Duality, descent and extensions

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Homotopical Algebra and its Applications
CIRM
25 June 2012

Joint work with...

- Brooke Shipley (model category foundations)
- Alexander Berglund (application to duality and descent)

Motivation

Understand relationships among

- Koszul duality of dg algebras
- homotopic Grothendieck descent along morphisms of dg algebras
- homotopic Hopf-Galois extensions of dg comodule algebras

Inspired by a remark of Jack Morava and by discussions with Michael Ching.

This talk concerns only the dg world, but many of the definitions and results extend to objects in any nice enough monoidal model category.

Outline

Conventions

- $\mathbb{k}$ is a commutative ring, and $\otimes = \otimes_{\mathbb{k}}$.
- dg=“differential graded”=$\exists$ underlying nonnegatively graded chain complex of $\mathbb{k}$-modules.

Corings and their comodules

Corings

Let A be a dg $\mathbb{k}$-algebra.

An A-coring is a comonoid in the monoidal category $({}_A\mathcal{M}_A, \otimes_A, A)$ of A-bimodules.

Corings

Let A be a dg $\Bbbk$-algebra.

An *A-coring* is a comonoid in the monoidal category $({}_A\mathcal{M}_A, \otimes_A, A)$ of A-bimodules, i.e., consists of

- an A-bimodule V,
- an A-bimodule morphism

$$\delta : V \to V \otimes_A V$$

  that is coassociative and counital with respect to an A-bimodule morphism.

$$\varepsilon : V \to A.$$

Comodules over corings

Let A be a dg $\Bbbk$-algebra, and let V be an A-coring.

Notation

- $\mathcal{M}_A =$ *the category of right A-modules.*
- $\mathcal{M}_A^V =$ *the category of V-comodules in $\mathcal{M}_A$.*

$$(M, \alpha) \in \mathcal{M}_A^V \Longrightarrow \begin{cases} M \in \mathcal{M}_A \\ \alpha : M \to M \otimes_A V \text{ morphism in } \mathcal{M}_A\text{, coaction} \end{cases}$$

Induced adjunctions

For any morphism of A-corings

$$g : V \to W,$$

there is an "extension of coefficients" adjunction

$$\begin{tikzcd} \mathcal{M}_A^V \arrow[r, shift left, "g_*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_A^W \arrow[l, shift left, "-\Box_W V"] \end{tikzcd}.$$

Homotopy theory of comodules

Theorem (H.-Shipley)

For any dg $\mathbb{k}$-algebra A, there is a combinatorial model category structure on $\mathcal{M}_A$ such that

- *the cofibrations are the injections, and*
- *the weak equivalences are the quasi-isomorphims.*

Proof by applying fancy machinery due to Beke, Lurie, and J. Smith to the usual projective model category structure on $\mathcal{M}_A$.

Homotopy theory of comodules

Theorem (H.-Shipley)

Suppose that $\mathbb{k}$ is semihereditary. Let A be a dg $\mathbb{k}$-algebra such that $H_1 A = 0$.

If V is an A-coring such that V is A-semifree on X, where

- $H_0(\mathbb{k} \otimes_A V) = \mathbb{k}$ *and* $H_1(\mathbb{k} \otimes_A V) = 0$*, and*
- X_n *is* $\mathbb{k}$*-free and finitely generated for all* n*,*

then $\mathcal{M}_A^V$ admits a model category structure such that

- *the cofibrations are the injections, and*
- *the weak equivalences are the quasi-isomorphims.*

Homotopy theory of comodules

- Special case of a general existence theorem for model category structure on categories of coalgebras over a comonad, in which the required factorizations

$$\bullet \overset{\sim}{\rightarrowtail} \bullet \twoheadrightarrow \bullet$$

  are constructed by induction on a filtration of weak equivalences by *n*-equivalences.
- Have considerable control over the fibrant objects in $\mathcal{M}_A^V$, e.g., under reasonable conditions, fibrant replacements given by cobar constructions.

