DUALITY, DESCENT AND EXTENSIONS

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Homotopical Algebra and its Applications
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Brooke Shipley (model category foundations)Alexander Berglund (application to duality and descent)

Joint work with...

MOTIVATION

Understand relationships among

- Koszul duality of dg algebras
- homotopic Grothendieck descent along morphisms of dg algebras
- homotopic Hopf-Galois extensions of dg comodule algebras

Inspired by a remark of Jack Morava and by discussions with Michael Ching.

This talk concerns only the dg world, but many of the definitions and results extend to objects in any nice enough monoidal model category.

OUTLINE

1 CORINGS AND THEIR COMODULES

2 THE HOPF-GALOIS FRAMEWORK

3 THE DUALITY THEOREMS

CONVENTIONS

- k is a commutative ring, and $\otimes = \otimes_k$.
- dg="differential graded"= \exists underlying nonnegatively graded chain complex of \Bbbk -modules.

CORINGS AND THEIR COMODULES

CORINGS

Let A be a dg k-algebra.

An A-coring is a comonoid in the monoidal category $({}_{A}\mathcal{M}_{A}, \otimes_{A}, A)$ of A-bimodules.

CORINGS

Let A be a dg k-algebra.

An A-coring is a comonoid in the monoidal category $({}_{A}\mathcal{M}_{A}, \otimes_{A}, A)$ of A-bimodules, i.e., consists of

- an A-bimodule V,
- an A-bimodule morphism

$$\delta: V \to V \otimes_A V$$

that is coassociative and counital with respect to an A-bimodule morphism.

$$\varepsilon: V \to A$$
.

COMODULES OVER CORINGS

Let A be a dg k-algebra, and let V be an A-coring.

NOTATION

- \mathcal{M}_A = the category of right A-modules.
- \mathcal{M}_{A}^{V} = the category of V-comodules in \mathcal{M}_{A} .

$$(M, \alpha) \in \mathcal{M}_A^V \Longrightarrow \left\{ egin{array}{l} M \in \mathcal{M}_A \\ \alpha : M \to M \otimes_A V ext{ morphism in } \mathcal{M}_A, ext{ coaction} \end{array}
ight.$$

INDUCED ADJUNCTIONS

For any morphism of A-corings

$$g: V \rightarrow W$$
,

there is an "extension of coefficients" adjunction

$$\mathbb{M}_{A}^{V} \xrightarrow{g_{*}} \mathbb{M}_{A}^{W}$$

THEOREM (H.-SHIPLEY)

For any dg k-algebra A, there is a combinatorial model category structure on \mathcal{M}_A such that

- the cofibrations are the injections, and
- the weak equivalences are the quasi-isomorphims.

Proof by applying fancy machinery due to Beke, Lurie, and J. Smith to the usual projective model category structure on \mathfrak{M}_A .

THEOREM (H.-SHIPLEY)

Suppose that k is semihereditary. Let A be a dg k-algebra such that $H_1A=0$.

If V is an A-coring such that V is A-semifree on X, where

- $H_0(\Bbbk \otimes_A V) = \Bbbk$ and $H_1(\Bbbk \otimes_A V) = 0$, and
- X_n is k-free and finitely generated for all n,

then \mathcal{M}_A^V admits a model category structure such that

- the cofibrations are the injections, and
- the weak equivalences are the quasi-isomorphims.

 Special case of a general existence theorem for model category structure on categories of coalgebras over a comonad, in which the required factorizations



are constructed by induction on a filtration of weak equivalences by *n*-equivalences.

• Have considerable control over the fibrant objects in \mathcal{M}_A^V , e.g., under reasonable conditions, fibrant replacements given by cobar constructions.

• If \mathcal{M}_A is endowed with the model structure of the first theorem, then

$$\mathcal{M}_{A}^{V} \xrightarrow{forget} \mathcal{M}_{A}$$

is a Quillen pair.

 If g: V → W is a morphism of A-corings satisfying the hypotheses above, then

$$\mathcal{M}_A^V \xrightarrow[-\square_W V]{g_*} \mathcal{M}_A^W$$

is a Quillen pair.

THE HOPF-GALOIS FRAMEWORK

HOPF-GALOIS DATA

$$\varphi: \mathbf{A} \to \mathbf{B}^{\circlearrowright H}$$

where

- H is a dg k-Hopf algebra with comultiplication
 Δ : H → H ⊗ H;
- A is a dg k-algebra seen as an H-comodule with trivial coaction;
- B is an H-comodule algebra, with H-coaction
 ρ: B → B ⊗ H;
- $\varphi : A \rightarrow B$ is a morphism of H-comodule algebras.

