Homotopic Hopf-Galois Extensions and Descent

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I. Descent from a homotopical viewpoint

A. Classical descent theory [Mesablishvili]

1. Grothendieck descent

$\varphi : A \to B$ ring homomorphism

$\Rightarrow$ adjunction $\quad -\otimes_A B : \underline{Mod}_A \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{Mod}_B : \varphi^*$

Informal descent problem (Realizability!)

(a) Given $N \in \underline{Mod}_B$, under what conditions $\exists\, M \in \underline{Mod}_A$ such that $N \cong M \otimes_A B$?

(b) Given $f : M \otimes_A B \to M' \otimes_A B$, under what conditions $\exists\, g : M \to M'$ hm of A-modules st $f = g \otimes_A B$?

The formal framework

- The descent co-ring associated to φ:

$$W_\varphi = (B \otimes_A B, \delta_\varphi, \varepsilon_\varphi)$$ (explain!)

- $\mathcal{D}(\varphi)$ = the category of descent data for φ
  - Ob $\mathcal{D}(\varphi)$ = W_φ-comodules in $\underline{Mod}_B$
  - Mor $\mathcal{D}(\varphi)$: preserve obvious structure
- Functors: Extension of scalars factors through $\mathcal{D}(\varphi)$

$$\begin{tikzcd}
\underline{Mod}_A \arrow[rr, "-\otimes_A B"] \arrow[dr, "Can_\varphi"'] & & \underline{Mod}_B \\
& \mathcal{D}(\varphi) \arrow[ur, "\mathcal{U}"'] &
\end{tikzcd}$$

$$M \longmapsto (M \otimes_A B, \rho_M)$$

Rmk: Can be seen as a Tannakian realization problem wrt fiber functor $-\otimes_A B$.

- **Defⁿ:** φ satisfies (effective) descent if Can_φ is fully faithful (resp. an equivalence of categories).

Formal descent problem

When does φ satisfy descent ($\Rightarrow$ answer to (b))? Effective descent ($\Rightarrow$ answer to (a) as well)?

Rmk: φ satisfies effective descent $\Rightarrow$ $\mathcal{U}: \mathcal{D}(\varphi) \longrightarrow \mathrm{Mod}_B$ solves the associated Tannakian realization problem, with associated "Galois co-group" W_φ.

2. Classical monadic descent: generalizing Grothendieck descent

Categorical preliminaries

A monad on a category $\mathcal{C}$ consists of

- an endofunctor $T: \mathcal{C} \longrightarrow \mathcal{C}$,
- a natural transformation $\mu: T \circ T \longrightarrow T$, and
- a natural transformation $\eta: \mathrm{Id}_{\mathcal{C}} \longrightarrow T$

such that

$$\begin{tikzcd}
T^3X \arrow[r, "\mu_{TX}"] \arrow[d, "T(\mu_X)"'] & T^2X \arrow[d, "\mu_X"] \\
T^2X \arrow[r, "\mu_X"] & TX
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
TX \arrow[d, "T\eta_X"'] \arrow[rd, "\mathrm{Id}_{TX}"] & \\
T^2X \arrow[r, "\mu_X"] & TX \\
TX \arrow[u, "\eta_{TX}"] \arrow[ru, "\mathrm{Id}_{TX}"'] &
\end{tikzcd}$$

commute $\forall X \in \mathrm{Ob}\,\mathcal{C}$.

Notation: $\mathbb{T} = (T, \mu, \eta)$

Dually, a comonad on $\mathcal{C}$ consists of $K: \mathcal{C} \to \mathcal{C}$, $\Delta: K \longrightarrow K^2$ and $\varepsilon: K \longrightarrow \mathrm{Id}_{\mathcal{C}}$ such that $\Delta_K \circ \Delta = K\Delta \circ \Delta$ and $\varepsilon\Delta = \mathrm{Id}_K$.

Notation: $\mathbb{K} = (K, \Delta, \varepsilon)$

Exercise: Let $L : \mathcal{B} \underset{\perp}{\rightleftarrows} \mathcal{C} : R$ be an adjunction. Show that $(RL, R\varepsilon_L, \eta)$ is a monad on $\mathcal{B}$ and that $(LR, L\eta_R, \varepsilon)$ is a comonad on $\mathcal{C}$, where $\eta : \mathrm{Id}_{\mathcal{B}} \to RL$ and $\varepsilon : LR \to \mathrm{Id}_{\mathcal{C}}$ are the unit and counit of the adjunction.