Homotopy theory of comodules

- If $\mathcal{M}_A$ is endowed with the model structure of the first theorem, then

$$\begin{tikzcd} \mathcal{M}_A^V \arrow[r, shift left, "forget"] \arrow[r, phantom, "\perp"] & \mathcal{M}_A \arrow[l, shift left, "-\otimes_A V"] \end{tikzcd}$$

is a Quillen pair.

- If $g : V \to W$ is a morphism of A-corings satisfying the hypotheses above, then

$$\begin{tikzcd} \mathcal{M}_A^V \arrow[r, shift left, "g_*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_A^W \arrow[l, shift left, "-\square_W V"] \end{tikzcd}$$

is a Quillen pair.

THE HOPF-GALOIS FRAMEWORK

HOPF-GALOIS DATA

$$\varphi : A \to B^{\circlearrowleft H}$$

where

- H is a dg $\Bbbk$-Hopf algebra with comultiplication $\Delta : H \to H \otimes H$;
- A is a dg $\Bbbk$-algebra seen as an H-comodule with trivial coaction;
- B is an H-comodule algebra, with H-coaction $\rho : B \to B \otimes H$;
- $\varphi : A \to B$ is a morphism of H-comodule algebras.

The "normal basis" example

Let C be a coaugmented dg coalgebra, M a right C-comodule and N a left C-comodule.

$\Omega(M; C; N)$ is the two-sided cobar construction on C with coefficients in M and N, i.e.,

$$\Omega(M; C; N) = (M \otimes Ts^{-1}\overline{C} \otimes N, d_\Omega),$$

where

- $\overline{C}$ is the coaugmentation coideal,
- s^{-1} denotes desuspension,
- $TX = \bigoplus_{n \geq 0} X^{\otimes n}$ for any graded $\mathbb{k}$-module X, and
- d_Ω is given in terms of the differentials on C, M and N and of the coactions and comultiplication.

The "normal basis" example

Let H be a dg $\Bbbk$-Hopf algebra, and let (E, γ) be an H-comodule algebra.

Proposition (H.-Levi)

There is Hopf-Galois data

$$\varphi_\gamma : \Omega(E; H; \Bbbk) \hookrightarrow \Omega(E; H; H)^{\circlearrowright H}$$

where the H-coaction on $\Omega(E; H; H)$ is given by applying Δ on the last tensor factor.

CONSTRUCTIONS ASSOCIATED TO $A \xrightarrow{\varphi} B^{\circlearrowleft H}$

Classically, the coinvariants of a coaction $M \to M \otimes C$ of a coalgebra C are defined to be

$$M^{co\,C} = M \square_C \mathbb{k}.$$

Analogously, the homotopy coinvariant algebra of $\rho : B \to B \otimes H$ is

$$B^{hco\,H} = \Omega(B; H; \mathbb{k}).$$

CONSTRUCTIONS ASSOCIATED TO $A \xrightarrow{\varphi} B^{\circlearrowright H}$

REMARK

The definition

$$B^{hco\,H} = \Omega(B; H; \mathbb{k})$$

is reasonable because

$$\begin{tikzcd} B \arrow[rr, "\rho"] \arrow[dr, "\simeq"'] & & B \otimes H \\ & \Omega(B; H; H) \arrow[ur] & \end{tikzcd}$$

is a fibrant replacement in the category $\mathcal{M}^H$ of H-comodules and

$$\Omega(B; H; \mathbb{k}) = \Omega(B; H; H)\square_H \mathbb{k}.$$

CONSTRUCTIONS ASSOCIATED TO $A \xrightarrow{\varphi} B^{\circlearrowright H}$

The Hopf B-coring associated to $\rho : B \to B \otimes H$:

$$(B \otimes H, \delta_\rho, \varepsilon_\rho)$$

where

- δ_ρ is the composite

$$B \otimes H \xrightarrow{B\otimes\Delta} B \otimes H \otimes H \cong (B \otimes H) \otimes_B (B \otimes H);$$

