On the Alexander Polynomial of a welded ribbon tangle

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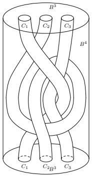
(joint work with Vincent Florens)

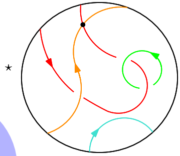
Panoramic view

Ribbon tubes: 2015, Audoux, Bellingeri, Meilhan, Wagner.

Ribbon tangles in B^4

Welded tangle diagrams in B^2





Alexander functor: functorial generalization of the Alexander polynomial

Alexander polynomial: combinatorial invariant

Classical tangles: 2012, Bigelow, Cattabriga, Florens. **Functoriality:** 2014, Florens, Massuyeau.

Virtual case: 2010, Archibald.

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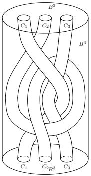

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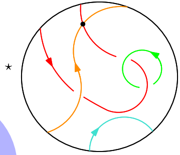

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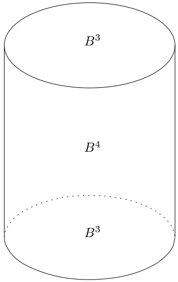
Welded ribbon tangles

- $B^4 = B^3 \times [0,1]$;



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- $B^4 = B^3 \times [0,1]$;
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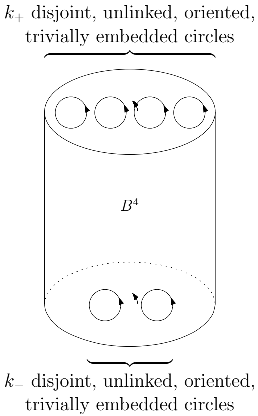
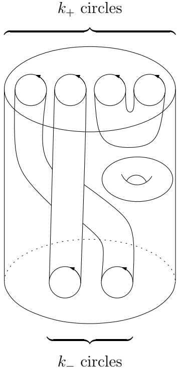


- $B^4 = B^3 \times [0,1]$;
- k_+ circles in the upper copy of B^3, and k_- circles in the lower copy of B^3;
- $A_1, \ldots, A_k$ embedded annuli and $E_1, \ldots, E_m$ embedded tori $E_1, \ldots, E_m$ s.t.:



$$k = \frac{k_+ + k_-}{2}$$

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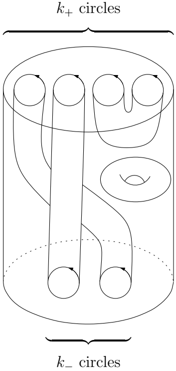

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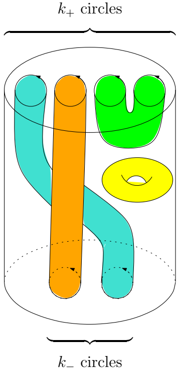

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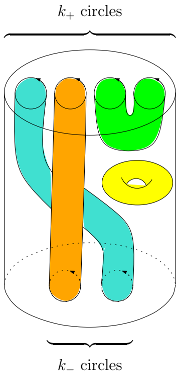

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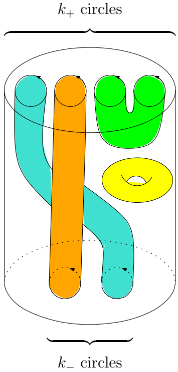


$$k = \frac{k_+ + k_-}{2}$$

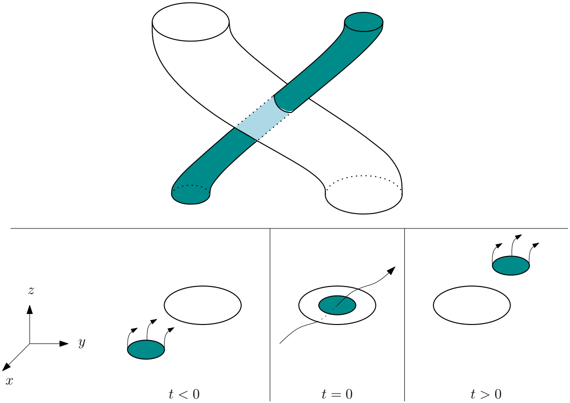
The set of ribbon tangles

rTA_n: set of ribbon tangles up to ambient isotopy fixing the boundary circles.

