Skein modules

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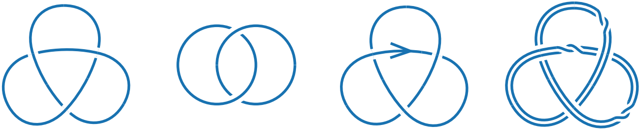
Basic definitions

- A knot is $S^1 \hookrightarrow \mathbb{R}^3$.
- A link is $S^1 \sqcup S^1 \cdots \sqcup S^1 \hookrightarrow \mathbb{R}^3$.
- Two knots K_1 and K_2 are equivalent if they are isotopic.
- A knot/link diagram is a regular projection of the knot/link to a plane $\Sigma \subset \mathbb{R}^3$ with information about under- and overcrossings.
- A knot is oriented if it is given an orientation.
- A framed knot is $S^1 \times I \hookrightarrow \mathbb{R}^3$.
- Let $\mathcal{L}$ be the set of all links in a 3-manifold M. A knot invariant is a map $\iota : \mathcal{L} \to U$ s.t. $K_1 \sim K_2 \Rightarrow \iota(K_1) = \iota(K_2)$.



The Kauffman bracket

The Kauffman bracket (the bracket polynomial) is a map

$$\langle \cdot \rangle : \mathcal{L}^{\mathrm{fr}} \longrightarrow \mathbb{Z}[A^{\pm 1}],$$

such that

- $\langle$  $\rangle = 1,$ (normalization)
- $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle,$ (skein relation)
- $\langle L \sqcup$  $\rangle = (-A^2 - A^{-2})\langle L \rangle.$ (framing relation)

Theorem [Kauffman '87]

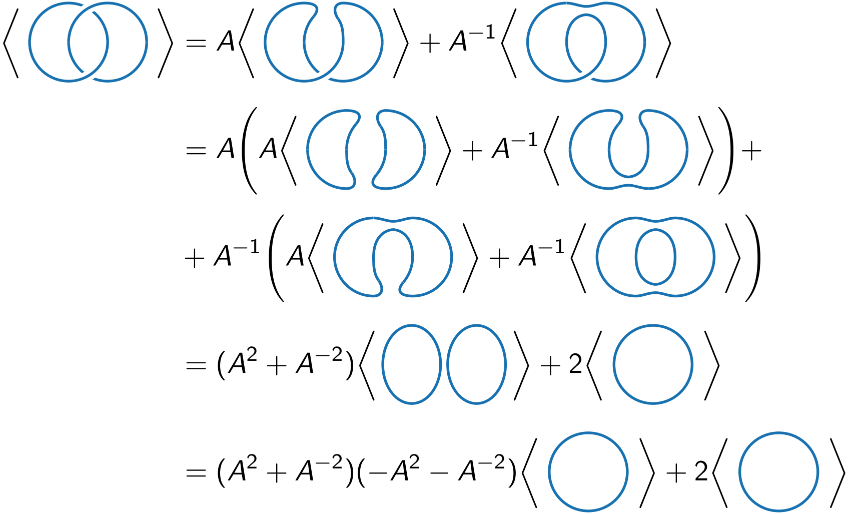
$\langle \cdot \rangle$ is in invariant of framed links.

Remark. The Kauffman bracket is closely related to the Jones polynomial

$$(-A^3)^{w(L)}\langle L \rangle \xlongequal{A \to t^{1/4}} V(L)$$

.

Example



$$= -A^4 - A^{-4}$$

The HOMFLYPT polynomial is a map

$$P : \mathcal{L}^{\mathrm{or}} \longrightarrow \mathbb{Z}[l^{\pm 1}, z^{\pm 1}],$$

such that

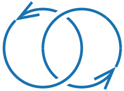
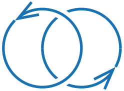
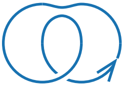
- $P(\bigcirc) = 1,$ (normalization)
- $l\,P($$) - l^{-1}P($$) = z\,P($$),$ (skein relation)

Theorem [HOMFLYPT '85]

P is in invariant of oriented links.

Remark. The polynomials $\langle \cdot \rangle$, V, ∇, and Δ can be computed from the HOMFLYPT polynomial.

Example

$P\Big($  $\Big) = l^2 P\Big($  $\Big) - lz\, P\Big($  $\Big)$

$= l^2(\frac{l - l^{-1}}{z})\, P\Big($  $\Big) - lz\, P\Big($  $\Big)$

$= l^3 z^{-1} - lz^{-1} - lz$

What about knots in other 3-manifolds?

- Classical knot invariants are often not very strong (if they are defined).
- Popularization of studying knot invariants for 3-manifolds:
  - Knot floer homology
  - Generalizations of Khovanov homology
  - Mixed links (for knots in 3-manifolds with surgery descriptions)
  - Mixed braids
    - generalized Temperley–Lieb algebras
    - generalized rook algebras
    - generalized Hecke algebras
  - Hyperbolic structures of knot complements
  - Generalization of knot polynomials
- Skein modules generalize knot invariants defined through skein relations.