- $\varepsilon_\rho = B \otimes \varepsilon : B \otimes H \to B$;

CONSTRUCTIONS ASSOCIATED TO $A \xrightarrow{\varphi} B^{\circlearrowright H}$

The Hopf B-coring associated to $\rho : B \to B \otimes H$:

$$(B \otimes H, \delta_\rho, \varepsilon_\rho)$$

where

- the left B-action on $B \otimes H$ is

$$B \otimes B \otimes H \xrightarrow{\mu_B \otimes H} B \otimes H;$$

- the right B-action on $B \otimes H$ is

$$B \otimes H \otimes B \xrightarrow{B \otimes H \otimes \rho} B \otimes H \otimes B \otimes H \cong B^{\otimes 2} \otimes H^{\otimes 2} \xrightarrow{\mu_B \otimes \mu_H} B \otimes H.$$

Constructions associated to $A \xrightarrow{\varphi} B^{\circlearrowleft H}$

The canonical B-coring associated to $\varphi : A \to B$:

$$(B \otimes_A B, \delta_\varphi, \varepsilon_\varphi)$$

where

- δ_φ is the composite

$$B \otimes_A B \cong B \otimes_A A \otimes_A B \xrightarrow{B \otimes_A \varphi \otimes_A B} B \otimes_A B \otimes_A B \cong (B \otimes_A B) \otimes_B (B \otimes_A B);$$

- $\varepsilon_\varphi = \bar{\mu}_B : B \otimes_A B \to B$ is induced by the multiplication map μ_B of B.

CONSTRUCTIONS ASSOCIATED TO $A \xrightarrow{\varphi} B^{\circlearrowright H}$

The Hopf-Galois map associated to $\varphi : A \to B$ and $\rho : B \to B \otimes H$

$$\begin{tikzcd}
B \otimes_A B \arrow[rr, "\beta_\varphi"] \arrow[dr, "B\otimes_A\rho"'] & & B \otimes H \\
& B \otimes_A B \otimes H \arrow[ur, "\bar{\mu}_B\otimes H"'] &
\end{tikzcd}$$

is a morphism of B-corings.
It therefore induces an “extension of coefficients” adjunction

$$\mathcal{M}_B^{B\otimes_A B} \underset{-\square_{B\otimes H}(B\otimes_A B)}{\overset{(\beta_\varphi)_*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{M}_B^{B\otimes H}.$$

CONSTRUCTIONS ASSOCIATED TO $A \xrightarrow{\varphi} B^{\circlearrowright H}$

A dual construction, perhaps somewhat more familiar and easier to understand...

Let G be a group acting on a set X, via

$$\rho : X \times G \to X : (x, a) \mapsto x \cdot a.$$

Let $Y = X_G$, the set of G-orbits. Take the pullback $X \times_Y X$ of the quotient map $X \to Y$ along itself.

There is a natural map

$$\beta : X \times G \to X \times_Y X : (x, a) \mapsto (x, x \cdot a).$$

CONSTRUCTIONS ASSOCIATED TO $A \xrightarrow{\varphi} B^{\circlearrowright H}$

$$
\begin{tikzcd}
\varphi : A \to B & \mathbf{Alg}^{\to} \\
\varphi : A \to B^{\circlearrowright H} \arrow[u, mapsto] \arrow[d, mapsto] & \mathbf{HopfGalois} \arrow[u, "\mathit{Forget}"] \arrow[d, "\mathit{Forget}"'] \\
B^{\circlearrowright H} & \mathbf{ComodAlg}
\end{tikzcd}
$$