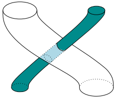
Ribbon singularity

Flatly transverse disk whose preimage are two disk:

- one in the interior of a filling,
- the other with interior included in the interior of a filling, and an essential curve as boundary.



Projecting a ribbon tangle's singularity in $B^3 = B^2 \times I$:



Warning

We loose the information about whether the "flying disk" was moving upward or downward!

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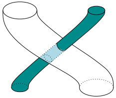

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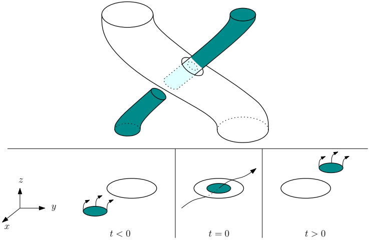
Convention

Erase a neighbourhood of the tube corresponding to the lower preimage disk.



Representing ribbon tangles

Any ribbon tangle can be represented by a broken surface diagram.

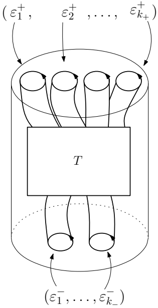
→ Yanagawa (1969): flat embeddings of 2-spheres in $\mathbb{R}^4$.

→ Audoux, Bellingeri, Meilhan, Wagner (2014): ribbon tubes.

Contents

The category Rib

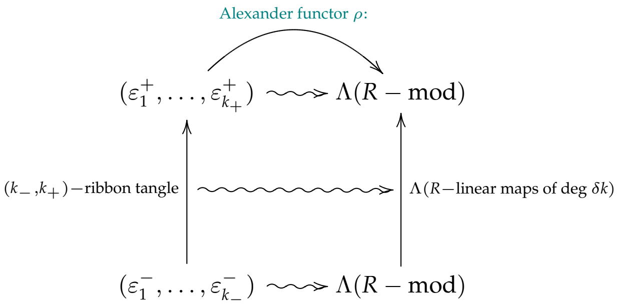
Objects: sequences of signs $(\varepsilon_1, \ldots, \varepsilon_k)$;
Morphisms: $(\varepsilon^-, \varepsilon^+)$-ribbon tangle with stacking as composition.



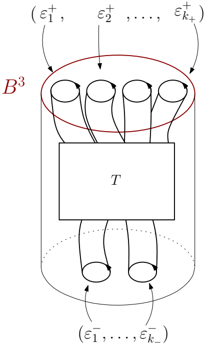
$\varepsilon_i^\pm \in \{-1, +1\}$

Construction of the functor

Objects of Rib	$\rightsquigarrow$	R-modules ($R = \mathbb{Z}[t, t^{-1}]$)
Morphisms of Rib	$\rightsquigarrow$	Linear maps of degree δk between exterior algebras of R-modules.

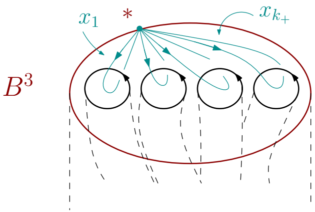


Induced coverings



$\varepsilon_i^\pm \in \{-1,+1\}$

$$(\varepsilon_1^+ , \quad \varepsilon_2^+ \qquad , \ldots , \qquad \varepsilon_{k_+}^+)$$

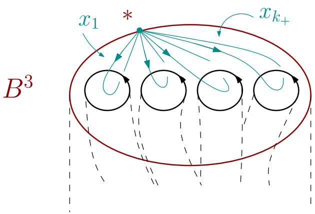


$$\chi_+ : \pi_1(B^3 \setminus \{C_1, \ldots, C_n\}, *) \to \mathbb{Z} = \langle t \rangle$$
$$x_i \mapsto t^{\varepsilon_i^+}$$

One variable or many variables?

It is possible to take $\mathbb{Z}^{k_+} = \langle t_1, \ldots, t_{k_+} \rangle$ in order to obtain a multivariable invariant (one variable for each component).

