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Mathematical Notes

Exponents for high-dimensional Gamma groups

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Abstract. The first purpose of this paper is to show the equivalence of the two classical definitions of the Gamma groups which appear in the Whitehead exact sequence involving the Hurewicz homomorphism: the Whitehead definition and that of Dold-Thom. Then we produce universal exponents for high-dimensional Gamma groups: if X is an $(m-1)$-connected CW-complex, the product of the exponents of the i-th stable homotopy groups of spheres, for $1 \leq i \leq n$, kills $\Gamma_{m+n}X$ if $n \leq m-2$. Finally, we generalize this result of the case where the groups $\pi_i X$ belong to a certain Serre class of abelian torsion groups for $1 \leq i < m$.

AMS Classification

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0 Introduction

For any 1-connected CW-complex X, J.H.C. Whitehead introduced in [9] the groups $\Gamma_n^W X$, for all $n \geq 2$, which appear in a certain exact sequence involving the homotopy groups of X and the integral homology groups of X:

$$\cdots \pi_{n+1}X \xrightarrow{h_{n+1}} H_{n+1}X \longrightarrow \Gamma_n^W X \longrightarrow \pi_n X \xrightarrow{h_n} H_N X \longrightarrow$$
$$\cdots \longrightarrow \Gamma_2^W X \longrightarrow \pi_2 X \xrightarrow{h_2} H_2 X \longrightarrow 0 \qquad (*)$$

where the h_* are the Hurewicz homomorphisms. Recall that $\Gamma_n^W X$ is defined as the image of the homomorphism $\pi_n X_{n-1} \longrightarrow \pi_n X_n$ induced by the inclusion of the $(n-1)$- skeleton X_{n-1} into X_n. Since X is 1-connected, it follows directly that the group $\Gamma_2^W X = 0$.

Another way to define such Gamma groups was given in [1] by A. Dold and R. Thom: They proved that $\pi_n SP^\infty X \cong H_n X$ where $SP^\infty X$ is the infinite symmetric product of X. The inclusion $X \hookrightarrow SP^\infty X$ has homotopy fiber ΓX and they set $\Gamma_n^{DT} X = \pi_n \Gamma X$: the homotopy long exact sequence of the fibration $\Gamma X \longrightarrow X \longrightarrow SP^\infty X$ is then an exact sequence similar to $(*)$.

What do we know about the Gamma groups? The answer is, not much! Let X be an $(m-1)$-connected CW-complex. If $m \geq 2$, then $\Gamma_m^W X = 0$, according

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to the Hurewicz theorem. If $m \geq 3$, $\Gamma^W_{m+1}X \cong \pi_m X/2\pi_m X$: this result is due to J.H.C. Whitehead (see [9], p. 291) and was shown in [8] for a special kind of spaces; a complete proof can be found in [2] (see p. 116, Theorem 2.4). Let us mention that G.W. Whitehead also proved the existence of an exact sequence $\pi_{m+2}X \longrightarrow H_{m+2}X \longrightarrow \pi_m X/2\pi_m X \longrightarrow \pi_{m+1}X \longrightarrow H_{m+1}X \longrightarrow 0$ for any $(m-1)$-connected CW-complex X with $m \geq 3$ (see [7] or [6], p. 555).

The purpose of the present paper is to show the equivalence of the two definitions of the Gamma groups given above and then the produce universal exponents for higher dimensional Gamma groups, more precisely for the $(m-2)$ first interesting Gamma groups for an $(m-1)$-connected CW-complex. This generalizes the fact that $2 \cdot \Gamma^W_{m+1}X = 0$ if $m \geq 3$.

In order to do this, we need a third definition of the Gamma groups denoted by $\Gamma_n X$. These Gamma groups should be seen as invariants of the space X which contain an important part of the homotopical information of X.

Our main result is the following assertion:

Theorem. *Let m and n be integers such that $1 \leq n \leq m-2$. If X is an $(m-1)$-connected CW-complex, then $(\Pi^n_{i=1}\varrho_i) \cdot \Gamma_{m+n}X = 0$, where ϱ_i denotes the exponent of the i-th stable homotopy group of spheres π^S_i.*

In the last part of the paper we extend this result of the case of more general spaces X which do not need to be highly connected, but for which we assume that their low-dimensional homotopy groups are torsion groups in a certain Serre class.

Theorem. *Let $m \geq 3$ and let P be a set of prime numbers containing all primes $p \leq (m+1)/2$. Let $\mathcal{C}$ be the Serre class of abelian P-torsion groups and X a 1-connected CW-complex such that $\pi_i X \in \mathcal{C}$ for $i < m$. Then*
(i) *$\Gamma_k X \in \mathcal{C}$ for $3 \leq k \leq 2m-2$.*
(ii) *$h_k : \pi_k X \longrightarrow H_k X$ is a $\mathcal{C}$-isomorphism for $3 \leq k \leq 2m-2$.*
(iii) *$h_{2m-1}X \longrightarrow H_{2m-1}X$ is a $\mathcal{C}$-epimorphism.*

The paper is organized as follows: The first section gives a new way of defining the Gamma groups of a 1-connected CW-complex and describes how they are involved in a Whitehead exact sequence (like (*)). Sections 2 and 3 shows that these new Gamma groups are isomorphic to the Dold-Thom Gamma groups and to the Whitehead Gamma groups. The main theorem is proved in Section 4. Finally Section 5 generalizes this result to the case of Serre clases of abelian groups as explained above.