From (co)monads to adjunctions

- $\mathbb{T}$ monad on $\mathcal{C}$ $\Rightarrow$ category $\mathcal{C}^{\mathbb{T}}$ of $\mathbb{T}$-algebras with objects: (A,m) where $m : TA \to A$ in $\mathcal{C}$ st

$$\begin{tikzcd}
T^2A \arrow[r, "Tm"] \arrow[d, "\mu_A"'] & TA \arrow[d, "m"] \\
TA \arrow[r, "m"] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{Id}_A"'] & TA \arrow[d, "m"] \\
 & A
\end{tikzcd}$$

commute.

Exercise: Show that (TX, μ_X) is a $\mathbb{T}$-algebra $\forall X \in \mathrm{Ob}\,\mathcal{C}$. (TX, μ_X) is the free $\mathbb{T}$-algebra on X.
Show moreover that there is an adjunction

$$\begin{tikzcd}
X \arrow[r, mapsto] & (TX, \mu_X) \\
F^{\mathbb{T}} : \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathbb{T}} : U^{\mathbb{T}} \arrow[l, shift left] \\
A & (A,m) \arrow[l, mapsto]
\end{tikzcd} \qquad (\star)$$

Exercise: Let $\varphi : A \to B$ be a ring hm. Let $\mathbb{T}_\varphi$ denote the monad on $\underline{\mathrm{Mod}}_A$ associated to the adjunction $(- \otimes_A B) \dashv \varphi^*$. Show that $(\underline{\mathrm{Mod}}_A)^{\mathbb{T}_\varphi} \cong \underline{\mathrm{Mod}}_B$.

Exercise: Show that the monad associated to $(\star)$ is $\mathbb{T}$.

Exercise: Let $\mathbb{K}^{\mathbb{T}}$ denote the comonad on $\mathcal{C}^{\mathbb{T}}$ associated to $(\star)$. Show that $K^{\mathbb{T}}(A,m) = (TA, \mu_A)$, $(\Delta^{\mathbb{T}})_{(A,m)} = T\eta_A : (TA, \mu_A) \to (T^2A, \mu_{TA})$ and $(\varepsilon^{\mathbb{T}})_{(A,m)} = m : (TA, \mu_A) \to (A,m)$.

Exercise: Calculate $\mathbb{K}^{\mathbb{T}_\varphi}$ explicitly.

Dually ...

- $\mathbb{K}$ comonad on $\mathscr{C}$ $\Rightarrow$ adjunction $\mathscr{C}_{\mathbb{K}} \underset{F_{\mathbb{K}}}{\overset{\mathcal{U}_{\mathbb{K}}}{\underset{\perp}{\rightleftarrows}}} \mathscr{C}$, where
  - $\mathscr{C}_{\mathbb{K}}$ is the category of $\mathbb{K}$-coalgebras, with objects (C, δ), $\delta : C \longrightarrow KC$ st $\Delta_C \circ \delta = T\delta \circ \delta$ and $\varepsilon_C \circ \delta = Id_C$;
  - $F_{\mathbb{K}} X = (KX, \Delta_X)$ the cofree $\mathbb{K}$-coalgebra on X
    $\mathcal{U}_{\mathbb{K}}(C, \delta) = C$

Exercise: Let $\mathbb{K}_\varphi$ denote the comonad associated to $(- \otimes_A B) \dashv \varphi^*$, for some ring homomorphism $\varphi : A \longrightarrow B$. Show that

$$(\underline{\mathrm{Mod}}_B)_{\mathbb{K}_\varphi} \cong \mathscr{D}(\varphi).$$

Informal monadic descent problem

(a) When is a $\mathbb{T}$-algebra isomorphic to a free $\mathbb{T}$-algebra?
(Realizability!)

(b) Given $f : (TX, \mu_X) \longrightarrow (TY, \mu_Y)$ in $\mathscr{C}^{\mathbb{T}}$, when is there $g : X \longrightarrow Y$ in $\mathscr{C}$ such that $f = Tg$?

The formal framework

$$\begin{tikzcd}
X \arrow[d, mapsto] & \mathscr{C} \arrow[rr, shift left, "F^{\mathbb{T}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, "Can^{\mathbb{T}}"'] & & \mathscr{C}^{\mathbb{T}} \arrow[ll, shift left, "U^{\mathbb{T}}"] \arrow[dl, shift left, "F_{\mathbb{K}^{\mathbb{T}}}"'] \arrow[dl, phantom, "\perp"] \\
(TX, \mu_X, T\eta_X) & & (\mathscr{C}^{\mathbb{T}})_{\mathbb{K}^{\mathbb{T}}} = \mathscr{D}(\mathbb{T}) \arrow[ul, shift right, "Prim^{\mathbb{T}} \; (*)"'] \arrow[ul, phantom, "\perp"] \arrow[ur, shift left, "\mathcal{U}_{\mathbb{K}^{\mathbb{T}}}"'] &
\end{tikzcd}$$

- the category of $\mathbb{T}$-descent data.