$(\varepsilon_1^+ , \quad \varepsilon_2^+ \qquad , \dots , \quad \varepsilon_{k_+}^+)$



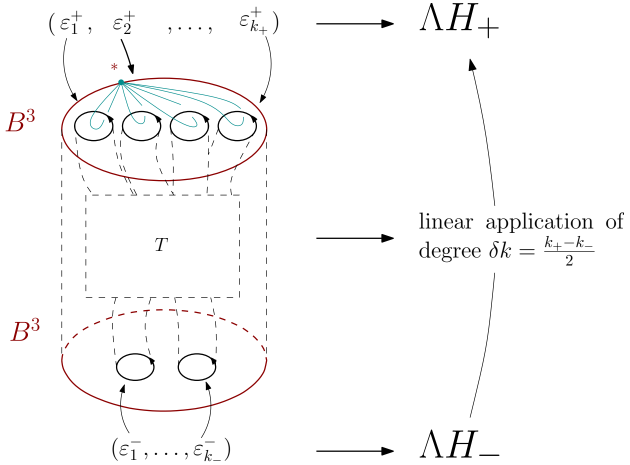
$$\chi_+ : \pi_1(B^3 \setminus \{C_1, \dots, C_n\}, *) \to \mathbb{Z} = \langle t \rangle$$
$$x_i \mapsto t^{\varepsilon_i^+}$$

This epimorphisms defines a covering $(B^3 \setminus \{C_1, \dots, C_n\})_+^\chi$.

The module H_+

We define H_+ to be $H_1\big((B^3 \setminus \{C_1, \dots, C_n\})_+^\chi, *; \mathbb{Z}[t, t^{-1}]\big)$.

The R-modules H_- and H_+



The R-module H

Let T be a ribbon tangle with n components, we denote:

- the exterior $X_T = B^4 \setminus \overset{\circ}{\text{Tub}}\,(T)$;
- $m_\pm$ the inclusion maps of the upper and lower copies of B^3 in X_T;
- the homology group $H_1(X_T) \simeq \mathbb{Z}^n$ is generated by the meridians of the annuli and tori;
- χ be the extention of the epimorphisms χ_+ and χ_-;
- $\hat{X}_T$ the maximal abelian cover defined.

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The module H

We define H to be $H_1(\hat{X}_T, *; \mathbb{Z}[t, t^{-1}])$, where $*$ is a basepoint on $\partial_* B^4$.

- $R = \mathbb{Z}[t, t^{-1}]$; M a R-module of finite type with a deficiency k presentation:
$$\langle \gamma_1, \ldots, \gamma_{p+k} \mid r_1, \ldots, r_p \rangle.$$
- Γ= free R-module engendred by $\langle \gamma_1, \ldots, \gamma_{p+k} \rangle$.
- $\hat{r} = r_1 \wedge \cdots \wedge r_p$ and $\hat{\gamma} = \gamma_1 \wedge \cdots \wedge \gamma_{p+k}$.

The Alexander function

$\varphi_{(M,k)} : \bigwedge^k M \to R$ is the R-linear application defined by:

$$u \wedge \hat{r} = \varphi(u) \cdot \hat{\gamma}$$

for each $u = u_1 \wedge \cdots \wedge u_k \in \bigwedge^k M$.

For k fixes, different deficiency k presentations give the Alexander functions which differ by a multiplicative unitary element of R.

The Alexander functor ρ

$\varphi_{(H,k)} : \bigwedge^k H \to R$ Alexander function, $i_\pm : H_\pm \to H$, $k = \frac{k_+ + k_-}{2}$, $\delta k = \frac{k_+ - k_-}{2}$.

Alexander invariant

$$\rho_\tau : \bigwedge(\rho_{i,\tau}) : \bigwedge H_- \to \bigwedge H_+$$

is defined as follows: for $u_- \in \bigwedge^i H_-$, $\rho_{i,\tau}(u_-)$ is the element of $\bigwedge^{i+\delta k} H_+$ that, for each $w_+ \in \bigwedge^{k-i} H_+$, satisfies:

$$\varphi(H,k)(i_-(u_-) \wedge i_+(w_+)) = \det_+(\rho_{i,\tau}(u_-) \wedge w_+).$$

where $\det_+ : \bigwedge^{k_+} H_+ \to R$ is a volume form on H_+.

$\to$ Classical case: Bigelow, Cattabriga et Florens (2012).

