n-Butterflies: Modeling Derived Morphisms of Strict n-Groups

Gregory (Ivan) Dungan II

Department of Mathematics
USMA, West Point

July 9, 2015

Outline

n-Homotopy Types

- A **homotopy n-type** is an object X of $Ho(\mathbf{Top}_Q)$ in which $\pi_k(X) = \mathbf{1}$ for $k > n$.

n-Homotopy Types

- A **homotopy n-type** is an object X of $Ho(\mathbf{Top}_Q)$ in which $\pi_k(X) = \mathbf{1}$ for $k > n$.
- The category of homotopy n-types is the full subcategory

$$\mathbf{H}\text{n}\mathbf{Typ} \subseteq Ho(\mathbf{Top}_Q).$$

n-Homotopy Types

- A **homotopy n-type** is an object X of $Ho(\mathbf{Top}_Q)$ in which $\pi_k(X) = \mathbf{1}$ for $k > n$.
- The category of homotopy n-types is the full subcategory

$$\mathbf{H}\mathrm{n}\mathbf{Typ} \subseteq Ho(\mathbf{Top}_Q).$$

- Moreover, $\mathbf{H}1\mathbf{Typ} \subseteq \mathbf{H}2\mathbf{Typ} \subseteq \mathbf{H}\mathrm{n}\mathbf{Typ} \subseteq Ho(\mathbf{Top}_Q)$.

Connected Homotopy 1-Types

- The functor $\pi_1 : \textbf{Top}^c \to \textbf{Grp}$ induces

$$\textbf{H}1\textbf{Typ}^c \simeq \textbf{Grp}$$

Connected Homotopy 1-Types

- The functor $\pi_1 : \textbf{Top}^c \to \textbf{Grp}$ induces

$$\textbf{H}1\textbf{Typ}^c \simeq \textbf{Grp}$$

- Groups model connected homotopy 1-types.

Connected Homotopy 1-Types

- The functor $\pi_1 : \textbf{Top}^c \to \textbf{Grp}$ induces

$$\textbf{H}1\textbf{Typ}^c \simeq \textbf{Grp}$$

- Groups model connected homotopy 1-types.
- $[X, Y]_{\textbf{Top}} \cong \textbf{Grp}(\pi_1(X), \pi_1(Y))$ where X, Y are connected homotopy 1-types.

Crossed Modules

- A **crossed module** $[G : \partial]$ is a homomorphism of groups $\partial : C_2 \to C_1$ with a right action x^a of G_1 on G_2 satisfying

$$\text{CM1} \quad \partial(x^a) = a^{-1}\partial(x)a$$

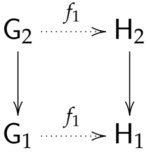
$$\text{CM2} \quad x^{\partial(y)} = y^{-1}xy$$

Crossed Modules

- A **crossed module** $[G : \partial]$ is a homomorphism of groups $\partial : C_2 \to C_1$ with a right action x^a of G_1 on G_2 satisfying

$$\text{CM1} \quad \partial(x^a) = a^{-1}\partial(x)a$$
$$\text{CM2} \quad x^{\partial(y)} = y^{-1}xy$$

- A morphism $f : [G, \partial] \to [H, \delta]$ is a commutative diagram



such that f_2 is f_1-equivariant.

Crossed Modules

- A **crossed module** $[\mathsf{G} : \partial]$ is a homomorphism of groups $\partial : \mathsf{C}_2 \to \mathsf{C}_1$ with a right action x^a of G_1 on G_2 satisfying

$$\begin{array}{ll} \text{CM1} & \partial(x^a) = a^{-1}\partial(x)a \\ \text{CM2} & x^{\partial(y)} = y^{-1}xy \end{array}$$

- A morphism $f : [\mathsf{G}, \partial] \to [\mathsf{H}, \delta]$ is a commutative diagram

$$\begin{array}{ccc} \mathsf{G}_2 & \xrightarrow{f_1} & \mathsf{H}_2 \\ \downarrow & & \downarrow \\ \mathsf{G}_1 & \xrightarrow{f_1} & \mathsf{H}_1 \end{array}$$

  such that f_2 is f_1-equivariant.
- Crossed modules with morphisms form a category **xm**.

Connected Homotopy 2-Types

Theorem (B. Noohi [?])

The Moerdijk-Svensson model structure on **xm** induces the equivalence

$$\mathbf{H2Typ}^c \simeq Ho(\mathbf{xm})$$

.

Connected Homotopy 2-Types

Theorem (B. Noohi [?])

The Moerdijk-Svensson model structure on **xm** induces the equivalence

$$\mathbf{H2Typ}^c \simeq Ho(\mathbf{xm})$$

.

- Crossed modules model connected homotopy 2-types.

Connected Homotopy 2-Types

Theorem (B. Noohi [?])

The Moerdijk-Svensson model structure on **xm** induces the equivalence

$$\mathbf{H2Typ}^{c} \simeq Ho(\mathbf{xm})$$

.

- Crossed modules model connected homotopy 2-types.
- The morphisms $[X, Y]_{\mathbf{xm}}$ model morphisms of connected homotopy 2-types.

Morphisms of **H2Typ**

- $[H, G]_{\mathbf{xm}} = \mathbf{xm}(Q, G)/\simeq$ where Q is a cofibrant replacement of H.

Morphisms of **H**2**Typ**

- $[H, G]_{\mathbf{xm}} = \mathbf{xm}(Q, G)/\simeq$ where Q is a cofibrant replacement of H.
- We would like to avoid computing cofibrant replacements.

Morphisms of **H**2**Typ**

- $[H, G]_{\mathbf{xm}} = \mathbf{xm}(Q, G)/\simeq$ where Q is a cofibrant replacement of H.
- We would like to avoid computing cofibrant replacements.

Theorem (B. Noohi [?])

There is a bijection

$$[H, G]_{\mathbf{xm}} \xrightarrow{\cong} \pi_0(B(H, G))$$

where B(H, G) is the groupoid of *butterflies*.

Morphisms of **H2Typ**

- $[H, G]_{\mathbf{xm}} = \mathbf{xm}(Q, G)/\simeq$ where Q is a cofibrant replacement of H.
- We would like to avoid computing cofibrant replacements.