THE "NORMAL BASIS" EXAMPLE

Let *C* be a coaugmented dg coalgebra, *M* a right *C*-comodule and *N* a left *C*-comodule.

 $\Omega(M; C; N)$ is the two-sided cobar construction on C with coefficients in M and N, i.e.,

$$\Omega(M; C; N) = (M \otimes Ts^{-1}\overline{C} \otimes N, d_{\Omega}),$$

where

- C is the coaugmentation coideal,
- s^{-1} denotes desuspension,
- $TX = \bigoplus_{n>0} X^{\otimes n}$ for any graded \mathbb{k} -module X, and
- d_{Ω} is given in terms of the differentials on C, M and N and of the coactions and comultiplication.

THE "NORMAL BASIS" EXAMPLE

Let H be a dg k-Hopf algebra, and let (E, γ) be an H-comodule algebra.

PROPOSITION (H.-LEVI)

There is Hopf-Galois data

$$\varphi_{\gamma}: \Omega(E; H; \mathbb{k}) \hookrightarrow \Omega(E; H; H)^{\circlearrowright H}$$

where the H-coaction on $\Omega(E; H; H)$ is given by applying Δ on the last tensor factor.

Classically, the coinvariants of a coaction $M \to M \otimes C$ of a coalgebra C are defined to be

$$M^{coC} = M \square_C \mathbb{k}$$
.

Analogously, the homotopy coinvariant algebra of $\rho: B \to B \otimes H$ is

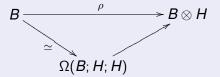
$$B^{hco\,H}=\Omega(B;H;\mathbb{k}).$$

REMARK

The definition

$$B^{hco\,H}=\Omega(B;H;\mathbb{k})$$

is reasonable because



is a fibrant replacement in the category \mathcal{M}^H of H-comodules and

$$\Omega(B; H; \mathbb{k}) = \Omega(B; H; H) \square_H \mathbb{k}.$$

The Hopf *B*-coring associated to $\rho: B \to B \otimes H$:

$$(B \otimes H, \delta_{
ho}, \varepsilon_{
ho})$$

where

ullet $\delta_{
ho}$ is the composite

$$B \otimes H \xrightarrow{B \otimes \Delta} B \otimes H \otimes H \cong (B \otimes H) \otimes_B (B \otimes H);$$

•
$$\varepsilon_{\rho} = B \otimes \varepsilon : B \otimes H \rightarrow B$$
;

Constructions associated to $A \stackrel{\varphi}{\rightarrow} B^{\ominus H}$

The Hopf *B*-coring associated to $\rho: B \to B \otimes H$:

$$(B \otimes H, \delta_{\rho}, \varepsilon_{\rho})$$

where

• the left *B*-action on $B \otimes H$ is

$$B \otimes B \otimes H \xrightarrow{\mu_B \otimes H} B \otimes H$$
;

• the right *B*-action on $B \otimes H$ is

$$B \otimes H \otimes B \xrightarrow{B \otimes H \otimes \rho} B \otimes H \otimes B \otimes H \cong B^{\otimes 2} \otimes H^{\otimes 2} \xrightarrow{\mu_B \otimes \mu_H} B \otimes H.$$

The canonical *B*-coring associated to $\varphi : A \rightarrow B$:

$$(B \otimes_A B, \delta_{\varphi}, \varepsilon_{\varphi})$$

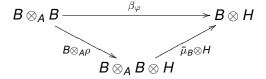
where

ullet δ_{arphi} is the composite

$$B \otimes_A B \cong B \otimes_A A \otimes_A B \xrightarrow{B \otimes_A \varphi \otimes_A B} B \otimes_A B \otimes_A B \cong (B \otimes_A B) \otimes_B (B \otimes_A B);$$

• $\varepsilon_{\varphi} = \bar{\mu}_B: B \otimes_A B \to B$ is induced by the multiplication map μ_B of B.

The Hopf-Galois map associated to $\varphi : A \to B$ and $\rho : B \to B \otimes H$



is a morphism of B-corings. It therefore induces an "extension of coefficients" adjunction

$$\mathcal{M}_{B}^{B\otimes_{A}B} \xrightarrow{\overset{(\beta_{\varphi})_{*}}{\bot}} \mathcal{M}_{B}^{B\otimes H}$$

A dual construction, perhaps somewhat more familiar and easier to understand...

Let G be a group acting on a set X, via

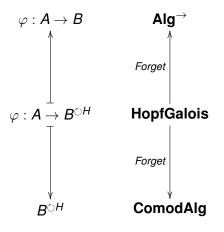
$$\rho: X \times G \rightarrow X: (x, a) \mapsto x \cdot a$$
.

