Spaces of long embeddings and iterated loop spaces

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joint w/ Dwyer

I. Introduction

Let $m \le n$, and fix a linear embedding $i : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$.

Notation:

Well understood by Smale-Hirsch.

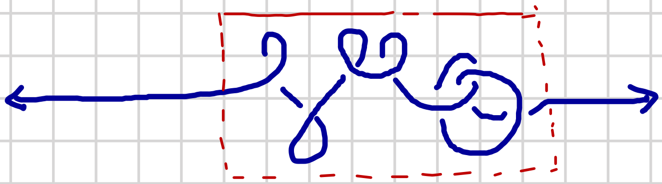
- $\mathrm{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) = \{ f : \mathbb{R}^m \to \mathbb{R}^n \mid f \text{ immersion}, \ \exists\, C \subset \mathbb{R}^m \text{ cpct st } f|_{\mathbb{R}^m \setminus C} = i|_{\mathbb{R}^m \setminus C} \}$

  $\cup|$

- $\mathrm{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) = \{ f \in \mathrm{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) \mid f \text{ embedding} \}$
- $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) = \mathrm{hfib}_i\left(\mathrm{Emb}_c(\mathbb{R}^m, \mathbb{R}^n) \hookrightarrow \mathrm{Imm}_c(\mathbb{R}^m, \mathbb{R}^n) \right)$

  – the space of long embeddings of $\mathbb{R}^m$ into $\mathbb{R}^n$; points are embeddings together w/ a fixed "tangential straightening", i.e., homotopy to i.

Special case: $m = 1 \Rightarrow \overline{\mathrm{Emb}}_c(\mathbb{R}^1, \mathbb{R}^n)$ is the space of long knots in $\mathbb{R}^n$.



Big question: What can we say about the homotopy type of $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$?

Idea: $\exists$ natural multiplication of long embeddings, by concatenation. This multiplication is even commutative up to homotopy, whence, if $n - m \ge 3$ and thus $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ is connected, it is at least a double loop space.

Main Theorem: [Dwyer-H.] For all $n \geq m+3$,

$$\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \simeq \Omega^{m+1} \underbrace{\mathrm{Map}^h_{\mathrm{Op}}(\mathcal{B}_m, \mathcal{B}_n)}.$$

(Case $m=1$ presented last year at Oberwolfach!)

(Also have a similar result for long links.)

– the derived mapping space of operad maps from the operad of little m-balls to the operad of little n-balls.

Goal today: Explain this statement, its meaning and something about the proof.

A first clarification: If $\mathcal{C}$ is a category endowed with a distinguished class $\mathcal{W}$ of weak equivalences, $\forall A, B \in \mathcal{C}$ $\exists$ $\mathrm{Map}^h(A,B) \in s\underline{\mathrm{Set}}/\underline{\mathrm{Top}}$ such that

$$\left.\begin{array}{l} A' \xrightarrow{\sim} A \\ B \xrightarrow{\sim} B' \end{array}\right\} \Rightarrow \mathrm{Map}^h(A,B) \xrightarrow{\sim} \mathrm{Map}^h(A',B').$$

I. From geometry to algebra

Goal: Describe an "algebraic" model for $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$, due to Arone and Turchin, given in terms of operads.

Operads and their bimodules

Intuition: Let X be a set. Just as the multiplicative structure on $\mathrm{Map}(X,X)$ given by composition motivates the def^n of a monoid, so does the multiplicative structure on $\{\mathrm{Map}(X^n, X)\}_{n\geq 0}$ motivate the def^n of an operad.

$$\left.\begin{array}{l} f: X^k \to X \\ g_i: X^{n_i} \to X,\ 1 \leq i \leq k \end{array}\right\} \Rightarrow f(g_1, \ldots, g_k): X^n \to X, \quad n = \sum_{i=1}^{k} n_i.$$

Definitions

- Symmetric sequence in sSet: $\mathcal{X} = \{\mathcal{X}(n)\}_{n\geq 0}$ w/ $\mathcal{X}(n) \circlearrowleft \Sigma_n \ \forall n$.

  the arity n part of $\mathcal{X}$

- Operad in sSet: $(\mathcal{P}, \mu, \eta)$, $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0} \in \underline{\text{SymSeq}}$