Theorem (B. Noohi [?])

There is a bijection

$$[H, G]_{\mathbf{xm}} \xrightarrow{\cong} \pi_0(B(H, G))$$

where $B(H, G)$ is the groupoid of *butterflies*.

- The connected components of $B(G, H)$ model morphisms of connected homotopy 2-types.

Butterflies

A butterfly from $[\mathsf{G} : \partial]$ to $[\mathsf{H} : \delta]$ is a commutative diagram

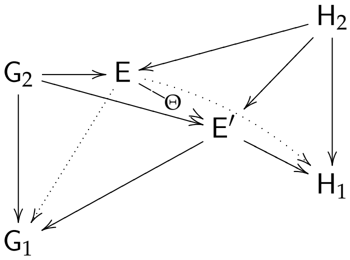
$$\begin{array}{ccccc} \mathsf{G}_2 & & & & \mathsf{H}_2 \\ & \searrow^{\alpha} & & {}^{\beta}\swarrow & \\ \partial\downarrow & & \mathsf{E} & & \downarrow\delta \\ & \swarrow_{p} & & {}_{f}\searrow & \\ \mathsf{G}_1 & & & & \mathsf{H}_1 \end{array}$$

where both diagonals are complexes, $\mathsf{H}_2 \to \mathsf{E} \to \mathsf{G}_1$ is short exact and for $x \in \mathsf{E}, g \in \mathsf{G}_2, h \in \mathsf{H}_2$

$$\alpha(g^{p(x)}) = x^{-1}\alpha(g)x \qquad\qquad \beta(h^{f(x)}) = x^{-1}\beta(h)x$$

Morphisms of Butterflies

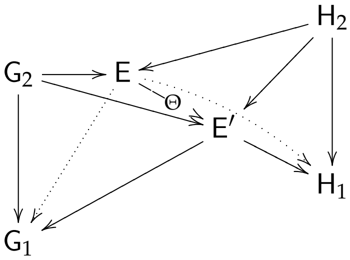
- A morphism of butterflies is an isomorphism $\Theta : E \to E'$ such that



commutes.

Morphisms of Butterflies

- A morphism of butterflies is an isomorphism $\Theta : E \to E'$ such that



  commutes.
- Butterflies from $[G : \partial]$ to $[H : \delta]$ with morphisms form a groupoid denoted by $B(G, H)$.

Question

- Can we model morphisms of other spaces up to homotopy type?

Question

- Can we model morphisms of other spaces up to homotopy type?
- In particular, is there an analog of butterflies for these spaces?

Crossed Complexes over a Groupoid

- A *crossed complex* $[\mathsf{G}, \delta]$ over a groupoid G_1 is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} \mathsf{G}_k \xrightarrow{\delta_k} \mathsf{G}_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} \mathsf{G}_2 \xrightarrow{\delta_2} \mathsf{G}_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{G}_0$$

Crossed Complexes over a Groupoid

- A *crossed complex* $[\mathsf{G}, \delta]$ over a groupoid G_1 is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} \mathsf{G}_k \xrightarrow{\delta_k} \mathsf{G}_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} \mathsf{G}_2 \xrightarrow{\delta_2} \mathsf{G}_1 \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} \mathsf{G}_0$$

1. For $k \geq 2$, $\mathsf{G}_k = \{\mathsf{G}_k(x)\}_{x \in \mathsf{G}_0}$ where $\mathsf{G}_k(x)$ is a group.

Crossed Complexes over a Groupoid

- A *crossed complex* $[\mathsf{G}, \delta]$ over a groupoid G_1 is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} \mathsf{G}_k \xrightarrow{\delta_k} \mathsf{G}_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} \mathsf{G}_2 \xrightarrow{\delta_2} \mathsf{G}_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{G}_0$$

1. For $k \geq 2$, $\mathsf{G}_k = \{\mathsf{G}_k(x)\}_{x \in \mathsf{G}_0}$ where $\mathsf{G}_k(x)$ is a group.
2. For $k \geq 3$, $\mathsf{G}_k(x)$ is abelian.

Crossed Complexes over a Groupoid

- A *crossed complex* $[\mathsf{G}, \delta]$ over a groupoid G_1 is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} \mathsf{G}_k \xrightarrow{\delta_k} \mathsf{G}_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} \mathsf{G}_2 \xrightarrow{\delta_2} \mathsf{G}_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{G}_0$$

1. For $k \geq 2$, $\mathsf{G}_k = \{\mathsf{G}_k(x)\}_{x \in \mathsf{G}_0}$ where $\mathsf{G}_k(x)$ is a group.
2. For $k \geq 3$, $\mathsf{G}_k(x)$ is abelian.
3. δ is a functor which respects G_0 such that $\delta \circ \delta = 1$.

Crossed Complexes over a Groupoid

UNITED STATES MILITARY ACADEMY WEST POINT

- A *crossed complex* $[\mathsf{G}, \delta]$ over a groupoid G_1 is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} \mathsf{G}_k \xrightarrow{\delta_k} \mathsf{G}_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} \mathsf{G}_2 \xrightarrow{\delta_2} \mathsf{G}_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{G}_0$$

1. For $k \geq 2$, $\mathsf{G}_k = \{\mathsf{G}_k(x)\}_{x \in \mathsf{G}_0}$ where $\mathsf{G}_k(x)$ is a group.
2. For $k \geq 3$, $\mathsf{G}_k(x)$ is abelian.
3. δ is a functor which respects G_0 such that $\delta \circ \delta = 1$.
4. G_1 acts on G_k on the right and satisfies:

Crossed Complexes over a Groupoid

- A *crossed complex* $[\mathsf{G},\delta]$ over a groupoid G_1 is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} \mathsf{G}_k \xrightarrow{\delta_k} \mathsf{G}_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} \mathsf{G}_2 \xrightarrow{\delta_2} \mathsf{G}_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{G}_0$$

1. For $k \geq 2$, $\mathsf{G}_k = \{\mathsf{G}_k(x)\}_{x \in \mathsf{G}_0}$ where $\mathsf{G}_k(x)$ is a group.
2. For $k \geq 3$, $\mathsf{G}_k(x)$ is abelian.
3. δ is a functor which respects G_0 such that $\delta \circ \delta = 1$.
4. G_1 acts on G_k on the right and satisfies:
    - 4.1 For $a \in \mathsf{G}_k(x), f \in \mathsf{G}_1(x,y)$, then $a^f \in \mathsf{G}_k(y)$.
    - 4.2 For $k \geq 2$, δ_k preserves the action.
    - 4.3 $\mathrm{Im}\delta_2$ acts by conjugation on G_2 and trivially on G_k for $k > 2$.

