Galois, Hopf, Grothendieck, Koszul and Quillen

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I. Galois meets Hopf

Recall: A field extension $\Bbbk \hookrightarrow \mathbb{E}$ is Galois if it's algebraic, normal and separable. If $[\mathbb{E}:\Bbbk]<\infty$, then the extension is Galois if it's algebraic and $\Bbbk = \mathbb{E}^{\mathrm{Aut}_{\Bbbk}(\mathbb{E})}$ ← the Galois group of the extension

First generalizations:

- [Auslander-Goldman, 1960]: generalization to commutative rings
- [Chase-Harrison-Rosenberg, 1965]: six characterizations of Galois extensions of commutative rings, including:

$R \hookrightarrow S$ is G-Galois for $G \leq \mathrm{Aut}_R(S)$, $|G|<\infty$

$$\Longleftrightarrow \quad R \xrightarrow{\cong} S^G \quad \text{and} \quad S \underset{R}{\otimes} S \xrightarrow{\cong} \prod_G S$$

$$s\otimes s' \longmapsto (s\cdot g(s'))_{g\in G}$$

Beyond group actions (and on to group schemes...)

- [Chase-Sweedler, 1969], [Kreimer-Takeuchi, 1981]:

$\Bbbk$ = commutative ring, $\otimes = \otimes_{\Bbbk}$

H = $\Bbbk$-bialgebra

B = $\Bbbk$-algebra with coaction $\rho: B \to B\otimes H$

$A = B^{coH} = \{b \mid \rho(b) = b\otimes 1\}$

↑ algebra homomorphism, coassociative, counital

The extension of $\Bbbk$-algebras $A \hookrightarrow B$ is H-Hopf-Galois if

$$B \otimes_A B \xrightarrow{B \otimes_A \rho} B \otimes_A B \otimes H \xrightarrow{\bar{\mu} \otimes H} B \otimes H$$

(the Galois map)

is an isomorphism.

Examples:

① $\Bbbk \hookrightarrow \mathbb{E}$ field extension, $G \leq \mathrm{Aut}_{\Bbbk}(\mathbb{E})$, $|G| < \infty$, $\mathbb{F} = \mathbb{E}^G$:

$$\mathbb{F} \hookrightarrow \mathbb{E} \text{ is } G\text{-Galois}$$

$$\Updownarrow$$

$\Bbbk^G = \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk)$

$$\mathbb{F} \hookrightarrow \mathbb{E} \text{ is } \Bbbk^G\text{-Hopf-Galois}$$

② X finite set, G finite group, $a : X \times G \to X$ action, $q : X \to X_G = Y$, $\Bbbk$ field:

$$X \times G \xrightarrow{\Delta \times G} X \times_Y X \times G \xrightarrow{X \times a} X \times_Y X \qquad (\star)$$

$\Rightarrow$ extension $\Bbbk^Y \xrightarrow{q^*} \Bbbk^X \circlearrowleft \Bbbk^G$ - coaction

q is a $\Bbbk^G$-Hopf-Galois extension

$$\Updownarrow$$

$(\star)$ is an isomorphism

$$\Updownarrow$$

a is a free G-action.

Fancier version:
- X = affine scheme
- G = affine alg gp scheme
- $X \to Y$ G-torsor

③ H a $\Bbbk$-bialgebra:

$$\Bbbk \xrightarrow{\eta} H \text{ is } H\text{-Hopf-Galois}$$

$$\Updownarrow$$

H is a Hopf algebra

More generally: $\left.\begin{array}{c}H \text{ Hopf algebra}\\ + \\ A\ \Bbbk\text{-algebra}\end{array}\right\} \Rightarrow A \hookrightarrow A\otimes H$ is H-Hopf-Galois of normal basis type

Why interesting?

- Faithfully flat HG-extensions over the coordinate ring of an affine group scheme correspond to G-principal bundles (torsors).
- Can study Hopf algebras via associated HG-extensions

II. Grothendieck

Framework

$\varphi: A \longrightarrow B$ ring homomorphism

$\Rightarrow$ adjunction $\quad -\otimes_A B : \underline{\mathrm{Mod}}_A \underset{\perp}{\rightleftarrows} \underline{\mathrm{Mod}}_B : \varphi^*$

Informal Grothendieck descent problem

Realizability problems!

(a) Given N_B, when $\exists\ M_A$ such that $N \cong M\otimes_A B$?

(b) Given $f: M\otimes_A B \longrightarrow M'\otimes_A B$, when $\exists\ g: M \longrightarrow M'$ homomorphism of A-modules st $f = g\otimes_A M$?

More formally

$\mathcal{D}(\varphi)$ = category of descent data associated to φ

Objects = pairs (N, ϑ) with $N \in \underline{\mathrm{Mod}}_B$,

$\vartheta: N \longrightarrow \varphi^*(N)\otimes_A B$ - coassociative, counital

$\exists$ factorization:

$$\begin{tikzcd}
\underline{\mathrm{Mod}}_A \arrow[rr, "- \otimes_A B"] \arrow[dr, "\mathrm{Can}"'] & & \underline{\mathrm{Mod}}_B \\
& \mathcal{D}(\varphi) \arrow[ur, "\mathrm{Forget}"'] &
\end{tikzcd}$$

$$M \longmapsto (M \otimes_A B, \theta_M)$$

$$\begin{tikzcd}
M \otimes_A B \cong M \otimes_A A \otimes_A B \arrow[rr, "M \otimes_A \varphi \otimes_A B"] \arrow[rr, bend right, "\theta_M"'] & & (M \otimes_A B) \otimes_A B
\end{tikzcd}$$

φ satisfies effective Grothendieck descent if $\mathrm{Can}: \underline{\mathrm{Mod}}_A \longrightarrow \mathcal{D}(\varphi)$ is an equivalence.

