v_2-periodicity of A_1

Philip Egger

Northwestern University

Young Topologist Meeting
EPF Lausanne, Switzerland
July 6, 2015

Acknowledgments

The following is joint work with Prasit Bhattacharya (Notre Dame postdoc) and Mark Mahowald.

Mark Mahowald (1931-2013)

- 84 descendants, including Mike Hopkins
- had an "organic" intuition for the homotopy groups of spheres
- 2 was his favorite prime

We are working in the stable homotopy category of spectra, where

- suspension is invertible
- fiber sequences are cofiber sequences
- $\pi_n X = \lim[S^{n+k}, \Sigma^k X]$

Families in $\pi_* S^0 \otimes \mathbb{Z}_{(p)}$

Want to construct infinite families in $\pi_* S^0 \otimes \mathbb{Z}_{(p)}$.

Idea

Take p-local finite CW-complexes X with top cell in dimension d and with self-maps

$$f : \Sigma^k X \to X$$

and the family

$$\phi_t : \; S^{kt} \longrightarrow \Sigma^{kt} X \xrightarrow{f^t} X \longrightarrow S^d$$

Remark

All the ϕ_t will be nontrivial, unless f is nilpotent.

Example: Greek letter families

- Consider the Smith-Toda complex $V(0) = S^0 \cup_p e^1$.
  If $p \geq 3$ and $q = 2p - 2$, then $V(0)$ admits a non-nilpotent self-map
  $$\alpha : \Sigma^q V(0) \to V(0),$$
  giving us a family of nontrivial elements of $\pi_{qt-1} S^0 \otimes \mathbb{Z}_{(p)}$
  $$\alpha_t : \ S^{qt} \longrightarrow \Sigma^{qt} V(0) \xrightarrow{\alpha^t} V(0) \longrightarrow S^1$$
- Consider the Smith-Toda complex $V(1) = cofiber(\alpha)$.
  If $p \geq 5$, then $V(1)$ admits a non-nilpotent self-map
  $$\beta : \Sigma^{q(p+1)} V(1) \to V(1),$$
  giving us a family of nontrivial elements
  $\beta_t \in \pi_{q(p+1)t-1} S^0 \otimes \mathbb{Z}_{(p)}$

Nilpotence I

Question

How does one detect nilpotence?

Recall the Brown-Peterson spectrum BP with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ and $|v_n| = 2(p^n - 1)$.

Theorem (Devinatz-Hopkins-Smith)

f is nilpotent if and only if $1_{BP} \wedge f$ is nilpotent.

Nilpotence II

Let $k(n)$ be the connected Morava K-theories with $k(n)_* = \mathbb{F}_p[v_n]$

Definition

A map $f : \Sigma^k X \to X$ is a v_n self-map if $k(n)_* f$ is multiplication by some power of v_n, and $k(m)_* f$ is nilpotent for every $m \neq n$.

Remark

The cofiber of a v_n self-map admits a v_{n+1} self-map.

What this all means in practice

- We'd like to find finite complexes admitting v_n self-maps
- We'd like to find the specific power of v_n

From now on, consider $p = 2$.

Two examples

Theorem (Davis-Mahowald)

Let $M(1) = S^0 \cup_2 e^1$, $M_\eta = S^0 \cup_\eta e^2$, and $Y = M(1) \wedge M_\eta$. Then there are v_1 self-maps

$$v_1^4 : \Sigma^8 M(1) \to M(1)$$

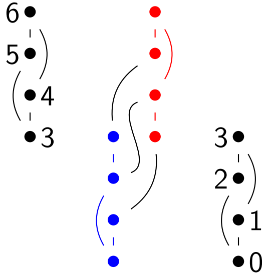
$$v_1 : \Sigma^2 Y \to Y$$

They construct the latter by constructing its cofiber A_1.

Why the name A_1?

Let A be the Steenrod algebra, $A(1) \subset A$ be generated by Sq^1, Sq^2.