Remark. Throughout the paper all homology groups are taken with integer coefficients and denoted $H_n X$.

1 A new definition of the Gamma groups

We first construct spaces $X\{n\}$, obtained from a 1-connected CW-complex X by attaching cells of dimension $\leq n+1$ in order to kill the homotopy groups $\pi_2 X, \ldots, \pi_n X$.

Let m be an integer ≥ 2, X an $(m-1)$-connected CW-complex; if J is a system of generators for $\pi_m X$ and $f : \bigvee_{\alpha\in J} S^m_\alpha \longrightarrow X$ a map such that $f|_{S^m_\alpha}$ represents α, we denote by Y^* the mapping cone of f.

Lemma 1.1. *(see [6], p. 556) Let m be an integer ≥ 2 and X an $(m-1)$-connected CW-complex. Then*
(i) *X^* is an m-connected space,*
(ii) *the pair (X^*, X) is m-connected.*

Remark 1.2. The space X^* is obtained from X by attaching $(m+1)$-cells, say $\{D^{m+1}_\alpha\}_{\alpha\in J}$. Thus $X^* = X \cup \bigvee_{\alpha\in J} D^{m+1}_\alpha$ and we obtain by excision that $H_q(X^*, X) \cong H_{q-1}(\bigvee_{\alpha\in J} S^m_\alpha) = 0$ for any $q \neq m+1$. In particular, $H_{m+2}X \cong H_{m+3}X^*$.

For a 1-connected CW-complex X, let us define $X\{1\} = X$ and inductively, since $X\{n\}$ is n-connected by the above lemma, $X\{n+1\} = X\{n\}^*$ for all integers $n \geq 2$.

Remark 1.3. A choice of a system of generators occurs in the definition of X^*, so the spaces $X\{n\}$ are not well-defined. However, we are only interested in a certain quotient of $\pi_{n+2}(X\{n\}, X)$ and we will prove that this is well-defined.

Definition 1.4. Let n be a positive integer, X a 1-connected CW-complex, let h_{n+2} denote the Hurewicz homomorphism $\pi_{n+2}X\{n\} \longrightarrow H_{n+2}X\{n\}$, and let $j_{n+2} : \pi_{n+2}X\{n\} \longrightarrow \pi_{n+2}(X\{n\}, X)$ be the homomorphism induced by the inclusion $j : (X\{n\}, *) \hookrightarrow (X\{n\}, X)$.

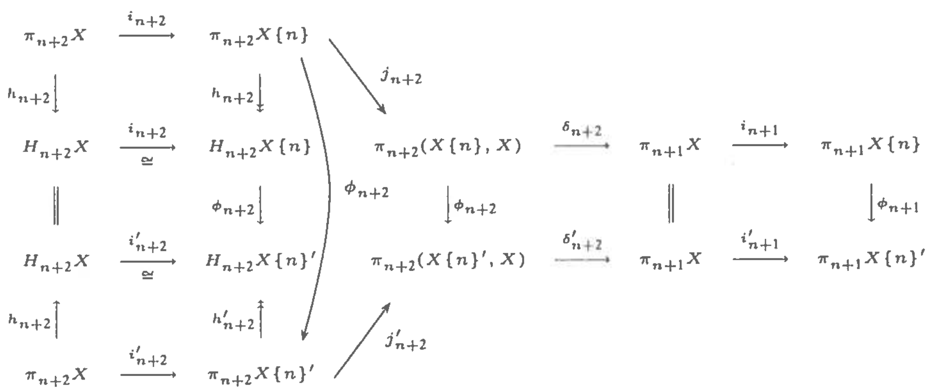
The define $\Gamma_{n+1}X = \pi_{n+2}(X\{n\}, X)/j_{n+2}(\mathrm{Ker}\, h_{n+2})$.

Proposition 1.5. *$\Gamma_{n+1}X$ is well-defined for all $n \geq 1$, i.e. $\Gamma_{n+1}X$ does not depend on the choice of the systems of generators for $\pi_2 X, \pi_3 X\{2\}, \ldots, \pi_n X\{n-1\}$.*

Proof. Let $X \subset \cdots \subset X\{n\}$ and $X \subset X\{2\}' \subset \cdots \subset X\{n\}'$ be two towers constructed with possibly different systems of generators. They define respectively $\Gamma_{n+1}X$ and $\Gamma'_{n+1}X$ as in Definition 1.4. First we use obstruction theory to extend the inclusion $X \hookrightarrow X\{n\}'$ to the space $X\{n\}$ and obtain a map $\phi : X\{n\} \longrightarrow X\{n\}'$. This map induces a homomorphism $\phi_{n+2} : \pi_{n+2}(X\{n\}, X) \longrightarrow \pi_{n+2}(X\{n\}', X)$

and since $\phi_{n+2}(j_{n+2}(\mathrm{Ker}\, h_{n+2})) \subset j'_{n+2}(\mathrm{Ker}\, h'_{n+2})$, we get a homomorphism $\Phi : \Gamma_{n+1}X \longrightarrow \Gamma'_{n+1}X$.

Consider the diagram below to prove that Φ is an isomorphism:



where the homomorphisms i_* and i'_* are induced by inclusions and where δ_* and δ'_* are the connecting homomorphisms of the appropriate homotopy sequences. In homology, the isomorphisms i_{n+2} and i'_{n+2} are isomorphisms by Remark 1.2. A simple diagram-chase shows that, for every $\xi \in \pi_{n+2}(X\{n\}', X)$, there exists an element $\omega \in \pi_{n+2}(X\{n\}, X)$ such that $\phi_{n+2}(\omega) - \xi \in j'_{n+2}(\mathrm{Ker}\, h'_{n+2})$.

