathbb{X} = (X, p|_X)$ has a well-defined parametrized cohomology, then the pair $\mathbb{Y} = (Y, p|_Y)$ has a well-defined reduced parametrized homology. Additionally, for all $j = 0, \ldots, n-1$:

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\begin{aligned}
\mathrm{Dgm}^{\backslash\backslash}(\widetilde{\mathrm{H}}_{n-j-1}\mathbb{Y}) &= \mathrm{Dgm}^{//}(\mathrm{H}^j\mathbb{X}),\\
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\end{aligned}$$

Killing / ceasing to exist duality

From the proof we can deduce the following duality: if a j-dimensional cohomology cycle in $\mathbb{X}$ is killed (ceases to exist) at endpoint p, then there is a corresponding $(n-j-1)$-dimensional homology cycle in $\mathbb{Y}$, which ceases to exist (is killed) beyond that same endpoint.

Situations for which the theorem applies

The conditions of the theorem are satisfied for $(X, p|_X)$, where:

- X is a compact manifold and $p|_X$ is Morse;
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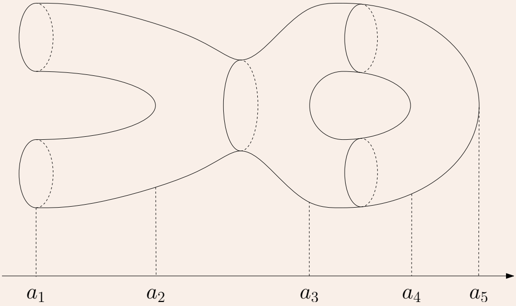
Revisiting the Example



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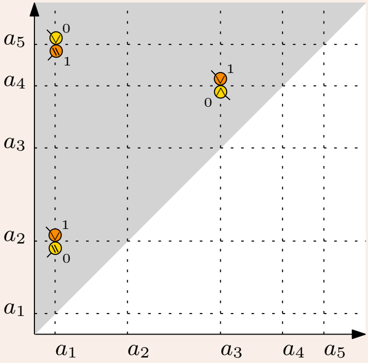

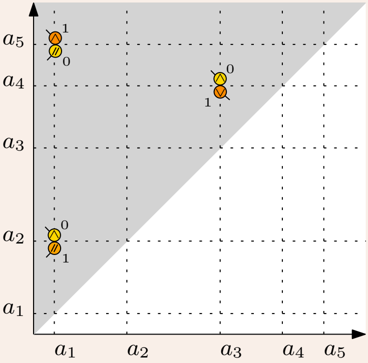

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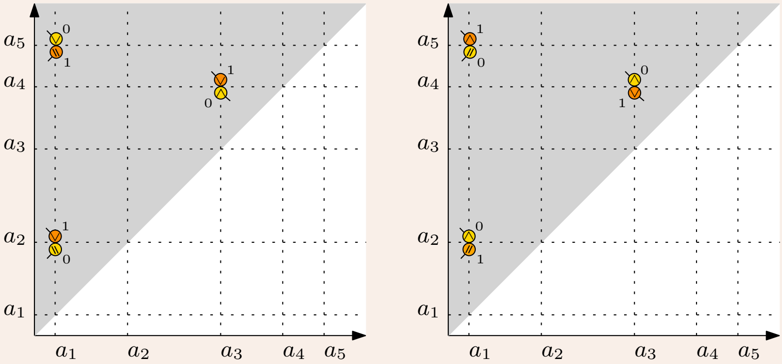

The parametrized cohomology of $\mathbb{X}$ is on the left and the one of $\mathbb{Y}$ on the right.

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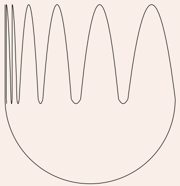

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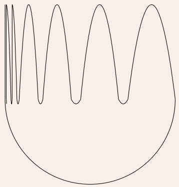

We can extend Alexander duality to the setting of parametrized Čech cohomology.

Alexander Duality for Parametrized Homology (Čech cohomology version)

Let $X \subset \mathbb{R}^n \times \mathbb{R}$ with $n \geq 2$, let $Y = (\mathbb{R}^n \times \mathbb{R}) \setminus X$, and let p be the projection onto the second factor. We assume that the levelsets $\mathbb{X}_a$ for $a \in \mathbb{R}$, and slices $\mathbb{X}_a^b$ for $a < b$ are compact. If $\mathbb{X} = (X, p|_X)$ has a well-defined parametrized cohomology, then the pair $\mathbb{Y} = (Y, p|_Y)$ has a well-defined reduced parametrized homology. Additionally, for all $j = 0, \ldots, n-1$:

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\end{aligned}$$

This version of the theorem holds for parametrized spaces $\mathbb{X} = (X, p|_X)$ where X is a compact polyhedron and $p|_X$ is a continuous map.

Test for coverage in a dynamic sensor network (Henry Adams)

'Claim': An evasion path exists in a sensor network if and only if there is a full-length interval $[1, 2m-1]$ in the $(d-1)$-dimensional barcode for K.

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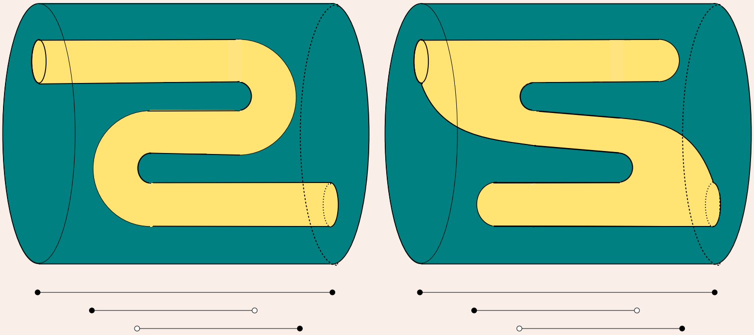

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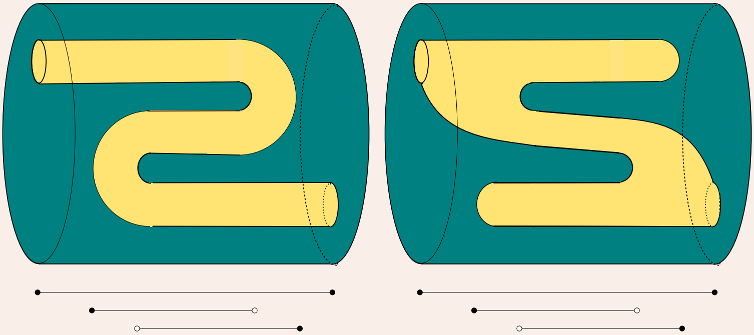

Claim: If an evasion path exists in a sensor network, then a full-length interval $[1, 2m-1]$ appears in the $(d-1)$-dimensional barcode for K.


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Dank U
Merci
mahalo
Köszi
спасибо
Grazie
Thank you
mauruuru
Takk
Gracias
Dziękuję
Děkuju
danke
Kiitos