Functoriality

ρ is a functor from the category of 3-dim cobordisms with a representation of their fundamental group to the category of $\mathbb{Z}$-graded R-modules.

$\to$ Classical case: Florens et Massuyeau (2014).

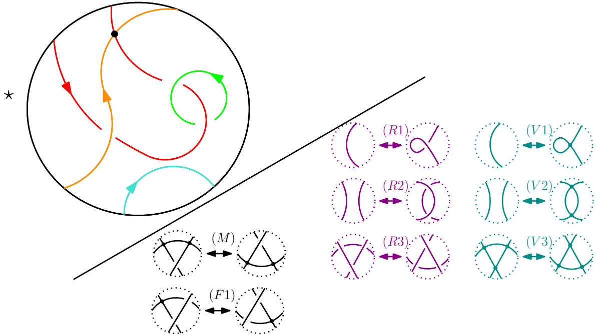
Contents

Welded diagrams

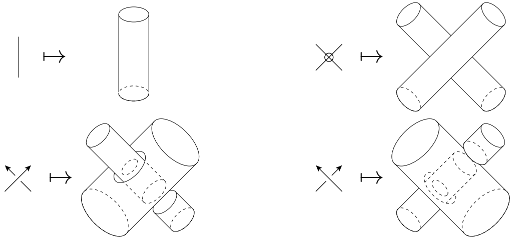
Welded k-tangle diagram T

Immersion of k oriented arcs and a certain number of circles in B^2 such that:

- $\partial I \subset \partial B_2$,
- double points: finite number, transverse, decorated as positive, negative or virtual, modulo generalized Reidemeister moves.



For every diagram, one can associate a broken surface, and hence a ribbon tangle by "blowing up" strings as follows:



The tube map

This assignment defines a map *Tube* : diagrams $\rightarrow rTA_n$.

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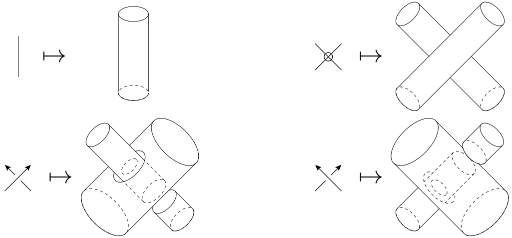

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This assignment defines a map *Tube* : diagrams $\to rTA_n$.

Proposition (Yanagawa, Satoh - Audoux, Bellingeri, Meilhan, Wagner)

The map *Tube* is surjective.

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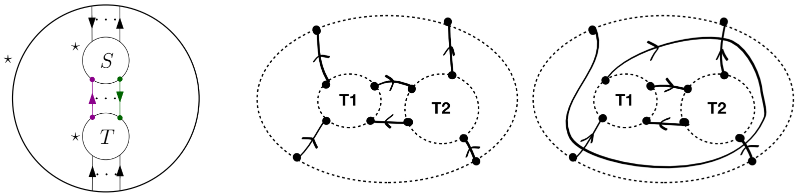
Welded tangle diagrams

Circuit algebra structure

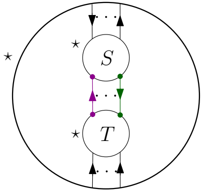


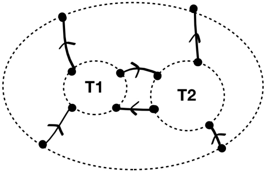
The Alexander "polynomial" for welded tangle diagrams

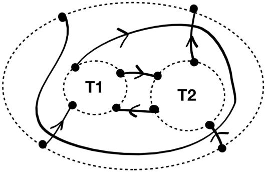
Welded tangle diagrams ⟶ ??

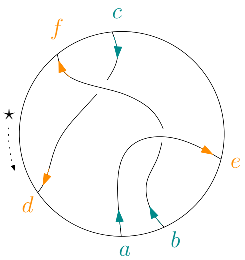
Circuit algebra structure

Another circuit algebra?