Moreover, if $\zeta \in \pi_{n+2}(X\{n\}, X)$ and $\phi_{n+2}(\zeta) \in j'_{n+2}(\mathrm{Ker}\, h'_{n+2})$, then $\zeta \in j_{n+2}(\mathrm{Ker}\, h_{n+2})$. So Φ is an isomorphism. □

We prove now that, for any 1-connected CW-complex X and for all $n \geq 1$, our Gamma groups $\Gamma_{n+1}X$ have the same property as the groups $\Gamma^W_{n+1}X$ or $\Gamma^{DT}_{n+1}X$:

Theorem 1.6. *Let X be a 1-connected CW-complex. Then there is a long exact sequence*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_{n+2}X & \xrightarrow{h_{n+2}} & H_{n+2}X & \xrightarrow{\epsilon_{n+2}} & \Gamma_{n+1}X & & \\
 & \xrightarrow{\partial_{n+1}} & \pi_{n+1}X & \xrightarrow{h_{n+1}} & H_{n+1}X & \xrightarrow{\epsilon_{n+1}} & \cdots & & \\
\cdots & \xrightarrow[\epsilon_3]{} & 0 & \xrightarrow[\partial_2]{} & \pi_2 X & \xrightarrow[h_2]{} & H_2X & \longrightarrow & 0
\end{array}$$

Proof. Let us define the homomorphisms ε_{n+2} and ∂_{n+1} for any integer $n \geq 1$ by considering the commutative ladder below where the rows are exact:

$$
\begin{array}{ccccccccc}
\pi_{n+2}X & \xrightarrow{i_{n+2}} & \pi_{n+2}X\{n\} & \xrightarrow{j_{n+2}} & \pi_{n+2}(X\{n\},X) & \xrightarrow{\delta_{n+2}} & \pi_{n+1}X & \xrightarrow{i_{n+1}} & \pi_{n+1}X\{n\} \\
{\scriptstyle h_{n+2}}\downarrow & & {\scriptstyle h_{n+2}}\downarrow & & {\scriptstyle h_{n+2}}\downarrow & & {\scriptstyle h_{n+1}}\downarrow & & {\scriptstyle h_{n+1}}\downarrow{\scriptstyle \cong} \\
H_{n+2}X & \underset{i_{n+2}}{\xrightarrow{\cong}} & H_{n+2}X\{n\} & \longrightarrow & \underbrace{H_{n+2}(X\{n\},X)}_{=0} & \longrightarrow & H_{n+1}X & \underset{i_{n+1}}{\longrightarrow} & H_{n+1}X\{n\}
\end{array}
$$

The homomorphism j_{n+2} induces $\bar{j}_{n+2} : \pi_{n+2}X\{n\} \longrightarrow \pi_{n+2}(X_n, X)/j_{n+2}$ (Ker h_{n+2}) $= \Gamma_{n+1}X$, hence,there exists a homomorphism $\alpha_{n+2} : H_{n+2}X\{n\} \longrightarrow \Gamma_{n+1}X$ such that $a_{n+2} \circ h_{n+2} = \bar{j}_{n+2}$.

We define $\varepsilon_{n+2} = -a_{n+2} \circ i_{n+2} : H_{n+2}X \longrightarrow \Gamma_{n+1}X$. Finally, $\partial_{n+1} : \Gamma_{n+1}X \longrightarrow \pi_{n+1}X$ is induced by $\delta_{n+2} : \pi_{n+2}(X\{n\}, X) \longrightarrow \pi_{n+1}X$. The result now follows easily (observe that $\Gamma_2 X = 0$). □

Proposition 1.7. *A map $f : X \longrightarrow Y$ between two 1-connect CW-complexes induces a homomorphism $f_{n+1} : \Gamma_{n+1}X \longrightarrow \Gamma_{n+1}Y$ and the following diagram is commutative for all $n \geq 1$:*

$$
\begin{array}{ccccccccc}
\pi_{n+2}X & \xrightarrow{h_{n+2}} & H_{n+2}X & \xrightarrow{\varepsilon_{n+2}} & \Gamma_{n+1}X & \xrightarrow{\delta_{n+1}} & \pi_{n+1}X & \xrightarrow{h_{n+1}} & H_{n+1}X \\
{\scriptstyle f_{n+2}}\downarrow & & {\scriptstyle f_{n+2}}\downarrow & & {\scriptstyle f_{n+1}}\downarrow & & {\scriptstyle f_{n+1}}\downarrow & & {\scriptstyle f_{n+1}}\downarrow \\
\pi_{n+2}Y & \underset{h_{n+2}}{\longrightarrow} & H_{n+2}Y & \underset{\varepsilon_{n+2}}{\longrightarrow} & \Gamma_{n+1}Y & \underset{\partial_{n+1}}{\longrightarrow} & \pi_{n+1}Y & \underset{i_{n+1}}{\longrightarrow} & H_{n+1}Y
\end{array}
$$

Proof. The n-connectivity of $X\{n\}$ and $Y\{n\}$ allows us to use obstruction theory to extend f to a map $X\{n\} \longrightarrow Y\{n\}$. □