Constructions associated to $A \xrightarrow{\varphi} B^{\circlearrowright H}$

$$
\begin{tikzcd}
\varphi : A \to B & \mathbf{Alg}^{\to} \arrow[dddrr, "Descent"] & \varphi : A \to B \arrow[dr, mapsto] & \\
 & & & B \otimes_A B \\
\varphi : A \to B^{\circlearrowright H} \arrow[uu, mapsto] \arrow[dd, mapsto] & \mathbf{HopfGalois} \arrow[uu, "Forget"'] \arrow[dd, "Forget"'] & & & \mathbf{CoRing} \\
 & & & B \otimes H & \\
B^{\circlearrowright H} & \mathbf{ComodAlg} \arrow[uurrr, "Hopf"] & B^{\circlearrowright H} \arrow[ur, mapsto] & &
\end{tikzcd}
$$

Constructions associated to $A \xrightarrow{\varphi} B^{\circlearrowright H}$

$$
\begin{tikzcd}
\varphi : A \to B \\
\varphi : A \to B^{\circlearrowright H} \arrow[u, mapsto] \arrow[d, mapsto] \\
B^{\circlearrowright H}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge]
\mathbf{Alg}^{\to} \arrow[drr, "\mathit{Descent}"] & & \\
\mathbf{HopfGalois} \arrow[u, "\mathit{Forget}"] \arrow[d, "\mathit{Forget}"'] \arrow[rr, shift left=2, ""{name=U, below}] \arrow[rr, shift right=2, ""{name=D}] & & \mathbf{CoRing} \\
\mathbf{ComodAlg} \arrow[urr, "\mathit{Hopf}"'] & &
\arrow[Rightarrow, from=U, to=D, "\beta_{(-)}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\varphi : A \to B \arrow[dr, mapsto] & \\
& B \otimes_A B \\
& B \otimes H \\
B^{\circlearrowright H} \arrow[ur, mapsto] &
\end{tikzcd}
$$

POSSIBLE PROPERTIES OF $A \xrightarrow{\varphi} B^{\circlearrowright H}$

Suppose henceforth that all categories in sight are model categories and all adjunctions are Quillen pairs.

We'll see explicit examples later where this is the case.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowleft H}$

H is a generalized Koszul dual of A if there is a Quillen equivalence

$$\begin{tikzcd} \mathcal{M}_A \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{M}^H \arrow[l, shift left] \end{tikzcd}.$$

Motivation (Lefèvre)

If $\Bbbk$ is a field, and $\tau : C \to A$ is an acyclic twisting cochain, there is a model category structure on the category of unbounded, cocomplete C-comodules such that the functor $\mathcal{M}^C \to \mathcal{M}_A$ induced by τ is the left member of a Quillen equivalence.

If A is a Koszul algebra, and C is its Koszul dual coalgebra, then the canonical twisting cochain $\tau : C \to A$ is acyclic, so $\mathcal{M}^C \to \mathcal{M}_A$ does indeed fit into a Quillen equivalence.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowright H}$

Example

Since the universal twisting cochain $H \to \Omega H$ is acyclic, H is always a generalized Koszul dual of ΩH.

Example

Let $\mathcal{B}$ denote the (reduced) bar construction. If A is a commutative dg algebra, then $\mathcal{B}A$ is naturally a commutative Hopf algebra.
Since the couniversal twisting cochain $\mathcal{B}A \to A$ is acyclic, $\mathcal{B}A$ is a generalized Koszul dual of A.

POSSIBLE PROPERTIES OF $A \xrightarrow{\varphi} B^{\circlearrowleft H}$

φ satisfies effective homotopic Grothendieck descent if

$$\begin{tikzcd} \mathcal{M}_A \arrow[r, shift left, "Can"] \arrow[r, phantom, "\perp"] & \mathcal{M}_B^{B\otimes_A B} \arrow[l, shift left, "Prim"] \end{tikzcd}.$$

is a Quillen equivalence, where $Can(M) = (M \otimes_A B, \psi_M)$, with ψ_M equal to the composite

$$M\otimes_A B \cong M\otimes_A A\otimes_A B \xrightarrow{M\otimes_A\varphi\otimes_A B} M\otimes_A B\otimes_A B \cong (M\otimes_A B)\otimes_B (B\otimes_A B).$$