Rmk: $(A, m, \delta) \in \mathscr{D}(\mathbb{T}) \Longleftrightarrow$
$$\begin{tikzcd}
TA \arrow[r, "T\delta"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2A \arrow[d, "\mu_A"] \\
A \arrow[r, "\delta"] \arrow[dr, "="'] & TA \arrow[d, "m"] \\
 & A
\end{tikzcd}$$

I.e., the multiplication m admits a section in $\mathscr{C}^{\mathbb{T}}$

(*) Exercise: Show that $Can^{\mathbb{T}}$ admits a right adjoint if $\mathscr{C}$ admits equalizers: $Prim^{\mathbb{T}}(A, m, \delta) = \lim \left(A \underset{\eta_A}{\overset{\delta}{\rightrightarrows}} TA \right)$.

Def^n: $\mathbb{T}$ satisfies descent (resp. effective descent) if $Can^{\mathbb{T}}$ is fully faithful (resp. an equivalence of categories

Formal descent problem

When does $\mathbb{T}$ satisfy descent ($\Rightarrow$ answer to (b))?
Effective descent ($\Rightarrow$ answer to (a) as well)?

(Why we care...)

Rmk: Descent data more highly structured than objects in $\mathcal{C}$ and therefore, in principle, easier to compute and classify. The descent framework is also well suited to the study of rigidity problems.

Sometimes we need to work object by object...

Defn: $\mathbb{T}$ satisfies descent at Y if

$$\mathrm{Can}^{\mathbb{T}} : \mathcal{C}(X,Y) \xrightarrow{\cong} \mathcal{D}(\mathbb{T})(\mathrm{Can}^{\mathbb{T}}X, \mathrm{Can}^{\mathbb{T}}Y) \ \forall X \in \mathrm{Ob}\,\mathcal{C}.$$

Exercise: An object Y in $\mathcal{C}$ is $\mathbb{T}$-injective if $\eta_Y : Y \to TY$ admits a retraction $\rho : TY \to Y$. In particular,

$$(A,m) \in \mathcal{C}^{\mathbb{T}} \Rightarrow A \ \mathbb{T}\text{-injective, e.g., } A = TX!$$

Show that: Y $\mathbb{T}$-injective $\Rightarrow$ $\mathbb{T}$ satisfies descent at Y, as long as $\mathcal{C}$ admits equalizers.

Hint: Use the bijection

$$\mathcal{D}(\mathbb{T})(\mathrm{Can}^{\mathbb{T}}X, \mathrm{Can}^{\mathbb{T}}Y) \cong \mathcal{C}(X, \mathrm{Prim}^{\mathbb{T}}\mathrm{Can}^{\mathbb{T}}Y).$$

Exercise: Characterize those A-modules at which $\mathbb{T}_\varphi$ satisfies descent, when $\varphi : A \to B$ is:

(1) $\mathbb{Z} \longrightarrow \mathbb{F}_p$ (reduction mod p)

(2) $R \hookrightarrow R[x]$, for any commutative ring R

Dually . . .

Comonadic codescent

Let $\mathbb{K}$ be a comonad on $\mathcal{C}$. Consider the diagram:

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{K}} \arrow[rr, shift left=2, "\mathcal{U}_{\mathbb{K}}"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift right=2, "F^{\mathbb{T}_{\mathbb{K}}}"'] & & \mathcal{C} \arrow[ll, shift left=2, "F_{\mathbb{K}}"] \arrow[dl, shift right=2, "Can^{co}_{\mathbb{K}}"'] \\
& (\mathcal{C}_{\mathbb{K}})^{\mathbb{T}_{\mathbb{K}}} \arrow[ul, shift right=2, "\mathcal{U}^{\mathbb{T}_{\mathbb{K}}}"'] \arrow[ul, phantom, "\top"] \arrow[ur, shift right=2, "Q_{\mathbb{K}}"'] \arrow[ur, phantom, "\top"] \arrow[d, phantom, sloped, "="] & \\
& \mathcal{D}^{co}(\mathbb{K}) &
\end{tikzcd}$$