2 Comparison with the Dold-Thom Gamma groups

Let $SP^{\infty}X$ be the infinite symmetric product of a 1-connected space X, ΓX the homotopy fiber of the inclusion $p : X \hookrightarrow SP^{\infty}X$ and $\Gamma^{DT}_{n+1}X = \pi_{n+1}\Gamma X$ for all $n \geq 1$. In this section we prove that the homotopy exact sequence of the fibration $\Gamma X \xrightarrow{k} X \xrightarrow{p} SP^{\infty}X$,

$$
\begin{aligned}
\cdots \longrightarrow \; & \pi_{n+2}X \xrightarrow{p_{n+2}} \pi_{n+2}SP^{\infty}X \\
& \cong H_{n+2}X \xrightarrow{d_{n+2}} \pi_{n+1}\Gamma X \xrightarrow{k_{n+1}} \pi_{n+1}X \longrightarrow \cdots,
\end{aligned}
$$

is exactly the same as the sequence given by Theorem 1.6. In order to compare $\Gamma_{n+1}X$ with $\Gamma^{DT}_{n+1}X$, it is sufficient to construct a suitable homomorphism $\Gamma_{n+1}X \longrightarrow \Gamma^{DT}_{n+1}X$.

Lemma 2.1. *Let n be an integer ≥ 2, and let $F \longrightarrow E \xrightarrow{p} B$ and $F' \longrightarrow E' \xrightarrow{p'} B'$ be two fibrations such that $F \subset F'$, $E \subset E'$ and $B \subset B'$. If this last inclusion induces an epimorphism $\pi_{n+1}B \longrightarrow \pi_{n+1}B'$, an isomorphism $\pi_n B \cong \pi_n B'$ and a monomorphism $\pi_{n-1}B \longrightarrow \pi_{n-1}B'$ then*

$$\pi_n(F', F) \cong \pi_n(E', E).$$

Proof. This is simply a standard diagram-chase. □

Corollary 2.2. *Let X be a 1-connected CW-complex. Then the map $k : \Gamma X \longrightarrow X$ induces an isomorphism for all $n \geq 1$*

$$k_{n+2} : \pi_{n+2}(\Gamma(X\{n\}), \Gamma X) \xrightarrow{\cong} \pi_{n+2}(X\{n\}, X)$$

Proof. This is a direct application of Lemma 2.1 for the fibrations:

$$\begin{array}{ccccc} \Gamma X & \xrightarrow{k} & X & \xrightarrow{p} & SP^{\infty}X \\ \cap & & \cap & & \cap \\ \Gamma(X\{n\}) & \xrightarrow[k]{} & X\{n\} & \xrightarrow[p]{} & SP^{\infty}(X\{n\}). \end{array}$$

Theorem 2.3. *If X is a 1-connected CW-complex, then $\Gamma_{n+1}X \cong \Gamma^{DT}_{n+1}X$ for all $n \geq 1$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \pi_{n+2}SP^{\infty}(X\{n\}) & & \Gamma_{n+1}X & & \\ \uparrow{\scriptstyle p_{n+2}} & & \uparrow{\scriptstyle \pi} & & \\ \pi_{n+2}X\{n\} & \xrightarrow{j_{n+2}} & \pi_{n+2}(X\{n\}, X) & \xrightarrow{\delta_{n+2}} & \pi_{n+1}X \\ \uparrow{\scriptstyle k_{n+2}} & & {\scriptstyle \cong}\uparrow{\scriptstyle k_{n+2}} & & \uparrow{\scriptstyle k_{n+1}} \\ \pi_{n+2}\Gamma(X\{n\}) & \xrightarrow[j_{n+2}]{} & \pi_{n+2}(\Gamma X\{n\}, \Gamma X) & \xrightarrow[\delta_{n+2}]{} & \Gamma^{DT}_{n+1}X \end{array}$$

where the rows and the left column are exact and π is the projection given by the definition of $\Gamma_{n+1}X$.

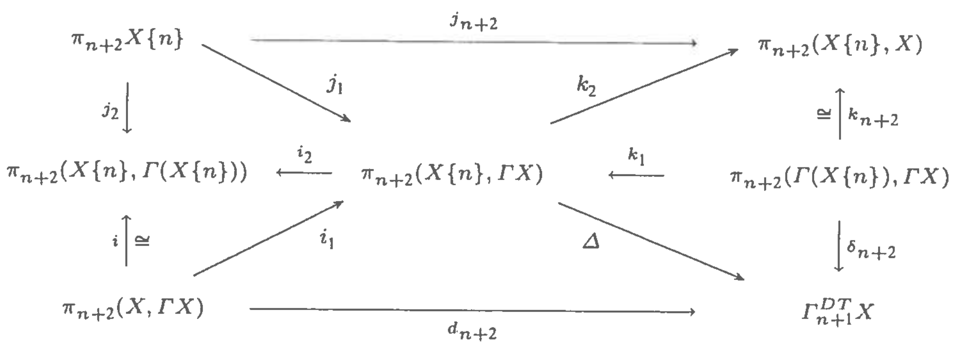
Define $\psi_{n+1} = \delta_{n+1} \circ k_{n+2}^{-1} \circ \pi^{-1} : \Gamma_{n+1}X \longrightarrow \Gamma_{n+1}^{DT}X$. It is easy to check that ψ_{n+1} is a well-defined homomorphism and, moreover, an isomorphism (its inverse is $\phi_{n+1} = \pi \circ k_{n+2} \circ \delta_{n+2}^{-1}$). $\square$