$\mathcal{M}_B^{B\otimes_A B}$ is exactly the category of descent data associated to φ.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowright H}$

The meaning of descent: which B-modules are weakly equivalent to $M \otimes_A B$ for some $M \in \mathcal{M}_A$?

$$
\begin{tikzcd}
\mathcal{M}_A \arrow[rr, shift left, "-\otimes_A B"] \arrow[rr, phantom, "\perp"] \arrow[d, shift right, "\mathit{Can}"'] \arrow[d, phantom, "\dashv"] & & \mathcal{M}_B \arrow[ll, shift left, "\varphi^*"] \\
\mathcal{M}_B^{B\otimes_A B} \arrow[u, shift right, "\mathit{Prim}"'] \arrow[urr, "\mathit{forget}"'] & &
\end{tikzcd}
$$

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowright H}$

Example

For any H-comodule algebra (E, γ), the inclusion

$$A = \Omega(E; H; \Bbbk) \hookrightarrow \Omega(E; H; H) = B$$

satisfies effective homotopic Grothendieck descent. In fact, $\mathit{Can} : \mathcal{M}_A \to \mathcal{M}_B^{B \otimes_A B}$ is an actual equivalence of categories.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowleft H}$

φ is a homotopic H-Hopf-Galois extension if

- the natural map $i_\varphi : A \to \Omega(B; H; \mathbb{k}) = B^{hco\,H}$ induces a Quillen equivalence

$$\begin{tikzcd} \mathcal{M}_A \arrow[r, shift left, "-\otimes_A B^{hco\,H}"] \arrow[r, phantom, "\perp"] & \mathcal{M}_{B^{hco\,H}} \arrow[l, shift left, "i_\varphi^*"] \end{tikzcd}.$$

- the "extension of coefficients" adjunction

$$\begin{tikzcd} \mathcal{M}_B^{B\otimes_A B} \arrow[r, shift left, "(\beta_\varphi)_*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_B^{B\otimes H} \arrow[l, shift left, "-\square_{B\otimes H}(B\otimes_A B)"] \end{tikzcd}.$$

  is a Quillen equivalence.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowright H}$

- Hopf-Galois extensions generalize ordinary Galois extensions of rings (where H is the dual of a group algebra).
- Faithfully flat Hopf-Galois extensions over the coordinate ring of an affine group scheme G correspond to G-principal fiber bundles.
- Homotopic Hopf-Galois extensions were first introduced by Rognes for ring spectra, e.g., $S \xrightarrow{\eta} MU^{\circlearrowright S[BU]}$.
- One can study Hopf algebras via their associated Hopf-Galois extensions.

Possible properties of $A \xrightarrow{\varphi} B^{\circlearrowright H}$

Example

For any H-comodule algebra (E, γ), the inclusion

$$A = \Omega(E; H; \Bbbk) \hookrightarrow \Omega(E; H; H)^{\circlearrowright H} = B^{\circlearrowright H}$$

is an H-Hopf-Galois extension. In fact,

$$\begin{tikzcd} \mathcal{M}_B^{B \otimes_A B} \arrow[r, shift left, "(\beta_\varphi)_*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_B^{B \otimes H} \arrow[l, shift left, "- \square_{B \otimes H} (B \otimes_A B)"] \end{tikzcd}.$$

is an actual equivalence of categories.

The duality theorems

HOPF-GALOIS $\Leftrightarrow$ GROTHENDIECK

THEOREM (BERGLUND-H.)

Let $A \xrightarrow{\varphi} B^{\circlearrowright H}$ be Hopf-Galois data. If $i_\varphi^* : \mathcal{M}_{B^{hco\,H}} \to \mathcal{M}_A$ is a Quillen equivalence, then

$$\varphi \text{ is a homotopic } H\text{-Hopf-Galois extension}$$

$$\Updownarrow$$

$$\varphi \text{ satisfies homotopic Grothendieck descent.}$$

Homotopic version of a "faithfully flat descent" result due to Schneider, as reformulated by Schauenburg.