⇐ ∃ if $\mathcal{C}$ admits coequalizers

where

- $Can^{co}_{\mathbb{K}}(X) = (KX, \Delta_X, K\varepsilon_X)$
- $(C, \delta, m) \in \mathcal{D}^{co}(\mathbb{K}) \iff$

$$\begin{tikzcd}
C \arrow[d, "\delta"'] \arrow[dr, "Id_C"] \arrow[dr, phantom, "\circlearrowleft", near start] & \\
KC \arrow[r, "m"] \arrow[d, "\Delta_C"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "\delta"] \\
K^2C \arrow[r, "Km"] & KC
\end{tikzcd}$$

I.e., m is a retraction of δ in $\mathcal{C}_{\mathbb{K}}$.

Def^n: $\mathbb{K}$ satisfies codescent (resp. effective codescent) if $Can^{co}_{\mathbb{K}}$ is fully faithful (resp. an equivalence of categories).

"Grothendieck codescent" [Tholen, et al]

Exercise: Let $p: E \to B$ be a continuous map (e.g., $\coprod_{\alpha \in J} \mathcal{U}_\alpha \to \bigcup_{\alpha \in J} \mathcal{U}_\alpha$). Consider the associated adjunction $p_* : \underline{Top}/E \rightleftarrows \underline{Top}/B : p^*$.

Let $\mathbb{T}_p$ and $\mathbb{K}_p$ be the associated monad and comonad. Show that

- $\underline{Top}/E \cong (\underline{Top}/B)_{\mathbb{K}_p}$
- $\mathcal{D}(\mathbb{K}_p) = (\underline{Top}/E)^{\mathbb{T}_{\mathbb{K}_p}}$: objects can be viewed as $f: X \to E$ + $\tau: X \times_B E \to X \times_B E$ satisfying cocycle condition + normalization : gluing data!

B. Homotopic descent

Goal: Develop descent and codescent theory in an "up-to-homotopy" form. A more elaborate version than that presented here, involving $\mathcal{V}$-model categories, has also been developed.

1) Model categories of algebras and coalgebras

Theorem: [Schwede-Shipley, 2000] Let $\mathcal{M}$ be a cofibrantly generated model category with sets $\mathcal{I}$ and $\mathcal{J}$ of generating cofibrations and generating acyclic cofibrations, respectively. Let $\mathbb{T}$ be a monad on $\mathcal{M}$ such that T commutes with filtered colimits. If : i) the domains of $F^{\mathbb{T}}(\mathcal{I})$ and $F^{\mathbb{T}}(\mathcal{J})$ are small with respect to $F^{\mathbb{T}}(\mathcal{I})$-cell and $F^{\mathbb{T}}(\mathcal{J})$-cell, respectively, and

ii) $U^{\mathbb{T}}(F^{\mathbb{T}}(\mathcal{J})\text{-cell}) \subseteq WE$,

then $\mathcal{M}^{\mathbb{T}}$ admits a cofibrantly generated model category structure with sets $F^{\mathbb{T}}(\mathcal{I})$ and $F^{\mathbb{T}}(\mathcal{J})$ of generating cofibrations. In particular,

$$WE_{\mathcal{M}^{\mathbb{T}}} = (U^{\mathbb{T}})^{-1}(WE_{\mathcal{M}}) \text{ and } Fib_{\mathcal{M}^{\mathbb{T}}} = (U^{\mathbb{T}})^{-1}(Fib_{\mathcal{M}}),$$

whence

$$F^{\mathbb{T}}: \mathcal{M} \rightleftarrows \mathcal{M}^{\mathbb{T}} : U^{\mathbb{T}}$$

is a Quillen pair.

Remark: Hypothesis (ii) can be replaced by:

(ii') every object of $\mathcal{C}$ is fibrant and every $\mathbb{T}$-algebra has a path object.

Before stating the existence theorem for model category structure on categories of coalgebras, need another category-theoretic notion.

Def^n: A category $\mathcal{C}$ is locally presentable if

i) $\mathcal{C}$ is small cocomplete;

ii) $\exists$ set $S \subseteq \mathrm{Ob}\,\mathcal{C}$ such that every object is the colimit of a diagram of objects in S;

iii) every object in S is small wrt $\mathrm{Mor}\,\mathcal{C}$;

iv) $\mathcal{C}(X,Y)$ is a set $\forall X, Y \in \mathcal{C}$.

Def^n: A model category is combinatorial if it is cofibrantly generated and locally presentable
[Smith]

Examples: ◦ sSet with either Kan or Joyal model structure

◦ dSet

◦ $Ch_R^{\geq 0}$ with the projective model structure

Non-example: Top is cofibrantly generated but not combinatorial.