Theorem 2.4. *Let X be a 1-connected CW-complex. Then there is a commutative diagram:*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_{n+2}X & \xrightarrow{h_{n+2}} & H_{n+2}X & \xrightarrow{\epsilon_{n+2}} & \Gamma_{n+1}X & \xrightarrow{\partial_{n+1}} & \pi_{n+1}X & \longrightarrow & \cdots \\
 & & \| & & \downarrow{\scriptstyle l_{n+2}}\,\cong & & \cong\,\downarrow{\scriptstyle \psi_{n+1}} & & \| & & \\
\cdots & \longrightarrow & \pi_{n+2}X & \xrightarrow[p_{n+2}]{} & \pi_{n+2}SP^{\infty}X & \xrightarrow[d_{n+2}]{} & \Gamma_{n+1}^{DT}X & \xrightarrow[k_{n+1}]{} & \pi_{n+1}X & \longrightarrow & \cdots
\end{array}$$

where l_{n+2} is the Dold-Thom isomorphism for all $n \geq 1$.

Proof. The difficult point is to check the commutativity of the middle square. Remember that the isomorphism $l_{n+2} : H_{n+2}X \longrightarrow \pi_{n+2}SP^{\infty}X$ is natural ([1] p. 274) and consider the situation in the – apparently – more complicated diagram:



where $j_{n+2}, j_1, j_2, i, i_1, i_2, k_{n+2}, k_1, k_2$ are induced by the obvious inclusions and $\delta_{n+2}, d_{n+2}, \triangle$ are connecting homomorphisms. Each triangle is commutative and the three sequences having $\pi_{n+2}(X\{n\}, \Gamma X)$ as central group are exact. Define $\alpha = d_{n+2} \circ i^{-1} \circ j_2$ and $\beta = \delta_{n+2} \circ k_{n+2}^{-1} \circ j_{n+2}$. We will prove that $\alpha = -\beta$.

Set $I = i_1 \circ i^{-1} \circ i_2$, $K = k_1 \circ k_{n+2}^{-1} \circ k_2$ and $J = I + K$. So $\alpha = \triangle \circ I \circ j_1$, $\beta = \triangle \circ K \circ j_1$ and $\alpha + \beta = \triangle \circ J \circ j_1$.

But $k_2 \circ J = k_2$ and $J \circ i_1 = i_1$. Hence $\mathrm{Im}(J - \mathrm{Id}) \subset \mathrm{Ker}\, k_2 = \mathrm{Im}\, i_1$ and $J \circ (J - \mathrm{Id}) = J - \mathrm{Id}$. Finally, because $J^2 = J$, we have $J = \mathrm{Id}$, i.e. $\alpha + \beta = 0$.

We will use this last equality to prove that $\psi_{n+1} \circ \varepsilon_{n+2} = d_{n+2} \circ l_{n+2}$. But let us first look at the following commutative diagram, which enables us fully to understand the homomorphism ε_{n+2}.

$$\begin{array}{ccccccc}
H_{n+2}X & \xrightarrow{i_{n+2}} & H_{n+2}X\{n\} & \xrightarrow{a_{n+2}} & \Gamma_{n+1}X & \xleftarrow{\pi} & \pi_{n+2}(X\{n\}, X) \\
{\scriptstyle l_{n+2}}\downarrow\cong & & \cong\downarrow & & \uparrow{\scriptstyle \tilde{j}_{n+2}} & & \uparrow{\scriptstyle j_{n+2}} \\
\pi_{n+2}(X, \Gamma X) & \xrightarrow[i]{} & \pi_{n+2}(X\{n\}, \Gamma(X\{n\})) & \xleftarrow[\tilde{j}_2]{\cong} & \pi_{n+2}X\{n\}/\mathrm{Ker}\, h_{n+2} & \xleftarrow[\pi']{} & \pi_{n+2}X\{n\}
\end{array}$$

Here the maps $\tilde{j}_*$ are induced by j_* on the quotient $\pi_{n+2}X\{n\}/\mathrm{Ker}\, h_{n+2}$ and π and π' are the obvious projections. Recall that $\psi_{n+1} = \delta_{n+2} \circ k_{n+2}^{-1} \circ \pi^{-1}$ and $\varepsilon_{n+2} = -a_{n+2} \circ i_{n+2}$ (see the proof of Theorem 1.6 for the definition of a_{n+2}).

Thus $\varepsilon_{n+2} = -\tilde{j}_{n+2} \circ \tilde{j}_2^{-1} \circ i \circ l_{n+2}$ and

$$\begin{aligned}
\psi_{n+1} \circ \varepsilon_{n+2} &= -(\delta_{n+2} \circ k_{n+2}^{-1} \circ \underbrace{\pi^{-1}) \circ (\tilde{j}_{n+2}}_{j_{n+2} \circ \pi'^{-1}} \circ \tilde{j}_2^{-1} \circ i \circ l_{n+2}) \\
&= -\beta \circ \underbrace{\pi'^{-1} \circ \tilde{j}_2^{-1}}_{j_2^{-1}} \circ i \circ l_{n+2} \\
&= \alpha \circ j_2^{-1} \circ i \circ l_{n+2} \\
&= d_{n+2} \circ l_{n+2} \,.
\end{aligned}$$

□

Remark 2.5. The strange "−" sign in the definition of ε_{n+2} is necessary for the commutativity of the diagram given by Theorem 2.4. The unexpected bonus is that the same definition also works in Section 3, in which we compare $\Gamma_{n+1}X$ with $\Gamma_{n+1}^{W}X$.