Let $Y = X_G$, the set of G-orbits. Take the pullback $X \times_Y X$ of the quotient map $X \to Y$ along itself.

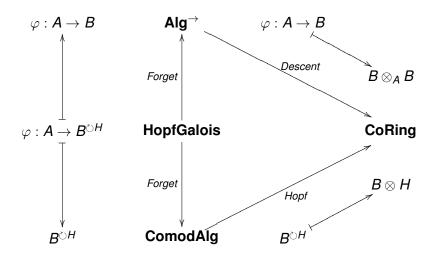
There is a natural map

$$\beta: X \times G \rightarrow X \times_Y X: (x, a) \mapsto (x, x \cdot a).$$

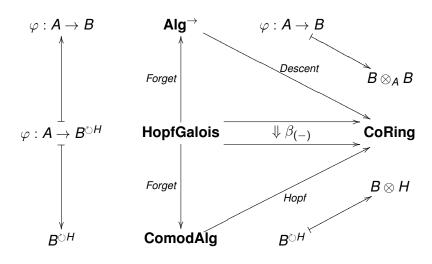
Constructions associated to $A \xrightarrow{\varphi} B^{\circlearrowright H}$



Constructions associated to $A \xrightarrow{\varphi} B^{\circlearrowright H}$



Constructions associated to $A \xrightarrow{\varphi} B^{\circlearrowright H}$



Suppose henceforth that all categories in sight are model categories and all adjunctions are Quillen pairs.

We'll see explicit examples later where this is the case.

H is a generalized Koszul dual of A if there is a Quillen equivalence

$$\mathcal{M}_{A} \xrightarrow{\underline{\hspace{1cm}}} \mathcal{M}^{H}.$$

MOTIVATION (LEFÈVRE)

If \Bbbk is a field, and $\tau: C \to A$ is an acyclic twisting cochain, there is a model category structure on the category of unbounded, cocomplete C-comodules such that the functor $\mathfrak{M}^C \to \mathfrak{M}_A$ induced by τ is the left member of a Quillen equivalence.

If A is a Koszul algebra, and C is its Koszul dual coalgebra, then the canonical twisting cochain $\tau:C\to A$ is acyclic, so $\mathbb{M}^C\to \mathbb{M}_A$ does indeed fit into a Quillen equivalence.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowright H}$

EXAMPLE

Since the universal twisting cochain $H \to \Omega H$ is acyclic, H is always a generalized Koszul dual of ΩH .

EXAMPLE

Let $\mathcal B$ denote the (reduced) bar construction. If A is a commutative dg algebra, then $\mathcal BA$ is naturally a commutative Hopf algebra.

Since the couniversal twisting cochain $\mathcal{B}A \to A$ is acyclic, $\mathcal{B}A$ is a generalized Koszul dual of A.

 φ satisfies effective homotopic Grothendieck descent if

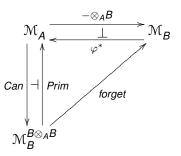
$$\mathcal{M}_A \xrightarrow{Can} \mathcal{M}_B^{B\otimes_A B}.$$

is a Quillen equivalence, where $Can(M) = (M \otimes_A B, \psi_M)$, with ψ_M equal to the composite

$$M \otimes_A B \cong M \otimes_A A \otimes_A B \xrightarrow{M \otimes_A \varphi \otimes_A B} M \otimes_A B \otimes_A B \cong (M \otimes_A B) \otimes_B (B \otimes_A B).$$

 $\mathcal{M}_{B}^{B\otimes_{A}B}$ is exactly the category of descent data associated to φ .

The meaning of descent: which B-modules are weakly equivalent to $M \otimes_A B$ for some $M \in \mathcal{M}_A$?



EXAMPLE

For any *H*-comodule algebra (E, γ) , the inclusion

$$A = \Omega(E; H; \mathbb{k}) \hookrightarrow \Omega(E; H; H) = B$$

satisfies effective homotopic Grothendieck descent. In fact, $Can : \mathcal{M}_A \to \mathcal{M}_B^{B \otimes_A B}$ is an actual equivalence of categories.