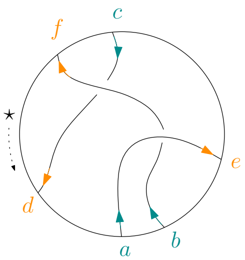


- $X^{in} = \{a, b, c\}$ $\rightarrow$ $H_{in} = \mathbb{Z}[t^{\pm 1}]$-module over X^{in}
- $X^{out} = \{d, e, f\}$ $\rightarrow$ $H_{out} = \mathbb{Z}[t^{\pm 1}]$-module over X^{out}

Alexander Half Densities

An *Alexander Half Density* of X^{in} and X^{out}, is an element of

$$\mathcal{D}(X^{in}, X^{out}) = \wedge^n (H_{in} \oplus H_{out}).$$



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The circuit algebra of Alexander Half Densities

Alexander Half Densities with composition (multilinear applications among AHD) form a circuit algebra.

T welded tangle diagram, we can associate a matrix of the form:

$$A(T) = \begin{array}{c} \\ \text{Internal arcs} \\ X^{out} \end{array} \begin{array}{c|c|c} \text{Internal arcs} & X^{in} & X^{out} \\ \left(\quad \right. & & \left. \quad \right) \\ \hline & & \end{array}$$

The Alexander polynomial of a welded tangle diagram

We define:

$$\mathcal{A}(T) = \sum_{i_1 < \cdots < i_k} (A(T)^{i_1, \ldots, i_k}) x_{i_1} \wedge \cdots \wedge x_{i_k} \in \bigwedge^k (H_{in} \oplus H_{out})$$

where $A(T)^{i_1 < \cdots < i_k}$ is the minor of A, with respect to the columns corresponding to internal arcs and arcs $i_1, \ldots, i_k$.

Properties

An invariant morphism of circuit algebras

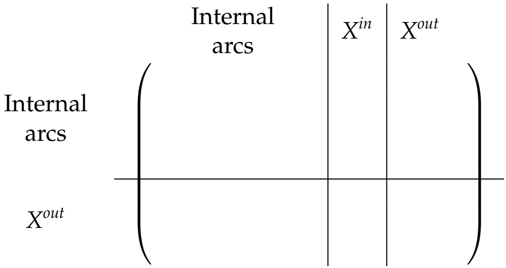
- $\mathcal{A}$ is a morphism between the circuit algebras $\mathcal{T}$ of welded tangle diagrams and $\mathcal{D}$ of Alexander Half Densities;
- $\mathcal{A}$ is an invariant for welded tangle diagrams defined modulo a unit of $\mathbb{Z}[t, t^{-1}]$.

$\rightarrow$ Virtual case: Jana Archibald (2010).

Contents

Proposition

The matrix $A(T)$ is a presentation matrix for H with deficiency k.



For a braid T: $k = k_- = k_+$, and $\delta k = \frac{k_+ - k_-}{2} = 0$.

- **Bigelow, Cattabriga, Florens:** $\rho_i : \bigwedge^i H_- \to \bigwedge^i H_+$ is the i-th external power of the Burau representation, modulo a multiplicative unit of R, that depends on the chosen presentation for H.
- Chosing $A(T)$ as presentation matrix, we get

$$\rho = \bigoplus_i \rho_i = -\bigoplus_i \bigwedge \rho_{Burau}.$$

Theorem (2015 - D., Florens)

Let τ be an $(\varepsilon^-, \varepsilon^+)$-ribbon tangles avec k_- et k_+ circles, $T(\tau)$ welded tangle diagram obtained by projection. There's a fonctorial isomorphism

$$\alpha \colon \bigwedge^k (H_{in} \oplus H_{out}) \to \mathrm{Hom}_{\delta k}(\bigwedge H_-, \bigwedge H_+)$$

that sends $\mathcal{A}(T(\tau))$ to $\rho(\tau)$.

Decomposition

Partition of the points of a welded tangle diagrams that tht Tube map will send to circles “at the top” and “at the bottom”.

Theorem (2015 - D., Florens)

Let T be a welded tangle diagram, and μ a decomposition. There's an isomorphism

$$\beta\colon\ \mathrm{Hom}_{\delta k}(\wedge H_-, \wedge H_+) \to \wedge^k(H_- \oplus H_+) \cong \wedge^k(H_{in} \oplus H_{out})$$

that send $\rho(\tau_\mu(T))$ to $\mathcal{A}(T)$. In particular, $\beta(\rho(\tau_\mu(T)))$ does not depend on the choice of μ.

Thank you for your attention.