Remark 2.6. The isomorphism $\psi_{n+1} : \Gamma_{n+1}X \xrightarrow{\cong} \Gamma_{n+1}^{DT}X$ is natural.

3 Comparison with the Whitehead Gamma groups

Let X be a 1-connected CW-complex and for all positive integers n, let X_n denote the n-skeleton of X. The inclusion $g : X_n \hookrightarrow X_{n+1}$ induces a homomorphism $g_{n+1} : \pi_{n+1}X_n \longrightarrow \pi_{n+1}X_{n+1}$. Then the Whitehead Gamma group $\Gamma^W_{n+1}X$ is defined as $\operatorname{Im} g_{n+1}$ and the Whitehead exact sequence is the sequence $(*)$ given in the introduction.

We obtain in this section the same results for $\Gamma^W_{n+1}X$ as for $\Gamma^{DT}_{n+1}X$ in Section 2. First, let us prove the next simple lemma to show how the skeleta of a CW-complex and the $\{n\}$-construction are related.

Lemma 3.1. *Let X be a 1-connected CW-complex, n an integer ≥ 1 and k an integer $\geq n+1$. Then $X\{n\}_k = X_k\{n\}$.*

Remark 3.2. The spaces $X\{n\}$ and $X_k\{n\}$ are not well-defined. The meaning of the equality above is the following: We can construct two particular $\{n\}$-spaces such that the equality holds.

Proof. Since, by induction over n, $\pi_n X\{n-1\} \cong \pi_n X\{n-1\}_k \cong \pi_n X_k\{n-1\}$ for $k \geq n+1$, we can choose the same system of generators J to build $X\{n\}$ and $X_k\{n\}$. So $X\{n\}_k = X_k\{n\}$, because we simply attach the same $(n+1)$-cells to both spaces. □

Lemma 3.3. *Let X be a 1-connected CW-complex and $n \geq 1$. Then there is an isomorphism $g_{n+2} : \pi_{n+2}(X_{n+1}\{n\}, X_{n+1}) \xrightarrow{\cong} \pi_{n+2}(X\{n\}, X)$.*

Proof. This follows from Lemma 3.1 and the Blakers-Massey theorem (cf [6], p. 366) for the triad $(X\{n\}; X, X\{n\}_{n+1})$.

Lemma 3.4. *Let X be a 1-connected CW-complex. Then $\Gamma_{n+1}X_{n+1} \cong \Gamma_{n+1}X$ for all $n \geq 1$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
0 = H_{n+2}X\{n\}_{n+1} & \xleftarrow{h_{n+2}} & \pi_{n+2}X\{n\}_{n+1} & \xrightarrow{j_{n+2}} & \pi_{n+2}(X\{n\}_{n+1}, X_{n+1}) \\
\downarrow & & \downarrow g_{n+2} & & \uparrow g_{n+2}\ \cong \\
H_{n+2}X\{n\} & \xleftarrow{h_{n+2}} & \pi_{n+2}X\{n\} & \xrightarrow{j_{n+2}} & \pi_{n+2}(X\{n\}, X)
\end{array}$$

Then $\Gamma_{n+1}X_{n+1} \cong \pi_{n+2}(X_{n+1}\{n\}, X_{n+1})/j_{n+2}(\operatorname{Ker} h_{n+2}) \cong \pi_{n+2}(X_{n+1}\{n\}, X_{n+1})\ /\operatorname{Im} j_{n+2} \cong \pi_{n+2}(X\{n\}, X)/\operatorname{Im}(j_{n+2} \circ g_{n+2}) \cong \pi_{n+2}(X\{n\}, X)/ j_{n+2}(\operatorname{Ker} h_{n+2}) = \Gamma_{n+1}X$, since one can check that $\operatorname{Im} g_{n+2}$ $\operatorname{Ker} h_{n+2}$. □

Theorem 3.5. *Let X be a 1-connected CW-complex. Then $\Gamma^W_{n+1}X \cong \Gamma_{n+1}X$ for all $n \geq 1$.*

Proof. Because of Lemma 3.4, it is sufficient to establish to isomorphism $\Gamma^W_{n+1}X \cong \Gamma_{n+1}X_{n+1}$.

The commutative square

$$\begin{array}{ccc} \pi_{n+1}X_{n+1} & \xrightarrow{j_{n+1}} & \pi_{n+1}(X_{n+1}, X_n) \\ {\scriptstyle h_{n+1}}\downarrow & & \downarrow{\scriptstyle \cong} \\ H_{n+1}X_{n+1} & \underset{j_{n+1}}{\hookrightarrow} & H_{n+1}(X\{n\}, X_n) \end{array}$$

Thus the group $\Gamma^W_{n+1}X = \operatorname{Im} g_{n+1} = \operatorname{Ker} j_{n+1} = \operatorname{Ker} h_{n+1} = \operatorname{Im}(\partial_{n+1} : \Gamma_{n+1}X_{n+1} \longrightarrow \pi_{n+1}X_{n+1}) \cong \Gamma_{n+1}X_{n+1}$ since $H_{n+2}X_{n+1} = 0$. The isomorphism $\Gamma^W_{n+1}X \longrightarrow \Gamma^W_{n+1}$ is then given by the composition $\Gamma^W_{n+1}X \hookrightarrow \pi_{n+1}X_{n+1} \xleftarrow{\partial_{n+1}} \Gamma_{n+1}X_{n+1} \xrightarrow{\cong} \Gamma_{n+1}X$. □

Remark 3.6. Lemma 3.4 could be seen as a direct application of Theorem 3.5, because, by definition, $\Gamma^W_{n+1}X_{n+1} = \Gamma^W_{n+1}X$.