 φ is a homotopic *H*-Hopf-Galois extension if

• the natural map $i_{\varphi}: A \to \Omega(B; H; \mathbb{k}) = B^{hco\,H}$ induces a Quillen equivalence

$$\mathcal{M}_{A} \xrightarrow{-\otimes_{A}B^{hco\,H}} \mathcal{M}_{B^{hco\,H}}.$$

the "extension of coefficients" adjunction

$$\mathcal{M}_{B}^{B\otimes_{A}B} \xrightarrow{(eta_{arphi})_{*}} \mathcal{M}_{B}^{B\otimes H}$$

is a Quillen equivalence.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowright H}$

- Hopf-Galois extensions generalize ordinary Galois extensions of rings (where H is the dual of a group algebra).
- Faithfully flat Hopf-Galois extensions over the coordinate ring of an affine group scheme G correspond to G-principal fiber bundles.
- Homotopic Hopf-Galois extensions were first introduced by Rognes for ring spectra, e.g., $S \xrightarrow{\eta} MU^{\odot S[BU]}$.
- One can study Hopf algebras via their associated Hopf-Galois extensions.

EXAMPLE

For any *H*-comodule algebra (E, γ) , the inclusion

$$A = \Omega(E; H; \mathbb{k}) \hookrightarrow \Omega(E; H; H)^{\circlearrowleft H} = B^{\circlearrowleft H}$$

is an H-Hopf-Galois extension. In fact,

$$\mathcal{M}_{B}^{B\otimes_{A}B} \xrightarrow{(\beta_{\varphi})_{*}} \mathcal{M}_{B}^{B\otimes H}$$

is an actual equivalence of categories.

THE DUALITY THEOREMS

HOPF-GALOIS ⇔ GROTHENDIECK

THEOREM (BERGLUND-H.)

Let $A \xrightarrow{\varphi} B^{\circlearrowright H}$ be Hopf-Galois data. If $i_{\varphi}^*: \mathcal{M}_{B^{hco}H} \to \mathcal{M}_A$ is a Quillen equivalence, then

 φ is a homotopic H-Hopf-Galois extension



 φ satisfies homotopic Grothendieck descent.

Homotopic version of a "faithfully flat descent" result due to Schneider, as reformulated by Schauenburg.

$Koszul \Rightarrow (Hopf-Galois \Leftrightarrow Grothendieck)$

THEOREM (BERGLUND-H.)

Let $A \xrightarrow{\varphi} B^{\circlearrowright H}$ be Hopf-Galois data. If $\Bbbk \xrightarrow{\simeq} B$, then

 $\emph{\emph{i}}_{arphi}^*: \mathfrak{M}_{\emph{\emph{B}}^{\textit{hco}\,\textit{H}}}
ightarrow \mathfrak{M}_{\emph{\emph{A}}}$ is a Quillen equivalence



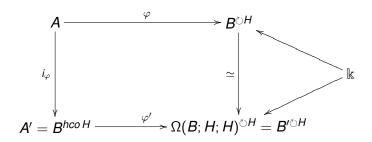
H is a generalized Koszul dual of A.

REMARK

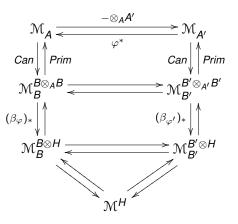
If $\mathbb{k} \xrightarrow{\simeq} B$, then H is always a generalized Koszul dual of $B^{hco\,H}$.

THE PROOF

Replace φ by a "normal basis" extension.



THE PROOF



THE PROOF

REMARK

From the previous diagram, we see that if H is a generalized Koszul dual of A, and φ satisfies homotopic Grothendieck descent, then $B \otimes_A B$ is a sort of "relative" generalized Koszul dual of A, which is what Jack intuited.

A SOURCE OF EXAMPLES

(Work in progress with A. Berglund, in the spirit of the "normal basis" basis example.)

Let K be a dg k-Hopf algebra and A a dg k-algebra.

A twisting cochain $\tau: K \to A$ is Hopf-Hirsch if $A \otimes_{\tau} K$ admits a multiplication such that

$$A \hookrightarrow A \otimes_{\tau} K \to K$$

is a sequence of algebra maps. Indeed there is a functor

$$\mathsf{Twist}_{\mathsf{HH}} \to \mathsf{HopfGalois} : (K \xrightarrow{\tau} A) \mapsto (A \to A \otimes_{\tau} K^{\circlearrowright K}).$$

A SOURCE OF EXAMPLES

Suppose that $H_1A=0=H_1K$, $H_0K=\Bbbk$ and K is \Bbbk -free of finite type.

If $\tau: K \to A$ is Hopf-Hirsch, then the extension

$$A \hookrightarrow A \otimes_{\tau} K^{\circlearrowright K}$$

is homotopic K-Hopf-Galois and satisfies homotopic Grothendieck descent.



Joyeux Anniversaire, cher Yves!