Free loop spaces and Koszul duality

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What?

We study the homology of free loop spaces $H_*(LX;\mathbb{k})$ where $LX = Map(S^1, X)$ is the unbased mapping space.

Why?

This has been studied for a long time since the Betti numbers of $H_*(LM; \Bbbk)$ have strong connections to the number of closed geodesics of a Riemannian manifold M (with sufficiently generic metric). Faster growing Betti numbers implies more closed geodesics.

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More recently, this is part of the study of *String topology* since a seminal paper of Chas and Sullivan in 1999. If M is a manifold, $H_*(LM;\Bbbk)$ has a lot of algebraic structure; it is an example of a topological field theory.

Problem?

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We use Koszul duality to attack this for manifolds that are highly connected in relation to their dimension, more specifically, $(n-1)$-connected manifolds of dimension at most $3n-2$. This is joint work with A. Berglund.

Crash course in Koszul duality theory

Definition

Let C be a differential graded coalgebra and let A be a differential graded algebra. The *convolution algebra* is a differential graded algebra with underlying module the graded module homomorphisms $Hom(C, A)$. The product is given by

$$f \star g = \mu_A \circ (f \otimes g) \circ \Delta_C,$$

where Δ_C is the comultiplication of C and μ_A is the multiplication of A. The differential is given by

$$\partial(f) = d_A \circ f - (-1)^f f \circ d_C.$$

A *twisting morphism* $C \to A$ is an element τ of degree -1 in the convolution algebra satisfying

$$\partial(\tau) + \tau \star \tau = 0.$$

Example

Examples of (differential graded) coalgebras are the chains (with coefficients in a field $\Bbbk$) on a space $C_*(X;\Bbbk)$ and the homology $H_*(X;\Bbbk)$. The comultiplication comes from the diagonal map $X \to X \times X$.

Example

Examples of (differential graded) algebras are the chains on a based loop space $C_*(\Omega X;\Bbbk)$ and the homology $H_*(\Omega X;\Bbbk)$. The product structure comes from concatenation of loops.

Example

There is a prototypical twisting morphism $\tau : C_*(X;\Bbbk) \to C_*(\Omega X;\Bbbk)$. A degree 1 simplex corresponding to a loop in X is sent to the corresponding degree 0 simplex of ΩX.

Definition

Given a twisting morphism τ, the *twisted tensor product* $C \otimes_\tau A$ is the tensor product of graded modules with the differential $d = d_{C\otimes A} + d_\tau$, where $d_{C\otimes A}$ is the usual differential on the tensor product of chain complexes and

$$d_\tau := (Id_C \otimes \mu_A) \circ (Id_C \otimes \tau \otimes Id_A) \circ (\Delta_C \otimes Id_A).$$

The *twisted convolution algebra* is the differential graded algebra

$$Hom^\tau(C, A) = \big(Hom(C, A), \star, \partial^\tau\big),$$

with differential $\partial^\tau = \partial + [\tau, -]$, where,

$$\partial(f) = d_A \circ f - (-1)^f f \circ d_C, \quad [\tau, f] = \tau \star f - (-1)^{|f|} f \star \tau.$$

A coalgebra C and an algebra A are called *Koszul dual* if there is a twisting morphism τ such that $C \otimes_\tau A$ is an acyclic complex. In this case we also call the coalgebra C and algebra A *Koszul.*

Definition

A space X is *formal* over $\mathbb{k}$ if there is a weak equivalence of differential graded coalgebras:

$$C_*(X;\mathbb{k}) \sim H_*(X;\mathbb{k}).$$

A space X is *coformal* over $\mathbb{k}$ if there is a weak equivalence of differential graded algebras:

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Theorem (Berglund)

If a simply connected space X of finite $\Bbbk$-type is both formal and coformal over $\Bbbk$, $H_*(X;\Bbbk)$ is Koszul dual to $H_*(\Omega X;\Bbbk)$.

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Theorem (Berglund-B.)

Let $n \geq 2$. An $(n-1)$-connected manifold of dimension at most $3n-2$ is formal and coformal over a field $\mathbb{k}$ if and only if $\dim(H_*(X;\mathbb{k})) \neq 3$.