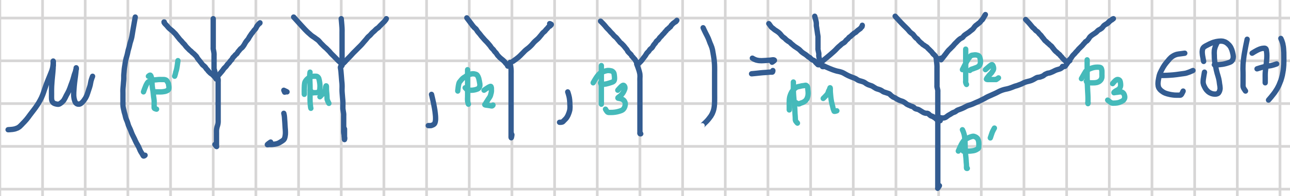
  $\mu \leftrightarrow$ collection of maps

  $$\mu_{k,\vec{n}} : \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \longrightarrow \mathcal{P}\left(\sum_{i=1}^{k} n_i\right)$$

  satisfying associativity, unitality, and equivariance conditions.

$\mathcal{P}$-algebra = sSet X + operad map $\mathcal{E}nd(X) \to \mathcal{P}$.
– representation!

Ex: $k=3$; $n_1=3$, $n_2=2$, $n_3=2$



Notation: $\underline{\mathcal{O}p}$ = category of operads

- Bimodules over an operad $\mathcal{P}$: $(\mathcal{X}, \rho, \lambda)$, $\mathcal{X} = \{\mathcal{X}(n)\}_{n\geq 0}$

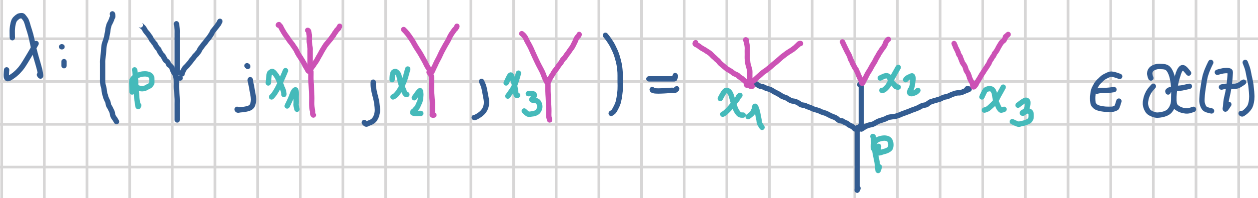
  $\lambda \longleftrightarrow$ collection $\lambda_{k,\vec{n}} : \mathcal{P}(k) \times \mathcal{X}(n_1) \times \cdots \times \mathcal{X}(n_k) \longrightarrow \mathcal{X}(n)$
  + conditions (highly nonlinear in $\mathcal{X}$!)

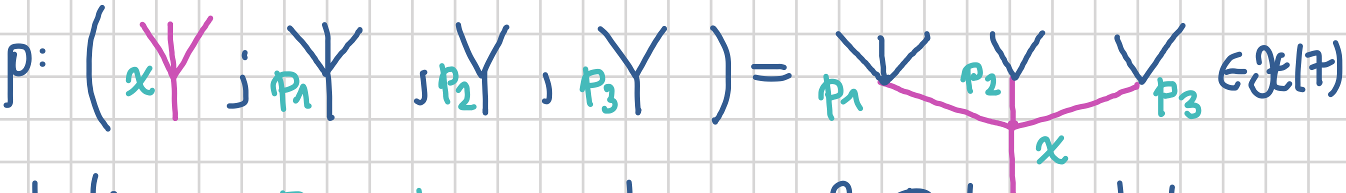
  compatible

  $\rho \longleftrightarrow$ collection $\rho_{k,\vec{n}} : \mathcal{X}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \longrightarrow \mathcal{X}(n)$
  + conditions

Rmk: Any operad is a bimodule over itself.