φ satisfies effective Grothendieck descent

$\Rightarrow$ have answers to ⓐ and ⓑ: can realize objects and morphisms in $\underline{\mathrm{Mod}}_B$ when they underlie objects and morphisms in $\mathcal{D}(\varphi)$.

III. Quillen

"Up-to-homotopy" versions of Hopf-Galois and Grothendieck theories.

Motivation:

[Rognes, 2008]: Galois theory of structured ring spectra

One important extension that is not Galois but is Hopf-Galois:

$$S \longrightarrow MU.$$

"Galois" and "Hopf-Galois" interpreted homotopically: iso $\rightsquigarrow$ w.e.

Grothendieck descent "up-to-homotopy" for morphisms of structured ring spectra also important, e.g., for studying completions.

Framework

- $(\mathcal{V}, \wedge, S)$ monoidal model category (nice enough)

Hopf-Galois data
$\left\{ \begin{array}{l} \circ\ \varphi : A \longrightarrow B \text{ morphism of monoids in } \mathcal{V} \\ \circ\ H \text{ bimonoid in } \mathcal{V} \\ \circ\ \rho : B \longrightarrow B \wedge H \text{ coaction st} \end{array} \right.$

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "A\wedge\eta"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\rho"] \\ A\wedge H \arrow[r, "\varphi\wedge H"] & B\wedge H \end{tikzcd}$$

$\underline{Alg}^H$

Notation: $A \xrightarrow{\varphi} B^{\circlearrowleft H}$

[Schwede-Shipley]: Well understood conditions under which $\exists$ Quillen model category structure on $\underline{Mod}_A$, $\underline{Mod}_B$, $\underline{Alg}$.

[H.-Shipley], [BHKKRS]:

New "left-induction" techniques $\Rightarrow$ reasonable conditions guaranteeing existence of Quillen model category structure on $\mathcal{D}(\varphi)$, $\underline{Alg}^H$.

Def^n: $A \xrightarrow{\varphi} B^{\circlearrowleft H}$ is a homotopic Hopf-Galois extension if:

- $A \xrightarrow{\simeq} B^{h\,coH}$ ← Need model category structure on $\underline{Alg}^H$ to define this.
- $B \underset{A}{\wedge} B \xrightarrow{B \underset{A}{\wedge} \rho} B \underset{A}{\wedge} B \wedge H \xrightarrow{\bar{\mu}\wedge H} B \wedge H$ is a weak equivalence.

$\underline{\text{Def}^n}$: $\varphi: A \longrightarrow B$ satisfies effective homotopic Grothendieck descent if

$$\mathrm{Can}: \underline{\mathrm{Mod}}_A \longrightarrow \mathcal{D}(\varphi)$$

is a Quillen equivalence.

$\underline{\text{Example}}$: $\Bbbk$ - commutative ring

H - 1-connected dg $\Bbbk$-bialgebra, degreewise $\Bbbk$-projective.

E - dg H-comodule algebra

[H.-Levi]: $\Omega(-;H;-): \underline{\mathrm{Alg}}^H \times {}^H\underline{\mathrm{Alg}} \longrightarrow \underline{\mathrm{Alg}}$

- the two-sided cobar construction

$\underline{\text{Proposition}}$: [Berglund - H.]

model for the inclusion of a homotopy fiber

$$\Omega(E;H;\Bbbk) \longrightarrow \Omega(E;H;H)$$

Homotopic normal basis extension

is homotopic H-Hopf-Galois and satisfies effective homotopic Grothendieck descent.

It's not a fluke that this morphism both is HG and satisfies descent!

IV. All together now! $\boxed{\mathcal{V} = \mathrm{Ch}_{\Bbbk}}$

$\underline{\text{Theorem}}$: Let H be as above. Let $\varphi: A \to B^{\partial H}$ be
[Berglund-H.] Hopf-Galois data such that $A \xrightarrow{\simeq} B^{h\,\mathrm{co}H}$.
Then:

φ is htpic H-Hopf-Galois $\Leftrightarrow$ φ satisfies effective htpic Grothendieck descent.

Proof by reducing to normal extension.

Remarks: ∘ Schneider proved a result with a similar flavor in the classical context.

∘ Rognes proved analogous results for commutative ring spectra.

And Koszul?

Recall: $\left.\begin{array}{l} A \text{ Koszul algebra} \\ \quad + \\ C = \text{Koszul dual coalgebra} \end{array}\right\} \Rightarrow \underline{\mathrm{Mod}}_A \underset{QE}{\simeq} \underline{\mathrm{Comod}}_C$.

Corollary: $\varphi: A \longrightarrow B^{\mathcal{D}H}$ as above.

If $B \simeq \Bbbk$, then H is a generalized Koszul dual of B, i.e.,

$$Ho(\underline{\mathrm{Mod}}_B) \simeq Ho(\underline{\mathrm{Comod}}_H).$$

Example: $E \simeq \Bbbk \Rightarrow H$ is a generalized Koszul dual of $\Omega(E; H; \Bbbk)$.