The same argument as in the proof of Theorem 2.4 yields the following:

Theorem 3.7. *Let X be a 1-connected CW-complex and $n \geq 1$. Then we have a commutative diagram:*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_{n+2}X & \xrightarrow{h_{n+2}} & H_{n+2}X & \longrightarrow & \Gamma^W_{n+1}X & \longrightarrow & \pi_{n+1}X & \longrightarrow & \cdots \\ & & \| & & \| & & \downarrow{\scriptstyle \cong} & & \| & & \\ \cdots & \longrightarrow & \pi_{n+2}X & \underset{h_{n+2}}{\longrightarrow} & H_{n+2}X & \underset{\varepsilon_{n+2}}{\longrightarrow} & \Gamma_{n+1}X & \underset{\partial_{n+1}}{\longrightarrow} & \pi_{n+1}X & \longrightarrow & \cdots \end{array}$$

Remark 3.8. The isomorphism $\Gamma_{n+1}X \cong \Gamma^W_{n+1}X$ is natural.

As corollary of Sections 2 and 3, we obtain this interesting result:

Corollary 3.9. *Let X be a 1-connected CW-complex. Then $\Gamma^{DT}_{n+1}X \cong \Gamma^W_{n+1}X$ for all $n \geq 1$.* □

Remark 3.10. This corollary implies in particular that $\pi_{n+1}\Gamma X \cong \pi_{n+1}\Gamma(X_{n+1})$ for every 1-connected CW-complex and all integers $n \geq 1$.

4 Exponents for the Gamma groups

This section is devoted to the proof of the main theorem of the present paper. First, we need two results about certain homotopy groups of highly connected spaces.

Lemma 4.1. *Let m and n be positive integers such that $n \leq m-2$ and let X be an $(m-1)$-connected CW-complex. If ϱ_k denotes the exponent of the k-th stable homotopy group of spheres π_k^S, then the group $\pi_{m+n+1}(X\{m+n-k\}, X\{m+n-k-1\})$ is killed by ϱ_k for all $1 \leq k \leq n$.*

Proof. Recall that a space $X\{m+n-k\}$ is obtained from $X\{m+n-k-1\}$ by attaching $(m+n-k+1)$-cells. More precisely, if $\pi_{m+n-k}X\{m+n-k-1\}$ is generated by J, then $X\{m+n-k\}$ is the mapping cone of a map $f : \bigvee_{\alpha\in J} S_\alpha^{m+n-k} \longrightarrow X\{m+n-k-1\}$, such that $f|_{S_\alpha^{m+n-k}}$ represents α.

We call the new cells $D_\alpha^{m+n-k+1}$ for $\alpha \in J$.

By the Blakers-Massey theorem ([6] p. 366) for the triad $(X\{m+n-k\}; X\{m+n-k-1\}, \bigvee_{\alpha\in J} D_\alpha^{m+n-k+1})$,

$$\begin{aligned}
&\pi_{m+n+1}(X\{m+n-k\}, X\{m+n-k-1\})\\
&\cong \pi_{m+n+1}\Big(\bigvee_{\alpha\in J} D_\alpha^{m+n-k+1}, \bigvee_{\alpha\in J} S_\alpha^{m+n-k}\Big)\\
&\cong \pi_{m+n}\Big(\bigvee_{\alpha\in J} S_\alpha^{m+n-k}\Big) \cong \bigoplus_{\alpha\in J} \pi_{m+n} S_\alpha^{m+n-k} \cong \bigoplus \pi_k^S
\end{aligned}$$

since $m+n-k \geq k+2$ by hypothesis. □

By induction over k we now deduce:

Corollary 4.2. *Let m and n be positive integers such that $n \leq m-2$ and let X be an $(m-1)$-connected CW-complex. If $k \leq n$, then $\pi_{m+n+1}(X\{m+n-1\}, X\{m+n-k-1\})$ is killed by the product $\prod_{i=1}^{k} \varrho_i$.* □

With these preparations we can now state the main theorem.

Theorem 4.3. *Let m and n be positive integers such that $n \leq m-2$. If X is an $(m-1)$-connected CW-complex, then*

$$\prod_{i=1}^{n} \varrho_i \cdot \Gamma_{m+n} X = 0$$

where ϱ_i denotes the exponent of ϱ_i^S.

Proof. Since X is $(m-1)$-connected, we can choose $X\{m-1\} = X$. By Corollary 4.2, $\prod_{i=1}^{n} \varrho_i$ kills $\pi_{m+n+1}(X\{m+n-1\}, X\{m-1\})$. The result now follows easily, since $\Gamma_{m+n} X$ is a quotient of this group. □

Remark 4.4. A prime number p divides the product $\prod_{i=1}^{n} \varrho_i$ if and only if $p \leq (n+3)/2$ ([5] p. 285). This fact will be used in the generalization to Serre classes (Section 5).

Remark 4.5. If we set $n = 1$ and $m \geq 3$, we obtain the well-known result that $\Gamma_{m+1} X$ is killed by $\varrho_1 = 2$.