Example: $k=3$; $n_1=3$, $n_2=2$, $n_3=2$





Notation: $\underline{\text{Bimod}}_{\mathcal{P}}$ = category of $\mathcal{P}$-bimodules

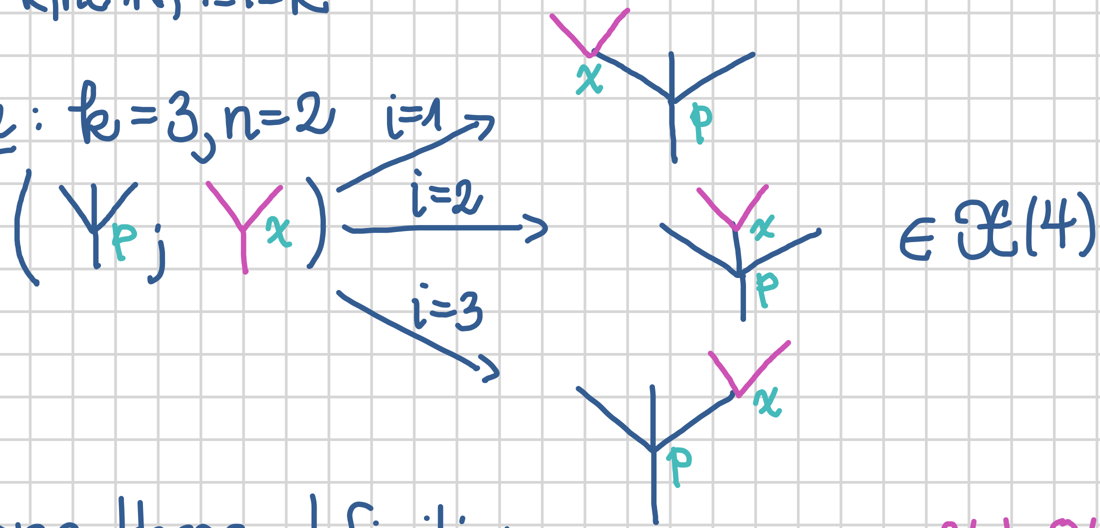
- Linear bimodules over an operad $\mathcal{P}$: $(\mathcal{X}, \ell, \rho)$

  ρ as before

  $\ell \leftrightarrow$ collection $\ell_{k,i,n} : \mathcal{P}(k) \times \mathcal{X}(n) \longrightarrow \mathcal{X}(n+k-1)$

  $k,n \in \mathbb{N}, 1 \le i \le k$

Example: $k=3, n=2$

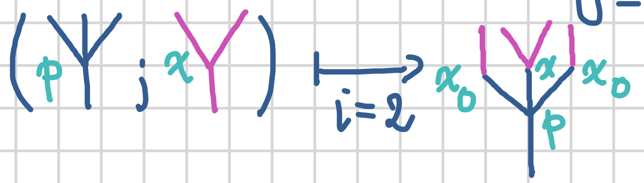


Relations among these definitions

1) operad hm $\mathcal{P} \xrightarrow{f} Q \Rightarrow Q$ a pointed $\mathcal{P}$-bimodule (via $f(1) \in Q(1)$)
2) $\mathcal{X}$ pointed $\mathcal{P}$-bimodule $\Rightarrow \mathcal{X}$ pointed linear $\mathcal{P}$-bimodule

   $\hookrightarrow x_0 \in \mathcal{X}(1)$

$\lambda \rightsquigarrow \ell$:



The essential operads today: $\mathcal{B}_m$, $m \ge 1$: the operad of little m-balls

Let B^m = open unit ball in $\mathbb{R}^m$.

- $\mathcal{B}_m(k) = \mathrm{sEmb}\left(\coprod_k B^m, B^m\right) \quad \forall k \in \mathbb{N}$

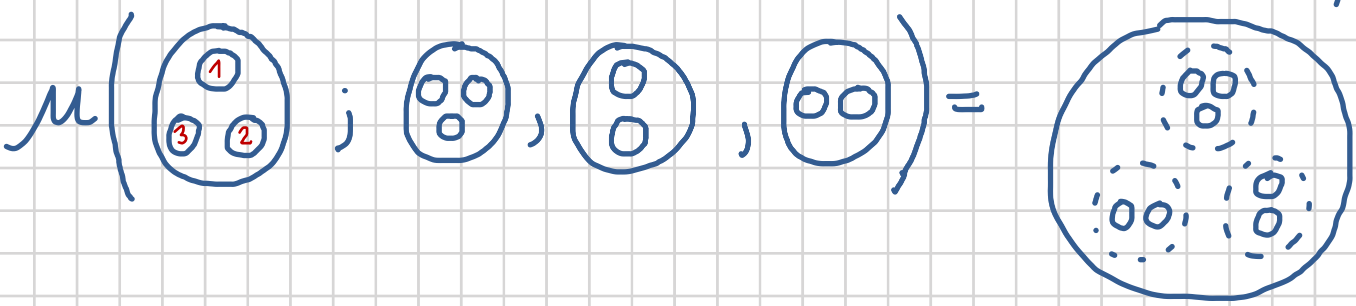
  space of standard embeddings, i.e., on each component have