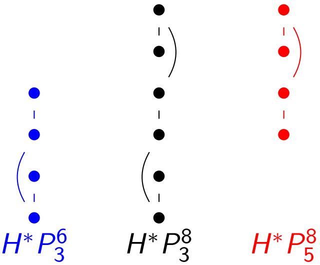
$$0 \to H^*\Sigma^3 Y \to H^*A_1 \to H^*Y \to 0$$



forces $H^*A_1 \cong A(1)$ as an $A(1)$-module.

Building A_1 from stunted projective spaces

Let $P_k^n = \mathbb{R}P^n/\mathbb{R}P^{k-1}$ be stunted projective spaces.



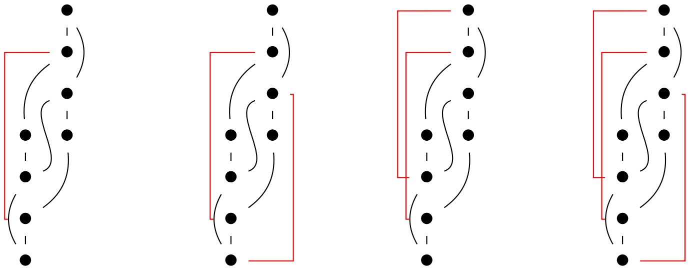
$$2: \ P_3^8 \longrightarrow P_5^8 \xrightarrow{g} P_3^6 \longrightarrow P_3^8 \ ,$$

and take

$$A_1 = cofiber(g).$$

There's more than one A_1

There are four different A-module structures on $A(1)$, depending on the action of Sq^4.

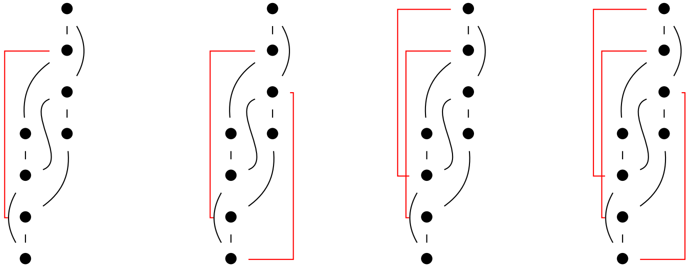


Theorem (Davis-Mahowald)

All four A-module structures of $A(1)$ are realizable topologically.

There's more than one A_1

There are four different A-module structures on $A(1)$, depending on the action of Sq^4.



$H^*A_1[00]$ $H^*A_1[10]$ $H^*A_1[01]$ $H^*A_1[11]$

Theorem (Davis-Mahowald)

All four A-module structures of $A(1)$ are realizable topologically.

Two v_2-periodicity theorems

Theorem (Behrens-Hill-Hopkins-Mahowald)

Let $M(1,4)$ be the cofiber of $v_1^4 : \Sigma^8 M(1) \to M(1)$. Then $M(1,4)$ admits the v_2 self-map

$$v_2^{32} : \Sigma^{192} M(1,4) \to M(1,4).$$

Theorem (Bhattacharya-E-Mahowald)

Let A_1 be the cofiber of one of the four $v_1 : \Sigma^2 Y \to Y$. Then A_1 admits the v_2 self-map

$$v_2^{32} : \Sigma^{192} A_1 \to A_1.$$

Idea of proof

We seek an element of

$$[\Sigma^{192} A_1, A_1] = [S^{192}, A_1 \wedge DA_1] = \pi_{192}(A_1 \wedge DA_1).$$

that maps to

$$v_2^{32} \in k(2)_{192}(A_1 \wedge DA_1).$$

Writing $X := A_1 \wedge DA_1$, we'd like to use the Adams spectral sequences

$$Ext_A^{s,t}(H^* X, \mathbb{F}_2) \Rightarrow \pi_{t-s} X$$

$$Ext_{E(Q_3)}^{s,t}(H^* X, \mathbb{F}_2) \Rightarrow k(2)_{t-s} X.$$

Problem

- $Ext_A^{s,t}(X)$ is too big to compute
- $Ext_{E(Q_3)}^{s,t}(X)$ is too small to be useful

Question

Is there a Goldilocks zone between the two?