Crossed Complexes over a Groupoid

- A *crossed complex* $[\mathsf{G},\delta]$ over a groupoid G_1 is a sequence

$$\cdots \xrightarrow{\delta_{k+1}} \mathsf{G}_k \xrightarrow{\delta_k} \mathsf{G}_{k-1} \xrightarrow{\delta_{k-1}} \cdots \xrightarrow{\delta_3} \mathsf{G}_2 \xrightarrow{\delta_2} \mathsf{G}_1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathsf{G}_0$$

1. For $k \geq 2$, $\mathsf{G}_k = \{\mathsf{G}_k(x)\}_{x\in \mathsf{G}_0}$ where $\mathsf{G}_k(x)$ is a group.
2. For $k \geq 3$, $\mathsf{G}_k(x)$ is abelian.
3. δ is a functor which respects G_0 such that $\delta \circ \delta = 1$.
4. G_1 acts on G_k on the right and satisfies:
   4.1 For $a \in \mathsf{G}_k(x), f \in \mathsf{G}_1(x,y)$, then $a^f \in \mathsf{G}_k(y)$.
   4.2 For $k \geq 2$, δ_k preserves the action.
   4.3 $\mathrm{Im}\delta_2$ acts by conjugation on G_2 and trivially on G_k for $k > 2$.

- $[\mathsf{G},\delta]$ is a *reduced crossed complex* if G_1 is a group.

Categories of Crossed Complexes

- A morphism $f : \mathsf{H} \to \mathsf{G}$ is the data:

Categories of Crossed Complexes

- A morphism $f : \mathsf{H} \to \mathsf{G}$ is the data:
  1. Set map $f_0 : \mathsf{H}_0 \to \mathsf{G}_0$
  2. Functors $f_k : \mathsf{H}_k \to \mathsf{G}_k$ over f_0 compatible with δ and the action.
- Crossed complexes with morphisms form a category **Xc**.

Categories of Crossed Complexes

- A morphism $f : \mathsf{H} \to \mathsf{G}$ is the data:
  1. Set map $f_0 : \mathsf{H}_0 \to \mathsf{G}_0$
  2. Functors $f_k : \mathsf{H}_k \to \mathsf{G}_k$ over f_0 compatible with δ and the action.
- Crossed complexes with morphisms form a category **Xc**.
- Reduced crossed complexes form a full subcategory **xc**.

Categories of Crossed Complexes

- A morphism $f : \mathsf{H} \to \mathsf{G}$ is the data:
  1. Set map $f_0 : \mathsf{H}_0 \to \mathsf{G}_0$
  2. Functors $f_k : \mathsf{H}_k \to \mathsf{G}_k$ over f_0 compatible with δ and the action.
- Crossed complexes with morphisms form a category **Xc**.
- Reduced crossed complexes form a full subcategory **xc**.
- *Reduced n-Crossed Complexes* n**xc** : $\mathsf{G}_k = \mathbf{1}$ for all $k > n$

Truncated Examples

- $\mathbf{xc}^1$: $\mathsf{G}_k = 0$ for $k \geq 1$

$$\mathbf{xc}^1 \simeq \mathbf{Grp} \simeq \mathbf{H}1\mathbf{Typ}^c$$

Truncated Examples

- $\mathbf{xc}^1 : \mathsf{G}_k = 0$ for $k \geq 1$

$$\mathbf{xc}^1 \simeq \mathbf{Grp} \simeq \mathbf{H}1\mathbf{Typ}^c$$

- $\mathbf{xc}^2 : \mathsf{G}_k = 0$ for $k \geq 2$

$$\mathbf{xc}^2 \simeq \mathbf{xm} \rightsquigarrow Ho(\mathbf{xc}^2) \simeq \mathbf{H}2\mathbf{Typ}^c$$

Examples

- Initial object in **Xc** is the empty crossed complex $\varnothing$

Examples

- Initial object in **Xc** is the empty crossed complex $\emptyset$
- Final object in **Xc**

$$1: \qquad \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

Examples

- Initial object in **Xc** is the empty crossed complex $\emptyset$
- Final object in **Xc**

$$1: \qquad \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

Examples

- Initial object in **Xc** is the empty crossed complex $\emptyset$
- Final object in **Xc**

$$1: \qquad \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

Examples

- Initial object in **Xc** is the empty crossed complex $\emptyset$
- Final object in **Xc**

$$1 : \qquad \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

- The *unit interval crossed complex* I:
  1. $\mathsf{I}_0 = \{0, 1\}$

Examples

- Initial object in **Xc** is the empty crossed complex $\emptyset$
- Final object in **Xc**

$$1: \qquad \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

- The *unit interval crossed complex* I:
    1. $\mathsf{I}_0 = \{0, 1\}$
    2. $\mathsf{I}_k(0) = \mathbf{1}$ and $\mathsf{I}_k(1) = \mathbf{1}$

Examples

- Initial object in **Xc** is the empty crossed complex $\varnothing$
- Final object in **Xc**

$$1: \qquad \cdots \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow \mathbf{1}$$

- The *unit interval crossed complex* I:
  1. $\mathsf{I}_0 = \{0, 1\}$
  2. $\mathsf{I}_k(0) = \mathbf{1}$ and $\mathsf{I}_k(1) = \mathbf{1}$
  3. $\mathsf{I}_1(0, 1) =< \iota >$ and $\mathsf{I}_1(1, 0) =< \iota^{-1} >$

Extended Dold-Kan Theorem

Theorem (N. Ashley)

The Moore complex gives an extension of the Dold-Kan correspondence to the category of reduced crossed complexes **xc**.