Corollary 4.6. *Let X be an $(m-1)$-connected CW-complex with $m \geq 3$. The Hurewicz homomorphism $h_{m+n} : \pi_{m+n} X \longrightarrow H_{m+n} X$ satisfies:*

(i) $\prod_{i=1}^{n} \varrho_i \cdot \operatorname{Ker} h_{m+n} = 0$ *for any integer* $n \leq m-2$.

(ii) $\prod_{i=1}^{n-1} \varrho_i \cdot \operatorname{Coker} h_{m+n} = 0$ *for any integer* $n \leq m-1$. □

5 Generalization to Serre classes

Serre introduced in [5] the notion of a Serre class of abelian groups. His goal was to generalize classical theorems, for example, the Hurewicz theorem:

Let $\mathcal{C}$ be a Serre class and X a 1-connected space such $\pi_i X \in \mathcal{C}$ for all $i \leq n-1$. Then the Hurewicz homomorphism $h_n : \pi_n X \longrightarrow H_n X$ is a $\mathcal{C}$-isomorphism, i.e. $\operatorname{Ker} h_n$ and $\operatorname{Coker} h_n$ are both in $\mathcal{C}$.

In this section we generalize Theorem 4.3 to the case of spaces X whose low-dimensional homotopy groups, say up to dimension $m-1$, belong to a certain Serre classe of abelian torsion groups (instead of being trivial).

Theorem 5.1. *Let $m \geq 2$ and let P be a set of prime numbers containing all primes $p \leq (m+1)/2$. Let $\mathcal{C}$ be the Serre class of abelian P-torsion groups and X a 1-connected CW-complex such that $\pi_i X \in \mathcal{C}$ for $i < m$. Then*
(i) $\Gamma_k X \in \mathcal{C}$ *for* $2 \leq k \leq 2m-2$.
(ii) $h_k : \pi_k X \longrightarrow H_k X$ *is a* $\mathcal{C}$*-isomorphism for* $2 \leq k \leq 2m-2$.
(ii) $h_{2m-1} : \pi_{2m-1} X \longrightarrow H_{2m-1} X$ *is a* $\mathcal{C}$*-epimorphism.*

Proof. Let us build $\widetilde{X}$ the $(m-1)$-connected cover of X, i.e. the fiber of the $(m-1)$-st Postnikov section of X, and consider the commutative diagram where the homomorphism j_* are induced by the inclusion $j : \widetilde{X} \hookrightarrow X$.

$$\begin{array}{ccccccccc}
\pi_{k+1}\widetilde{X} & \xrightarrow{h_{k+1}} & H_{k+1}\widetilde{X} & \xrightarrow{\epsilon_{k+1}} & \Gamma_k\widetilde{X} & \xrightarrow{\partial_k} & \pi_k\widetilde{X} & \xrightarrow{h_k} & H_k\widetilde{X} \\
\downarrow{\scriptstyle j_{k+1}} & & \downarrow{\scriptstyle j_{k+1}} & & \downarrow{\scriptstyle j_k} & & \downarrow{\scriptstyle j_k} & & \downarrow{\scriptstyle j_k} \\
\pi_{k+1}X & \xrightarrow[h_{k+1}]{} & H_{k+1}X & \xrightarrow[\epsilon_{k+1}]{} & \Gamma_k X & \xrightarrow[\partial_k]{} & \pi_k X & \xrightarrow[h_k]{} & H_k X
\end{array}$$

Since $j_k : \pi_k\widetilde{X} \longrightarrow \pi_k X$ is a $\mathcal{C}$-isomorphism for all integers $k \geq 2$, we can use the $\mathcal{C}$-version of the Whitehead theorem ([5] Théorème 3, p. 276) and deduce that $j_k : H_j\widetilde{X} \longrightarrow H_k X$ is also a $\mathcal{C}$-isomorphism for all integers $k \geq 2$. By the $\mathcal{C}$-version of the five lemma, $j_k : \Gamma_k\widetilde{X} \longrightarrow X$ is a $\mathcal{C}$-isomorphism for all integers $k \geq 2$.

We deduce now from Theorem 4.3 that the product $\prod_{i=1}^{n} \varrho_i$ kills the group $\Gamma_{m+n}\widetilde{X}$ if $n \leq m-2$. Hence $\Gamma_{m+n}\widetilde{X}$ and $\Gamma_{m+n}X \in \mathcal{C}$, since a prime p divides $\prod_{i=1}^{n} \varrho_i$ if and only if $p \leq (n+3)/2 \leq (m+1)/2$ (cf Remark 4.4). Assertions (ii) and (iii) are easy consequences of (i). □

Remark 5.2. The same statement is true if $\mathcal{C}$ is the Serre class of abelian groups of finite exponent.

To show how strong this theorem is, we state an example.

Example 5.3. Consider the 1-connected space $BSL\mathbb{Z}^+$ obtained by performing the plus construction on the classifying space of the infinite special linear group over the integers, and let $\mathcal{C}$ be the Serre class of 2 and 3-torsion groups. It is known that $K_2\mathbb{Z}$ and $K_3\mathbb{Z} \in \mathcal{C}$ and that $K_4\mathbb{Z} = 0$ (see [3] and [4]).

The previous theorem (with $m = 5$) implies:

$h_k : K_n\mathbb{Z} \longrightarrow H_n(SL\mathbb{Z})$ is a $\mathcal{C}$-isomorphism for all $2 \leq n \leq 8$.

$h_9 : K_9\mathbb{Z} \longrightarrow H_9(SL\mathbb{Z})$ is a $\mathcal{C}$-epimorphism.

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