$$\begin{array}{ccc} B^m & \xrightarrow{f} & B^m \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xrightarrow{g} & \mathbb{R}^m \end{array}$$

  where g is a std iso: $g = (\text{mult par } \lambda > 0) \circ (\text{transl})$

- $\Sigma_k \curvearrowright \mathcal{B}_m(k)$ by permuting input components.

• The operad multiplication: E.g., if $m=2$, $k=3$, $n_1=3, n_2=2=n_3$:



Rmk: The inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ induces an operad hm $\mathcal{B}_m \xrightarrow{\varphi_{m,n}} \mathcal{B}_n$, giving $\mathcal{B}_n$ natural structures in $\mathrm{Bimod}_{\mathcal{B}_m}$ and $\mathrm{LBimod}_{\mathcal{B}_m}$.

The passage from geometry to algebra

Theorem: [Sinha, Arone-Turchin, Turchin]
(m=1, 2006) (2011) (2012)

$$n \geq m+3 \Rightarrow \overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) = \mathrm{Map}^h_{\mathrm{LBimod}_{\mathcal{B}_m}}(\mathcal{B}_m, \mathcal{B}_n).$$

(Proof by methods from Weiss embedding calculus.)

II. Proof of the Main Theorem

Theorem above $\Rightarrow$ Main Theorem follows from ...

Theorem A: $\mathrm{Map}^h_{\mathrm{LBimod}_{\mathcal{B}_m}}(\mathcal{B}_m, \mathcal{B}_n) \simeq \Omega^{m+1} \mathrm{Map}^h_{\mathrm{Op}}(\mathcal{B}_m, \mathcal{B}_n)_{\varphi_{m,n}}$

This theorem is proved in two steps.

Theorem A': $\mathrm{Map}^h_{\mathrm{Bimod}_{\mathcal{B}_m}}(\mathcal{B}_m, \mathcal{B}_n) \simeq \Omega \mathrm{Map}^h_{\mathrm{Op}}(\mathcal{B}_m, \mathcal{B}_n)_{\varphi_{m,n}}$.

Theorem A'': $\mathrm{Map}^h_{\mathrm{LBimod}_{\mathcal{B}_m}}(\mathcal{B}_m, \mathcal{B}_n) \simeq \Omega^m \mathrm{Map}^h_{\mathrm{Bimod}_{\mathcal{B}_m}}(\mathcal{B}_m, \mathcal{B}_n)$

First step (cf. [Dwyer-H., 2012])

Theorem B: Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a monoidal category with "appropriately compatible" model category structure. Let $\varphi: R \to S$ be a "nice" monoid homomorphism.

There is a fibration sequence

$$\Omega \mathrm{Map}^h_{\mathrm{Mon}}(R,S)_\varphi \longrightarrow \mathrm{Map}^h_{\mathrm{Bimod}_R}(R,S) \longrightarrow \mathrm{Map}^h_{\mathcal{M}}(\mathbb{I},S).$$

fiber over $\eta: \mathbb{I} \to S$.

Corollary 1: Theorem A' holds.

Proof: Apply Theorem B to $\mathcal{M}$ = cat. of symm. sequences, $\otimes$ = composition product, $\mathbb{I} = (\emptyset, *, \emptyset, \ldots)$, $\varphi = \varphi_{m,n}$. Observe that $\mathrm{Map}^h_{\mathcal{M}}(\mathbb{I}, \mathcal{B}_n) \simeq \mathcal{B}_n(1) \simeq *$. //

Corollary 2: Theorem A'' holds for $m=1$.

Proof: $\mathrm{Bimod}_{\mathcal{B}_1} \simeq \{$graded monoids in $\mathrm{Mod}_{\mathcal{B}_1}\}$

Apply Theorem B to this category + graded tensor product. Again get contractible base. //

Rmk: The hard work in these proofs consists in checking "appropriate compatibility" and "niceness".