$$\begin{array}{ccc}
s\mathbf{AbGrp} & \xrightarrow[\simeq]{Norm(-)} & \mathbf{Ch}_{\geq 0}(\mathbb{Z}) \\
\big\downarrow & & \big\downarrow \\
\mathbf{Grp}^{T}\mathbf{Cmplx} & \xrightarrow[\simeq]{Moore(-)} & \mathbf{xc} \\
\big\downarrow & & \\
s\mathbf{Grp} & &
\end{array}$$

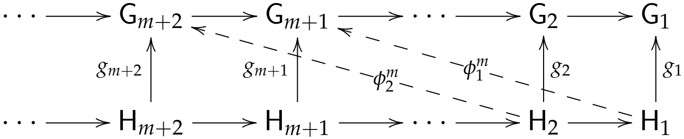
k-Fold Left Homotopy

A m-**fold left homotopy** $(g, \phi_k^m : \mathsf{H}_k \to \mathsf{G}_{k+m}) : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ is a morphism

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathsf{G}_{m+2} & \longrightarrow & \mathsf{G}_{m+1} & \longrightarrow & \cdots & \longrightarrow & \mathsf{G}_2 & \longrightarrow & \mathsf{G}_1 \\
 & & \uparrow{\scriptstyle g_{m+2}} & & \uparrow{\scriptstyle g_{m+1}} & & & & \uparrow{\scriptstyle g_2} & & \uparrow{\scriptstyle g_1} \\
\cdots & \longrightarrow & \mathsf{H}_{m+2} & \longrightarrow & \mathsf{H}_{m+1} & \longrightarrow & \cdots & \longrightarrow & \mathsf{H}_2 & \longrightarrow & \mathsf{H}_1
\end{array}$$

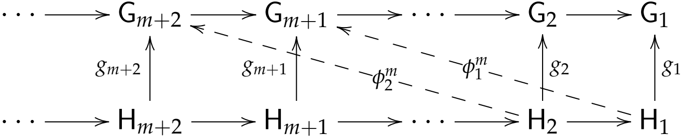
k-Fold Left Homotopy

A m-**fold left homotopy** $(g, \phi_k^m : \mathsf{H}_k \to \mathsf{G}_{k+m}) : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ is a morphism



k-Fold Left Homotopy

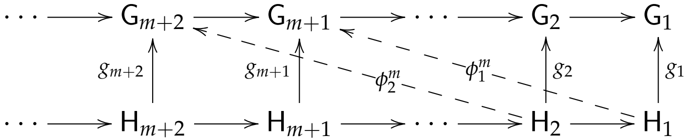
A m-**fold left homotopy** $(g, \phi_k^m : \mathsf{H}_k \to \mathsf{G}_{k+m}) : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ is a morphism



1. $\phi_1^m(ab) = \phi_1^m(a)^{g_1(b)}\phi_1^m(b)$ for $a, b \in \mathsf{H}_1$;

k-Fold Left Homotopy

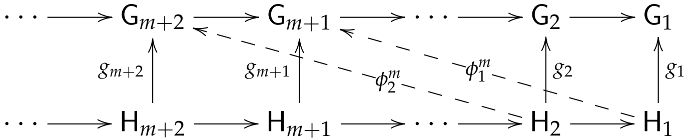
A m-**fold left homotopy** $(g, \phi_k^m : \mathsf{H}_k \to \mathsf{G}_{k+m}) : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ is a morphism



1. $\phi_1^m(ab) = \phi_1^m(a)^{g_1(b)}\phi_1^m(b)$ for $a, b \in \mathsf{H}_1$;
2. $\phi_k^m(xy) = \phi_k^m(x)\phi_k^m(y)$ for $x, y \in \mathsf{H}_k$ where $k \geq 2$;

k-Fold Left Homotopy

A m-**fold left homotopy** $(g, \phi_k^m : \mathsf{H}_k \to \mathsf{G}_{k+m}) : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ is a morphism



1. $\phi_1^m(ab) = \phi_1^m(a)^{g_1(b)}\phi_1^m(b)$ for $a, b \in \mathsf{H}_1$;
2. $\phi_k^m(xy) = \phi_k^m(x)\phi_k^m(y)$ for $x, y \in \mathsf{H}_k$ where $k \geq 2$;
3. ϕ_k^m is g_1-equivariant for $k \geq 2$.

Internal Hom and Tensor

Theorem (R. Brown, P. Higgins [?])

For crossed complexes H, G, there is a crossed $\mathbf{XC}(\mathsf{H},\mathsf{G})$ given by

$$\mathbf{XC}(\mathsf{H},\mathsf{G})_0 = \mathbf{Xc}(\mathsf{H},\mathsf{G})$$
$$\mathbf{XC}(\mathsf{H},\mathsf{G})_k = \{k - \text{fold left homotopies}\}$$

Internal Hom and Tensor

Theorem (R. Brown, P. Higgins [?])

For crossed complexes H, G, there is a crossed $\mathbf{XC}(\mathsf{H},\mathsf{G})$ given by

$$\mathbf{XC}(\mathsf{H},\mathsf{G})_0 = \mathbf{Xc}(\mathsf{H},\mathsf{G})$$
$$\mathbf{XC}(\mathsf{H},\mathsf{G})_k = \{k-\text{fold left homotopies}\}$$

Theorem (R. Brown, P. Higgins [?])

For every $\mathsf{C}, \mathsf{D}, \mathsf{E} \in \mathbf{Xc}$,

$$\mathbf{Xc}(\mathsf{C} \otimes \mathsf{D}, \mathsf{E}) \cong \mathbf{Xc}(\mathsf{C}, \mathbf{XC}(\mathsf{D}, \mathsf{E}))$$

which makes $(\mathbf{Xc}, \otimes, \mathbf{1})$ a closed symmetric monoidal category.

n-Homotopy Groups

For a crossed complex $[\mathsf{G} : \delta]$ and $x \in \mathsf{G}_0$

- Connected Components: $\pi_0(\mathsf{G}) = \pi_0(\mathsf{G}_1)$;

n-Homotopy Groups

For a crossed complex $[\mathsf{G} : \delta]$ and $x \in \mathsf{G}_0$

- Connected Components: $\pi_0(\mathsf{G}) = \pi_0(\mathsf{G}_1)$;
- Fundamental Homotopy Group: $\pi_1(\mathsf{G}, x) = \text{coker}\delta_2(x)$;

n-Homotopy Groups

For a crossed complex $[\mathsf{G} : \delta]$ and $x \in \mathsf{G}_0$

- Connected Components: $\pi_0(\mathsf{G}) = \pi_0(\mathsf{G}_1)$;
- Fundamental Homotopy Group: $\pi_1(\mathsf{G}, x) = \text{coker}\delta_2(x)$;
- n-Homotpy Group: $\pi_n(\mathsf{G}, x) = H_n(\mathsf{G}(x))$.