Definition (skein module)

Let:

- M be an orientable 3-manifold,
- R be a ring with 1,
- $\mathcal{L}$ be the set of isotopy classes of oriented links in M,
- $R\cdot\mathcal{L}$ be the free R-module generated by $\mathcal{L}$,

The n^{th} skein module of M is

$\mathcal{S}_n(M)(r_0, r_1, \dots, r_{n-1}) = R\cdot\mathcal{L} \Big/ r_0\cdot\Big[$  $\Big] + r_1\cdot\Big[$  $\Big] + r_2\cdot\Big[$  $\Big] + \cdots + r_{n-1}\cdot\Big[$  $\Big]$,

where $r_0, r_1, r_2, \dots, r_{n-1} \in R$.

Remark. $[L] \in \mathcal{S}_n(M)$ is a knot invariant.

Simple examples

- $\mathcal{S}_n(M)(0,\dots,0) = R\cdot\mathcal{L}$.
- The signed skein module

$$\mathcal{S}_\pm(M) := R\cdot\mathcal{L}\Big/$$ [] $-$ [],

is a free module generated by homotopy classes of closed curves in M. $\mathcal{L}\cup\{\emptyset\}$ admits a natural multiplication $[L_1]\cdot[L_2] = [L_1 \sqcup L_2]$, thus $\mathcal{S}_\pm(M)$ extends to an R-algebra $\mathcal{S}_\pm^{alg}(M)$ isomorphic $\mathbf{S}R\hat{\pi}_1(M)$, the symmetric tensor algebra over the conjugacy classes of $\pi_1(M)$.

In fact, it can be shown that:

$$\mathcal{S}_\pm(M) \cong \mathcal{S}_n(M)(1,0,-1) = R\cdot\mathcal{L}\Big/$$ [] $-$ []

- $\mathcal{S}_\pm(S^3) \cong R$

The Kauffman bracket skein module

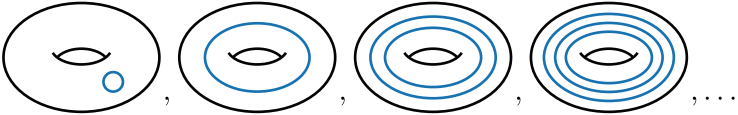
Let $R = \mathbb{Z}[A^{\pm 1}]$ and let $\mathcal{L}^{\mathrm{fr}}$ be the set of isotopy classes of framed links in M.
The Kauffman bracket skein module of M is

$$KBSM(M) := \mathcal{S}_{2,\infty}(M) = R \cdot \mathcal{L}^{\mathrm{fr}} \Big/ \begin{matrix} [\times] - A[\asymp] - A^{-1}[)(] \\ [L \sqcup \bigcirc] - (-A^2 - A^{-2})[\bigcirc] \end{matrix} .$$

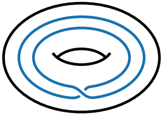
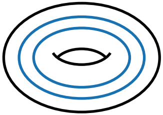
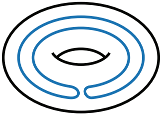
Remark. If for a given manifold M the basis is known and the possible torsion is understood, then $[L] \in KBSM(M)$ can be used as a computable knot invariant.
If $KBSM(M)$ has torsion, the expression of $[L] \in KBSM(M)$ is not unique.

KBSM (known results)

- *KBSM*(S^3) is freely generated by $\{$  $\}$ (trivial)
- *KBSM*(T) is freely generated by

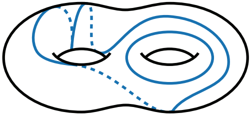


(Turaev '88). Example:

 $= A$  $+ A^{-1}$ 

KBSM (conjectures)

- *KBSM*($F \times I$), where F is a surface, is freely generated by simple closed curves on F (Przytycki '92).



- *KBSM*($S^2 \times S^1$) is infinitely generated and has torsion.

$$KBSM(S^2 \times S^1) \cong \mathbb{Z}[A^{\pm 1}] \oplus \bigoplus_{i=0}^{\infty} \mathbb{Z}[A^{\pm 1}]/(1 - A^{2i+4})$$

(Hoste & Przytycki '95)

Open questions:

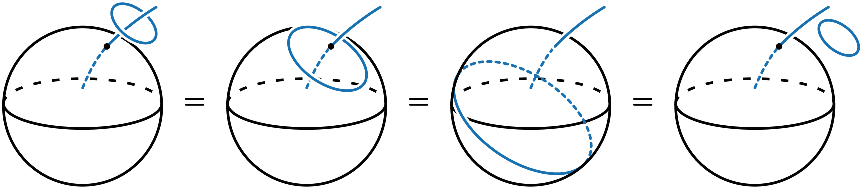
- $S^1 \times S^1 \times S^1$, $F \times S^1$
- homology spheres, complements of knots, Seifert manifolds,...
- connected sums

KBSM (properties)

Theorem [Przytycki]

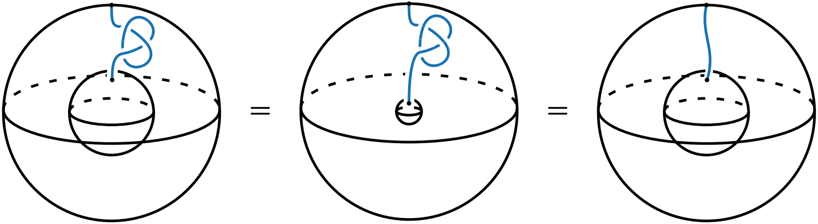
If there exists an embedded non-seperating 2-sphere in M, then $KBSM(M)$ has torsion.