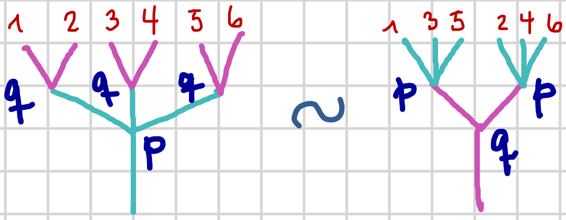
Second step: Induct up from $m=1$!

Def^n: The Boardman-Vogt tensor product of operads $\mathcal{P}$ and $\mathcal{Q}$ is the operad

$$\mathcal{P} \otimes \mathcal{Q} = \mathcal{P} \amalg \mathcal{Q} \,/\, \text{exchange rel'ns}$$

$\mathcal{P}\otimes\mathcal{Q}$-algebras = objects w/ commuting $\mathcal{P}$- and $\mathcal{Q}$-alg structures.

i.e.,



The key to the induction ...

Theorem: [Fiedorowicz-Vogt, Lurie]

$$\underbrace{\mathcal{B}_1 \otimes^h \mathcal{B}_1 \otimes^h \cdots \otimes^h \mathcal{B}_1}_{m \text{ facteurs}} \simeq \mathcal{B}_m .$$

How to exploit this

- $\exists$ Quillen adjunction $\mathcal{L}_{\mathcal{P}} : \underline{\mathrm{LBimod}}^*_{\mathcal{P}} \rightleftarrows \underline{\mathrm{Bimod}}^*_{\mathcal{P}} : \mathcal{U}$
- Proof of Thm A' for $m=1 \Leftrightarrow \mathcal{L}_{\mathcal{B}_1}(\mathcal{B}_1) \simeq S^1 \cdot \mathcal{B}_1$ "suspension"
- $\exists$ lift of $\otimes$ to bimodules, i.e.,
  $- \tilde{\otimes} - : \underline{\mathrm{Bimod}}_{\mathcal{P}} \times \underline{\mathrm{Bimod}}_{\mathcal{Q}} \longrightarrow \underline{\mathrm{Bimod}}_{\mathcal{P} \otimes \mathcal{Q}}$ (lots of nice properties...)

Proposition: $\exists$ homotopy pushout in $\underline{\mathrm{Bimod}}_{\mathcal{P} \otimes \mathcal{Q}}$:

(Both $\mathcal{P}$ & $\mathcal{Q}$ playing 3 different roles in this diagram.)

$$\begin{array}{ccc} \mathcal{L}_{\mathcal{P}}(\mathcal{P}) \tilde{\otimes} \mathcal{Q} \cup \mathcal{P} \tilde{\otimes} \mathcal{L}_{\mathcal{Q}}(\mathcal{Q}) & \longrightarrow & \mathcal{P} \otimes \mathcal{Q} \\ \downarrow & & \downarrow \\ \mathcal{L}_{\mathcal{P}}(\mathcal{P}) \tilde{\otimes} \mathcal{L}_{\mathcal{Q}}(\mathcal{Q}) & \longrightarrow & \mathcal{L}_{\mathcal{P} \otimes \mathcal{Q}}(\mathcal{P} \otimes \mathcal{Q}) \end{array}$$

Prop + F-V Thm $\Rightarrow$ $\mathcal{L}_{\mathcal{B}_m}(\mathcal{B}_m) \simeq S^m \cdot \mathcal{B}_m$, by induction on m.

Proof of Thm A":

$$\mathrm{Map}^h_{\underline{\mathrm{LBimod}}_{\mathcal{B}_m}}(\mathcal{B}_m, \mathcal{B}_n) \simeq \mathrm{Map}^h_{\underline{\mathrm{Bimod}}_{\mathcal{B}_m}}(\mathcal{L}_{\mathcal{B}_m}(\mathcal{B}_m), \mathcal{B}_n)$$

$$\simeq \mathrm{Map}^h_{\underline{\mathrm{Bimod}}_{\mathcal{B}_m}}(S^m \cdot \mathcal{B}_m, \mathcal{B}_n) \simeq \Omega^m \mathrm{Map}^h_{\underline{\mathrm{Bimod}}_{\mathcal{B}_m}}(\mathcal{B}_m, \mathcal{B}_n). \quad //$$