Weak Equivalences and Fibrations

- A **weak equivalence** is a morphism $f : \mathsf{H} \to \mathsf{G}$ in **Xc** which induces

$$\pi_0(\mathsf{H}) \cong \pi_0(\mathsf{G})$$
$$\pi_k(\mathsf{H}, x) \cong \pi_k(\mathsf{G}, f_0(x))$$

Weak Equivalences and Fibrations

- A **weak equivalence** is a morphism $f : \mathsf{H} \to \mathsf{G}$ in **Xc** which induces

$$\pi_0(\mathsf{H}) \cong \pi_0(\mathsf{G})$$
$$\pi_k(\mathsf{H}, x) \cong \pi_k(\mathsf{G}, f_0(x))$$

- A **fibration** is a morphism $f : \mathsf{H} \to \mathsf{G}$ in **Xc** such that
  1. $f_1 : \mathsf{H}_1 \to \mathsf{G}_1$ is a fibration of groupoids;

Weak Equivalences and Fibrations

- A **weak equivalence** is a morphism $f : \mathsf{H} \to \mathsf{G}$ in **Xc** which induces

$$\pi_0(\mathsf{H}) \cong \pi_0(\mathsf{G})$$
$$\pi_k(\mathsf{H}, x) \cong \pi_k(\mathsf{G}, f_0(x))$$

- A **fibration** is a morphism $f : \mathsf{H} \to \mathsf{G}$ in **Xc** such that
  1. $f_1 : \mathsf{H}_1 \to \mathsf{G}_1$ is a fibration of groupoids;
  2. $f(x)_k : \mathsf{H}_k(x) \to \mathsf{G}_k(f_0(x))$ is a surjection for all $x \in \mathsf{H}_0$ and $k \geq 2$.

Model Structure

Theorem (R. Brown, M. Golasinski [?])

Weak equivalences and fibrations form a closed model structure on **Xc**.

Model Structure

Theorem (R. Brown, M. Golasinski [**?**])

Weak equivalences and fibrations form a closed model structure on **Xc**.

- The homotopy category of **Xc** has as morphisms

$$[H, G]_{\mathbf{Xc}} = \mathbf{Xc}(Q, G)/\simeq$$

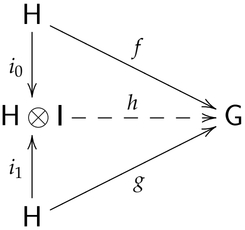
where Q is a cofibrant replacement of H.

Homotopy

For $f, g : \mathsf{H} \to \mathsf{G}$, a *homotopy* from f to g is a morphism

$$h : \mathsf{H} \otimes \mathsf{I} \to \mathsf{G}$$

such that



commutes.

Relation to 1-Fold Left Homotopy

Theorem (A. Tonks [?])

- Let $f, g : \mathsf{H} \to \mathsf{G}$ be morphisms of reduced crossed complexes. Defining a homotopy $h : f \simeq g$ is equivalent to defining a 1-fold left homotopy

$$(g, \phi_k : \mathsf{H}_k \to \mathsf{G}_{k+1})$$

Relation to 1-Fold Left Homotopy

Theorem (A. Tonks [?])

- Let $f, g : \mathsf{H} \to \mathsf{G}$ be morphisms of reduced crossed complexes. Defining a homotopy $h : f \simeq g$ is equivalent to defining a 1-fold left homotopy

$$(g, \phi_k : \mathsf{H}_k \to \mathsf{G}_{k+1})$$

- Moreover, f is determined by

$$f_1(a) = g_1(a)\delta_2(\phi_1(a))$$
$$f_k(x) = g_k(x)\delta_{k+1}(\phi_k(x))\phi_{k-1}(\partial_k(x))$$

Relation to 1-Fold Left Homotopy

Theorem (A. Tonks [?])

- Let $f, g : \mathsf{H} \to \mathsf{G}$ be morphisms of reduced crossed complexes. Defining a homotopy $h : f \simeq g$ is equivalent to defining a 1-fold left homotopy
$$(g, \phi_k : \mathsf{H}_k \to \mathsf{G}_{k+1})$$
- Moreover, f is determined by
$$f_1(a) = g_1(a)\delta_2(\phi_1(a))$$
$$f_k(x) = g_k(x)\delta_{k+1}(\phi_k(x))\phi_{k-1}(\partial_k(x))$$

- In other words, the quotient set $[\mathsf{H}, \mathsf{G}]_{\mathbf{Xc}} = \mathbf{Xc}(\mathsf{Q}, \mathsf{G})/\simeq$ can be described using 1-fold left homotopies.

Definition

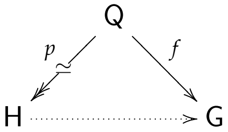
- We would like to model $[H, G]_{\mathbf{xc}} = \mathbf{Xc}(Q, G)/\simeq$.

Definition

- We would like to model $[H, G]_{\mathbf{xc}} = \mathbf{Xc}(Q, G)/\simeq$.
- Define *derived morphisms* to be the elements of the set $\mathbf{Xc}(Q, G)$.

Definition

- We would like to model $[\mathsf{H},\mathsf{G}]_{\mathbf{xc}} = \mathbf{Xc}(\mathsf{Q},\mathsf{G})/\simeq$.
- Define *derived morphisms* to be the elements of the set $\mathbf{Xc}(\mathsf{Q},\mathsf{G})$.
- Derived morphisms can be viewed as fractions:



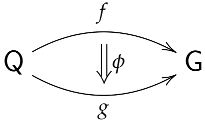
where $p : \mathsf{Q} \to \mathsf{H}$ is a cofibrant replacement of H.