We can see the torsion:



KBSM (properties)

Torsion in $S^2 \times S^1$ is easily observable via the "lightbulb trick":



KBSM (properties)

Theorem [Przytycki]

If there exists an embedded non-seperating torus in M, then $KBSM(M)$ has torsion.

Conjecture 1

$KBSM(M)$ has torsion if there exists an embedded non-seperating closed surface in M.

Conjecture 2

$KBSM(M)$ has torsion iff there exists an embedded non-seperating closed surface in M.

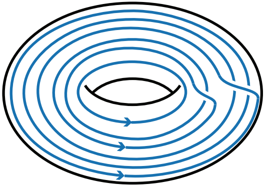
Let $R = \mathbb{Z}[l^{\pm 1}, z^{\pm 1}]$ and let $\mathcal{L}$ be the set of isotopy classes of oriented links in M.
The HOMFLYPT skein module of M is

$HSM(M) := \mathcal{S}_3(M) = R \cdot \mathcal{L} \Big/ z\Big[$  $\Big] - l\Big[$  $\Big] + l^{-1}\Big[$  $\Big].$

Remark. $HSM(M)$ is stronger than $KBSM(M)$ and generally harder to compute.

HSM (known results)

- $HSM(S^3)$ is freely generated by $\{$  $\}$ (trivial).
- $HSM(T)$, $T = D^2 \times S^1$, is freely generated by elements of type:



I.e. $HSM(T)$ is an algebra which, as an R-module, is a free module isomorphic to the symmetric tensor algebra, $\mathbf{S}R\hat{\pi}_1^O(T)$, where $\hat{\pi}_1^O(T)$ denotes the conjugacy classes of nontrivial elements of $\pi_1(T)$.

- $HSM(L(p,1))$ is freely infinitely generated (Gabrovšek, Mroczkowski '14)

HSM (known results)

Conjecture 1

$HSM(M)$ has torsion if there exists an embedded non-seperating closed surface in M.

Conjecture 2

$HSM(M)$ has torsion iff there exists an embedded non-seperating closed surface in M.

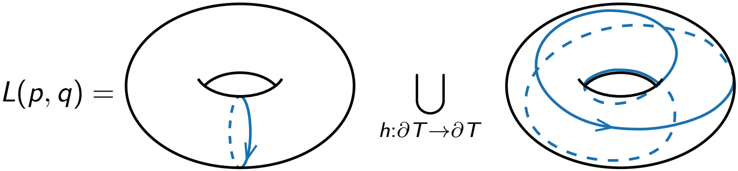
Conjecture 3

If $M \subset \mathbb{Q}HS$ and it does not contain a closed, oriented non-seperating surface then $HSM(M)$ is free and isomorphic to the symmetric tensor algebra over the module spanned by conjugacy classes of nontrivial elements of the fundamental group, $HSM^{alg}(M) \cong \mathbf{S}R\hat{\pi}_1^O(T)$.

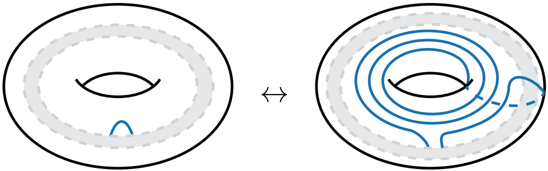
Open questions: HSM of any other 3-manifold.

Idea of calculation of $HSM(L(p,1))$

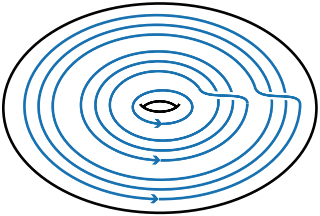
The lens space $L(p,q)$ has a Heegaard decomposition of genus 1:



We can isotope every knot to the first torus.
We get an extra slide move:



For a link $L = L_1 \cup L_2 \cup \ldots \cup L_n$, we can isotope each component so that it winds up less than p times around the torus. That is $0 \leq [L_i] < p$ in $\pi_1(L(p,1)) \cong \mathbb{Z}/p$, $\forall i$. We end up in a (ordered) basis $t_{n_1} t_{n_2} t_{n_3} \ldots$, t_{n_i}'s are ascending diagrams with $wind(n_i) < p$.



$t_1 t_3 t_3$

Difficult step: proving that the basis is invariant under the Reidemeister moves and the slide move.