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Remark

Over the rationals, other spaces that are both formal and coformal include spheres, suspensions, loop spaces and configuration spaces of points in $\mathbb{R}^n$. The property of being formal and coformal is also preserved by products and wedges.

Free loop space homology and Hochschild cohomology

Remark

The set of twisting morphisms $Tw(C, A)$ determines a bifunctor. It is representable in both arguments

$$Tw(C, A) \cong Hom_{dgAlg}(\Omega C, A) \cong Hom_{dgCoalg}(C, BA).$$

We call B and Ω the bar and cobar constructions respectively.

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The bar contruction BA can be explicitly described as $\bigoplus_{k>0}(sA)^{\otimes k}$ where s is raising the degree by 1. The comultiplication is given by a sum over all deconcatenations and $d_{BA}(sa_1 \otimes \cdots \otimes a_k)$ is given by

$$\sum \pm sa_1 \otimes \cdots \otimes s(a_i a_j) \otimes \cdots \otimes sa_k + \sum \pm sa_1 \otimes \cdots \otimes s(d_A(a_i)) \otimes \cdots \otimes sa_k.$$

There is a twisting morphism $\tau : BA \to A$ given by projecting onto the $k = 1$ part.

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Definition

$Hom^{\tau}(BA, A)$ is called the *Hochschild (co)complex* of the algebra A. The homology $HH^*(A)$ of this complex is called the *Hochschild cohomology* of A (with coefficients in itself).

Theorem (Berglund-B.)

Put $C := H_*(X)$ *and* $A := H_*(\Omega X)$*. If a space* X *is formal and coformal over a field* $\Bbbk$ *there are isomorphisms*

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Proof idea.

It is well known that $H_*(LX) \cong HH^*(C_*(\Omega X))$. Hochschild cohomology respects weak equivalences:

$$HH^*(C_*(\Omega X)) \cong HH^*(A) = H_*(\mathrm{Hom}^\tau(BA, A))$$

Since C and A are Koszul dual we have a deformation retract

$$h \circlearrowright BA \underset{f}{\overset{g}{\rightleftarrows}} C$$

which yield the following deformation retract after taking $\mathrm{Hom}(-, A)$ and twisting.

$$h' \circlearrowright \mathrm{Hom}^\tau(BA, A) \underset{g'}{\overset{f'}{\rightleftarrows}} \mathrm{Hom}^\tau(C, A) \ .$$

□

Applications

Theorem (Berglund-B.)

Let $n \geq 2$ and suppose that M is an $(n-1)$-connected closed manifold of dimension $d \leq 3n-2$ such that $\dim H^*(M) > 4$. Choose a basis $x_1, \ldots, x_r$ for the indecomposables of $H^*(M)$ and let $c_{ij} = \langle x_i x_j, [M] \rangle$. The homology of the based loop space $H_*(\Omega M)$ is freely generated as an associative algebra by classes $u_1, \ldots, u_r$, with $|u_i| = |x_i| - 1$, modulo the single quadratic relation

$$\sum_{i,j} (-1)^{|x_i|} c_{ji} u_i u_j = 0.$$

There is a Koszul twisting morphism $H_*(M) \to H_*(\Omega M)$ given by $x_i^* \mapsto u_i$. There is an explicit complex computing $H_*(LM)$ given by

$$H^0(M) \otimes H_*(\Omega M) \xrightarrow{[\tau,-]} H^{0<i<d}(M) \otimes H_*(\Omega M) \xrightarrow{[\tau,-]} H^d(M) \otimes H_*(\Omega M),$$

where $[\tau, -]$ is the commutator with $\tau := \sum_i x_i \otimes u_i$.

Corollary

Let $\Bbbk$ *be any field and let* M *be an* $(n-1)$*-connected closed manifold of dimension at most* $3n-2$ *(*$n \geq 2$*) with* $\dim H^*(M;\Bbbk) > 4$*. Then the sequence* $\dim(H_n(LM;\Bbbk))$ *grows exponentially.*

Corollary

For a generic metric on M*, the number of geometrically distinct closed geodesics of length* $\leq T$ *grows exponentially in* T*.*

Thank you for listening!