Derived Groupoid

- The derived groupoid $\underline{\mathbf{Rhom}}(\mathsf{H},\mathsf{G})$ is defined by $\underline{\mathbf{Rhom}}(\mathsf{H},\mathsf{G})_0 = \mathbf{Xc}(\mathsf{Q},\mathsf{G})$ and morphisms of the form

Derived Groupoid

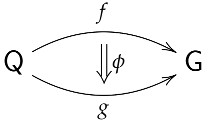
- The derived groupoid $\underline{\textbf{Rhom}}(\mathsf{H}, \mathsf{G})$ is defined by $\underline{\textbf{Rhom}}(\mathsf{H}, \mathsf{G})_0 = \textbf{Xc}(\mathsf{Q}, \mathsf{G})$ and morphisms of the form



where ϕ is a 1-fold left homotopy.

Derived Groupoid

- The derived groupoid $\underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G})$ is defined by $\underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G})_0 = \mathbf{Xc}(\mathsf{Q}, \mathsf{G})$ and morphisms of the form



where ϕ is a 1-fold left homotopy.

- By definition, there is a bijection

$$[\mathsf{H}, \mathsf{G}]_{\mathbf{xc}} \cong \pi_0(\underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G})).$$

Model of Derived Morphisms

- The main result:

Model of Derived Morphisms

- The main result:

Theorem (D.)

Let H, G be reduced n-crossed complexes. Then there is an equivalence of categories

$$\underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G}) \simeq n\mathsf{B}(\mathsf{H}, \mathsf{G})$$

where $n\mathsf{B}(\mathsf{H}, \mathsf{G})$ is the groupoid of n-butterflies.

Model of Derived Morphisms

- The main result:

Theorem (D.)

Let H, G be reduced n-crossed complexes. Then there is an equivalence of categories

$$\underline{\mathbf{Rhom}}(\mathsf{H}, \mathsf{G}) \simeq n\mathsf{B}(\mathsf{H}, \mathsf{G})$$

where $n\mathsf{B}(\mathsf{H}, \mathsf{G})$ is the groupoid of n-butterflies.

Corollary

Let H, G be reduced n-crossed complexes. Then there is a bijection

$$[\mathsf{H}, \mathsf{G}]_{\mathbf{xc}} \cong \pi_0(n\mathsf{B}(\mathsf{H}, \mathsf{G}))$$

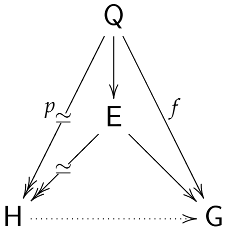
where $n\mathsf{B}(\mathsf{H}, \mathsf{G})$ is the groupoid of n-butterflies.

Algebraic Replacement

- Goal: avoid computing a cofibrant replacement of H.

Algebraic Replacement

- Goal: avoid computing a cofibrant replacement of H.
- Instead, find a crossed complex E which satisfies



∇ Factorization

- For a derived morphism $f : \mathsf{Q} \to \mathsf{G}$, consider the morphism

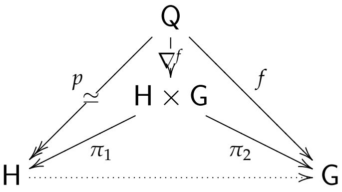
$$\nabla^f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}.$$

∇ Factorization

- For a derived morphism $f : \mathsf{Q} \to \mathsf{G}$, consider the morphism

$$\nabla^f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}.$$

- Then the following diagram commutes.

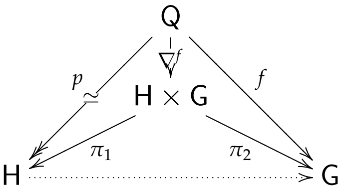


∇ Factorization

- For a derived morphism $f : \mathsf{Q} \to \mathsf{G}$, consider the morphism

$$\nabla^f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}.$$

- Then the following diagram commutes.



- But not necessarily a fraction!

n-Pushout below ∇_n^f

- For a derived morphism $f : [\mathsf{Q} : \xi] \to [\mathsf{G} : \delta]$ in n**xc**, there is a reduced n-crossed complex

$$\mathsf{H}_n \times \mathsf{G}_n \longrightarrow \mathsf{Q}_{n-1} \times^{\nabla_n^f} \mathsf{H}_n \times \mathsf{G}_n \longrightarrow \mathsf{Q}_{n-2} \longrightarrow \mathsf{Q}_{n-3} \longrightarrow \cdots$$

n-Pushout below ∇_n^f

- For a derived morphism $f : [\mathsf{Q} : \xi] \to [\mathsf{G} : \delta]$ in n**xc**, there is a reduced n-crossed complex

$$\mathsf{H}_n \times \mathsf{G}_n \longrightarrow \mathsf{Q}_{n-1} \times^{\nabla_n^f} \mathsf{H}_n \times \mathsf{G}_n \longrightarrow \mathsf{Q}_{n-2} \longrightarrow \mathsf{Q}_{n-3} \longrightarrow \cdots$$

where in the $n = 2$ case replace product with semidirect product.

n-Pushout below ∇_n^f

- For a derived morphism $f : [\mathsf{Q} : \xi] \to [\mathsf{G} : \delta]$ in $n\mathbf{xc}$, there is a reduced n-crossed complex

$$\mathsf{H}_n \times \mathsf{G}_n \longrightarrow \mathsf{Q}_{n-1} \times^{\nabla_n^f} \mathsf{H}_n \times \mathsf{G}_n \longrightarrow \mathsf{Q}_{n-2} \longrightarrow \mathsf{Q}_{n-3} \longrightarrow \cdots$$

  where in the $n = 2$ case replace product with semidirect product.
- We will call this crossed complex the *n-pushout below* ∇_n^f and denote it by $[\mathsf{Q}^f : \xi^f]$.