Problem

- $Ext_A^{s,t}(X)$ is too big to compute
- $Ext_{E(Q_3)}^{s,t}(X)$ is too small to be useful

Question

Is there a Goldilocks zone between the two?

Answer

The inclusion $E(Q_3) \to A$ factors through $A(2) \subset A$, generated by Sq^1, Sq^2, Sq^4, so

$$Ext_A^{s,t}(X) \to Ext_{A(2)}^{s,t}(X) \to Ext_{E(Q_3)}^{s,t}(X)$$

factors. $Ext_{A(2)}^{s,t}(X)$ is the Goldilocks group.

Outline of proof

$$Ext_A^{s,t}(X) \to Ext_{A(2)}^{s,t}(X) \to Ext_{E(Q_3)}^{s,t}(X)$$

1. Show that $v_2^8, v_2^{16} \in Ext_{A(2)}^{s,t}(X)$ are not permanent cycles.
2. Show that $v_2^{32} \in Ext_{A(2)}^{s,t}(X)$ is a nonzero permanent cycle.
3. Show that $v_2^{32} \in Ext_{A(2)}^{s,t}(X)$ lifts to $v_2^{32} \in Ext_A^{s,t}(X)$.
4. Show that $v_2^{32} \in Ext_A^{s,t}(X)$ is a nonzero permanent cycle.

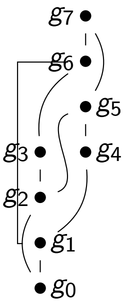
The proof is computational.

Bob Bruner's Ext software

Much of the proof uses Bob Bruner's Ext program.

1. You input a module M over A (or $A(2)$).
2. For $0 \leq s \leq 40$ (modifiable) and t in a user-defined range, the program computes $Ext_A^{s,t}(M, \mathbb{F}_2)$.
3. The program makes pretty charts of $Ext_A^{s,t}(M, \mathbb{F}_2)$.
4. And much more!

Example of module input: $H^*A_1[00]$



$$Sq^1(g_0) = g_1 \Rightarrow 0 \quad 1 \quad 1 \quad 1$$

$$Sq^4(g_1) = g_6 \Rightarrow 1 \quad 4 \quad 1 \quad 6$$

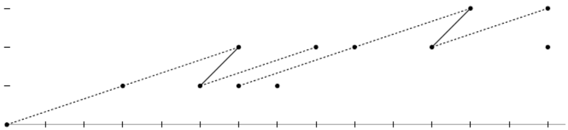
Example of module input: $H^*A_1[00]$

0	1	1	1
0	2	1	2
0	3	1	3
0	6	1	7
1	2	1	4
1	3	1	5
1	4	1	6
1	5	1	7
2	1	1	3
2	2	1	5
3	2	1	6
3	3	1	7
4	1	1	5
5	2	1	7
6	1	1	7

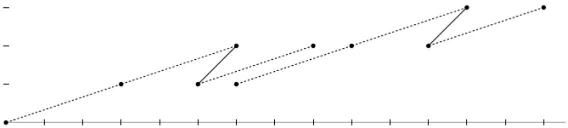
Example of charts: $H^*A_1[00]$

s: vertical axis, $t-s$: horizontal axis
solid line: multiplication by $h_1 \in Ext^{1,2}(\mathbb{F}_2)$ (Hopf map η)
dotted line: multiplication by $h_2 \in Ext^{1,4}(\mathbb{F}_2)$ (Hopf map ν)

$Ext_A^{s,t}(A_1[00])$



$Ext_{A(2)}^{s,t}(A_1[00])$



Thanks for listening!