KOSZUL $\Rightarrow$ (HOPF-GALOIS $\Leftrightarrow$ GROTHENDIECK)

THEOREM (BERGLUND-H.)

Let $A \xrightarrow{\varphi} B^{\circlearrowleft H}$ *be Hopf-Galois data.*
If $\Bbbk \xrightarrow{\simeq} B$, *then*

$$i_\varphi^* : \mathcal{M}_{B^{hco\,H}} \to \mathcal{M}_A \text{ is a Quillen equivalence}$$

$$\Updownarrow$$

$$H \text{ is a generalized Koszul dual of } A.$$

REMARK

If $\Bbbk \xrightarrow{\simeq} B$, then H is always a generalized Koszul dual of $B^{hco\,H}$.

THE PROOF

Replace φ by a "normal basis" extension.

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dd, "i_\varphi"'] & & B^{\circlearrowright H} \arrow[dd, "\simeq"'] & \\
& & & \Bbbk \arrow[ul] \arrow[dl] \\
A' = B^{hco\,H} \arrow[rr, "\varphi'"] & & \Omega(B;H;H)^{\circlearrowright H} = B'^{\circlearrowright H} &
\end{tikzcd}$$

THE PROOF

$$
\begin{tikzcd}
\mathcal{M}_A \arrow[rr, shift left, "-\otimes_A A'"] \arrow[d, shift right, "Can"'] & & \mathcal{M}_{A'} \arrow[ll, shift left, "\varphi^*"] \arrow[d, shift right, "Can"'] \\
\mathcal{M}_B^{B\otimes_A B} \arrow[u, shift right, "Prim"'] \arrow[rr, shift left] \arrow[d, shift right, "(\beta_\varphi)_*"'] & & \mathcal{M}_{B'}^{B'\otimes_{A'} B'} \arrow[u, shift right, "Prim"'] \arrow[ll, shift left] \arrow[d, shift right, "(\beta_{\varphi'})_*"'] \\
\mathcal{M}_B^{B\otimes H} \arrow[u, shift right] \arrow[rr, shift left] \arrow[dr, shift right] & & \mathcal{M}_{B'}^{B'\otimes H} \arrow[u, shift right] \arrow[ll, shift left] \arrow[dl, shift left] \\
& \mathcal{M}^H \arrow[ul, shift right] \arrow[ur, shift left] &
\end{tikzcd}
$$

The proof

Remark

From the previous diagram, we see that if H is a generalized Koszul dual of A, and φ satisfies homotopic Grothendieck descent, then $B \otimes_A B$ is a sort of "relative" generalized Koszul dual of A, which is what Jack intuited.

A source of examples

(Work in progress with A. Berglund, in the spirit of the “normal basis” basis example.)

Let K be a dg $\mathbb{k}$-Hopf algebra and A a dg $\mathbb{k}$-algebra.

A twisting cochain $\tau : K \to A$ is Hopf-Hirsch if $A \otimes_\tau K$ admits a multiplication such that

$$A \hookrightarrow A \otimes_\tau K \to K$$

is a sequence of algebra maps.
Indeed there is a functor

$$\mathbf{Twist_{HH}} \to \mathbf{HopfGalois} : (K \xrightarrow{\tau} A) \mapsto (A \to A \otimes_\tau K^{\circlearrowright K}).$$

A source of examples

Suppose that $H_1 A = 0 = H_1 K$, $H_0 K = \Bbbk$ and K is $\Bbbk$-free of finite type.

If $\tau : K \to A$ is Hopf-Hirsch, then the extension

$$A \hookrightarrow A \otimes_\tau K^{\circlearrowright K}$$

is homotopic K-Hopf-Galois and satisfies homotopic Grothendieck descent.

Joyeux Anniversaire, cher Yves!