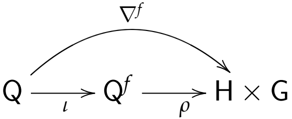
n-Pushout below ∇_n^f

- Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism n**xc** and Q a cofibrant replacement of H. Then we have the factorization:

$$\mathsf{Q} \xrightarrow[\iota]{} \mathsf{Q}^f \xrightarrow[\rho]{} \mathsf{H} \times \mathsf{G}, \qquad \nabla^f : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}$$

n-Pushout below ∇_n^f

- Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism n**xc** and Q a cofibrant replacement of H. Then we have the factorization:



Theorem (D.)

The morphism $\iota : \mathsf{Q} \to \mathsf{Q}^f$ is a weak equivalence.

n-Pushout below ∇_n^f

- Let $f : [\mathsf{H} : \partial] \to [\mathsf{G} : \delta]$ be a morphism n**xc** and Q a cofibrant replacement of H. Then we have the factorization:

$$\mathsf{Q} \xrightarrow[\iota]{} \mathsf{Q}^f \xrightarrow[\rho]{} \mathsf{H} \times \mathsf{G}, \qquad \nabla^f = \rho \circ \iota : \mathsf{Q} \to \mathsf{H} \times \mathsf{G}$$

Theorem (D.)

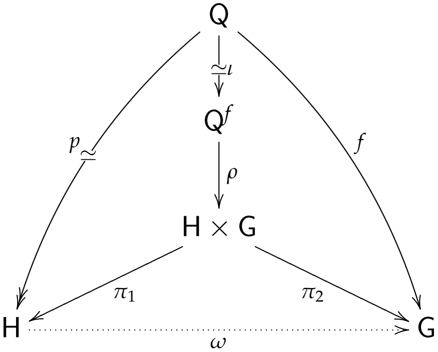
The morphism $\iota : \mathsf{Q} \to \mathsf{Q}^f$ is a weak equivalence.

Theorem (D.)

The morphism $cotr_{n-1}(\iota) : cotr_{n-1}(\mathsf{Q}) \to cotr_{n-1}(\mathsf{Q}^f)$ is an isomorphism in degree $n-1$ and the identity for $k < n-1$.

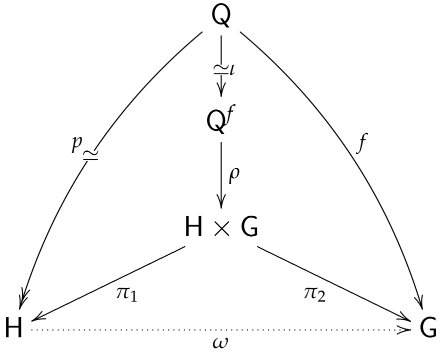
Induced Fraction

- The following diagram commutes.



Induced Fraction

- The following diagram commutes.

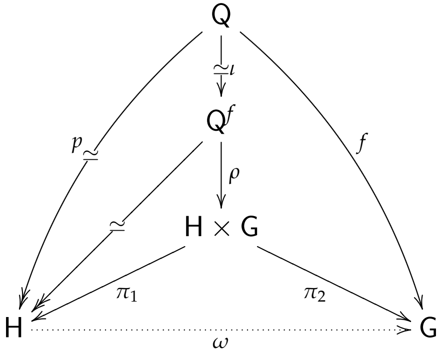


Theorem (D.)

The induced morphism $Q^f \longrightarrow G \times H \xrightarrow{\pi_1} G$ is a trivial fibration.

Induced Fraction

- The following diagram commutes.

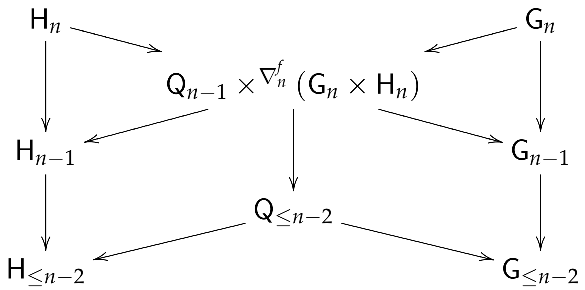


Theorem (D.)

The induced morphism $Q^f \longrightarrow G \times H \xrightarrow{\pi_1} G$ is a trivial fibration.

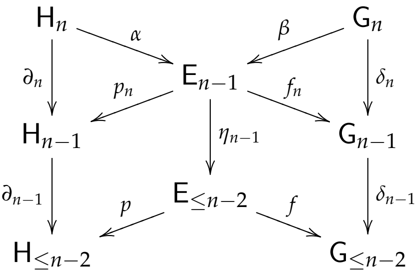
Unfolding Q^f

- By unfolding the map $Q^f \xrightarrow{\rho} H \times G$, we have a commutative diagram



Definition

A n-**Butterfly** from H to G is



where $[\mathsf{E} : \eta] \xrightarrow{p} [\mathsf{H}_{\leq n-1} : \partial]$ and $[\mathsf{E} : \eta] \xrightarrow{f} [\mathsf{G}_{\leq n-1} : \partial]$
are morphisms of reduced $(n-1)$-crossed complexes;

Definition Continued

- the induced sequences

$$1 \longrightarrow \mathsf{G}_n \xrightarrow{\beta} \mathsf{E}_{n-1} \xrightarrow{u_n} \ker \eta_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1} \longrightarrow 1$$

$$\mathsf{E}_k \xrightarrow{u_k} \ker \eta_{k-1} \times_{\ker \partial_{k-1}} \mathsf{H}_k \longrightarrow 1$$

for $k \leq n-2$ are exact;

Definition Continued

- the induced sequences

$$1 \longrightarrow \mathsf{G}_n \xrightarrow{\beta} \mathsf{E}_{n-1} \xrightarrow{u_n} \ker \eta_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1} \longrightarrow 1$$

$$\mathsf{E}_k \xrightarrow{u_k} \ker \eta_{k-1} \times_{\ker \partial_{k-1}} \mathsf{H}_k \longrightarrow 1$$

  for $k \leq n-2$ are exact;
- the compositions $\eta_{n-1} \circ (\alpha \times \beta)$ and $f_n \circ \alpha$ are complexes

Definition Continued

- the induced sequences

$$1 \longrightarrow \mathsf{G}_n \xrightarrow{\beta} \mathsf{E}_{n-1} \xrightarrow{u_n} \ker \eta_{n-2} \times_{\ker \partial_{n-2}} \mathsf{H}_{n-1} \longrightarrow 1$$

$$\mathsf{E}_k \xrightarrow{u_k} \ker \eta_{k-1} \times_{\ker \partial_{k-1}} \mathsf{H}_k \longrightarrow 1$$

  for $k \leq n-2$ are exact;
- the compositions $\eta_{n-1} \circ (\alpha \times \beta)$ and $f_n \circ \alpha$ are complexes
- α, β satisfy the compatibility conditions

$$\alpha\left(x^{p_1(a)}\right) = \alpha(x)^a \qquad \text{and} \qquad \beta\left(y^{f_1(a)}\right) = \beta(y)^a$$

Folding a n-Butterfly

Theorem (D.)

Let $([\mathsf{E},\eta],p,f,\alpha,\beta)$ be a n-butterfly from G to H. Then the induced morphism

$$\begin{array}{ccc} \mathsf{H}_n \times \mathsf{G}_n & \xrightarrow{\pi_1} & \mathsf{H}_n \\ \Big\downarrow{\scriptstyle \alpha\times\beta} & & \Big\downarrow{\scriptstyle \partial_n} \\ \mathsf{E}_{\leq n-1} & \xrightarrow[p]{} & \mathsf{H}_{\leq n-1} \end{array}$$

of reduced n-crossed complexes is a trivial fibration.

Folding a n-Butterfly

Theorem (D.)

Let $([\mathsf{E},\eta],p,f,\alpha,\beta)$ be a n-butterfly from G to H. Then the induced morphism

$$\begin{array}{ccc} \mathsf{H}_n \times \mathsf{G}_n & \xrightarrow{\pi_1} & \mathsf{H}_n \\ \downarrow{\scriptstyle \alpha\times\beta} & & \downarrow{\scriptstyle \partial_n} \\ \mathsf{E}_{\leq n-1} & \xrightarrow[p]{} & \mathsf{H}_{\leq n-1} \end{array}$$

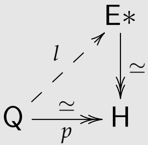
of reduced n-crossed complexes is a trivial fibration.

- We denote the folded n-butterfly on the left by E^*.

n-Butterfly over Q

Corollary

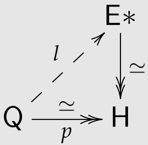
Let $p : \mathsf{Q} \to \mathsf{H}$ be a cofibrant replacement of H. Then there exists a lift l



n-Butterfly over Q

Corollary

Let $p : \mathsf{Q} \to \mathsf{H}$ be a cofibrant replacement of H. Then there exists a lift l

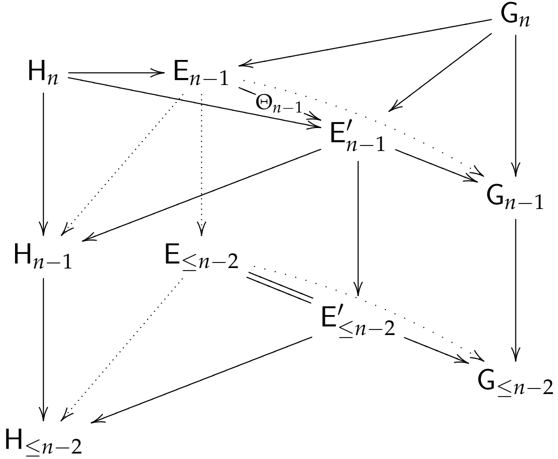


Definition

Let Q be a cofibrant replacement of H. A *n-butterfly over* Q is an n-butterfly with a lift l such that $cotr_{n-1}(l) : cotr_{n-1}(\mathsf{Q}) \to cotr_{n-1}(\mathsf{E}^*)$ is an isomorphism in degree $n-1$ and the identity for $k < n-1$.

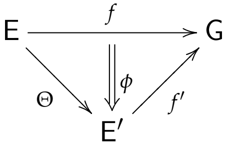
Morphisms of n-Butterflies

A morphism of n-butterflies over Q from H to G is a diagram



n-Butterflies Groupoid

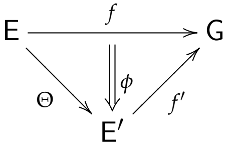
- where Θ is an isomorphism in degree $n-1$, the identity for $k < n-1$, and makes the diagram



commute up to a left 1-fold homotopy ϕ.

n-Butterflies Groupoid

- where Θ is an isomorphism in degree $n-1$, the identity for $k < n-1$, and makes the diagram



commute up to a left 1-fold homotopy ϕ.

Theorem (D.)

The n-butterflies from H to G over Q with the morphisms form a groupoid denoted by $n\mathrm{B}(\mathrm{H}, \mathrm{G})$.

Property of Morphisms of n-Butterflies

Corollary

Let $(\Theta, \phi) : ([\mathsf{E}, \eta], p, f, \alpha, \beta) \to ([\mathsf{E}', \eta'], p', f', \alpha', \beta')$ be a morphism of n-butterflies. Then the induced morphism $\mathsf{E}^* \to (\mathsf{E}')^*$ of reduced n-crossed complexes is a weak equivalence.

Property of Morphisms of n-Butterflies

Corollary

Let $(\Theta, \phi) : ([\mathsf{E}, \eta], p, f, \alpha, \beta) \to ([\mathsf{E}', \eta'], p', f', \alpha', \beta')$ be a morphism of n-butterflies. Then the induced morphism $\mathsf{E}^* \to (\mathsf{E}')^*$ of reduced n-crossed complexes is a weak equivalence.

Theorem (D.)

Let H, G be reduced n-crossed complexes. Then there is an equivalence of categories

$$\underline{\textbf{Rhom}}(\mathsf{H}, \mathsf{G}) \simeq n\mathsf{B}(\mathsf{H}, \mathsf{G}).$$

Moreover, there is a bijection

$$[\mathsf{H}, \mathsf{G}]_{\mathbf{xc}} \cong \pi_0(n\mathsf{B}(\mathsf{H}, \mathsf{G})).$$

Thank you. Questions?