Proposition: [Adámek-Rosický] If $\mathcal{C}$ is locally prentable and $\mathbb{T}$ is a a monad on $\mathcal{C}$ such that T preserves filtered colimits, then $\mathcal{C}^{\mathbb{T}}$ is also locally presentable.

Corollary: Under the hypotheses of the Schwede-Shipley theorem, if $\mathcal{M}$ is combinatorial, so is $\mathcal{M}^{\mathbb{T}}$.

What about categories of coalgebras?

Observations: 1) For any comonad $\mathbb{K}$ on a category $\mathcal{C}$, colimits in $\mathcal{C}_{\mathbb{K}}$ are created in $\mathcal{C}$. In particular, if $\mathcal{C}$ is cocomplete, so is $\mathcal{C}_{\mathbb{K}}$. (Exercise!)

2) Since $F_{\mathbb{K}}$ is a right adjoint, it's easy to calculate limits in $\mathcal{C}_{\mathbb{K}}$ of diagrams in the image of $F_{\mathbb{K}}$, e.g., $F_{\mathbb{K}} X \times F_{\mathbb{K}} Y \cong F_{\mathbb{K}}(X \times Y)$. But other limits??

Lemma: [Adámek-Rosický] Let $\mathbb{K}$ be a comonad on a well-powered (e.g., locally presentable) category $\mathcal{C}$.
If K preserves monomorphisms, then $\mathcal{C}_{\mathbb{K}}$ is complete.

Lemma: [Barr-Wells] Let $\mathbb{K}$ be a comonad on a complete category $\mathcal{C}$. If K commutes with countable inverse limits, then $\mathcal{C}_{\mathbb{K}}$ is complete.

The following construction dual to ( )-cell is crucial to proving the existence of model category structure for categories of coalgebras.

Def^n: Let $\mathcal{X}$ be a class of morphisms in a complete category $\mathcal{C}$. An $\mathcal{X}$-Postnikov tower is the composition

$$\lim_{\beta<\lambda} Y_\beta \longrightarrow Y_0$$

of a tower $Y_\bullet : \lambda^{op} \longrightarrow \mathcal{C}$ for some ordinal λ, i.e.,

$$\cdots \longrightarrow Y_{\beta+1} \xrightarrow{q_{\beta+1}} Y_\beta \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{q_2} Y_1 \xrightarrow{q_1} Y_0,$$

where for every $\beta<\lambda$, $\exists\, p_{\beta+1} : X_{\beta+1} \longrightarrow X_\beta \in \mathcal{X}$ and $k_\beta : Y_\beta \longrightarrow X_\beta$ such that

$$\begin{tikzcd} Y_{\beta+1} \arrow[r] \arrow[d, "q_{\beta+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{\beta+1} \arrow[d, "p_{\beta+1}"] \\ Y_\beta \arrow[r, "k_\beta"] & X_\beta \end{tikzcd}.$$

If β is a limit ordinal, then $Y_\beta = \lim_{\alpha<\beta} Y_\alpha$.

Notation:
- $\mathrm{Post}_{\mathcal{X}} = \{\mathcal{X}\text{-Postnikov towers}\}$.
- $\mathcal{Y} \subseteq \mathrm{Mor}\,\mathcal{C} \Rightarrow \hat{\mathcal{Y}} = $ retract-closure of $\mathcal{Y}$.

Def^n: A Postnikov presentation of a class $\mathcal{Y} \subseteq \mathrm{Mor}\,\mathcal{C}$ is a class $\mathcal{X}$ such that $\mathcal{Y} = \widehat{\mathrm{Post}}_{\mathcal{X}}$.

Theorem: [Bayeh-H-Karpova-Kedziorek-Riehl-Shipley]
Let $\mathcal{M}$ be a combinatorial model category such that $\mathrm{Fib}_{\mathcal{M}} \cap \mathrm{WE}_{\mathcal{M}}$ admits a Postnikov presentation via $\mathcal{Z} \subseteq \mathrm{Mor}\,\mathcal{M}$. Let $\mathbb{K}$ be a comonad on $\mathcal{M}$ such that K preserves filtered colimits and monomorphisms or countable inverse limits.
If $\mathcal{U}_{\mathbb{K}}(\mathrm{Post}_{F_{\mathbb{K}}\mathcal{Z}}) \subseteq \mathrm{WE}_{\mathcal{M}}$, then $\mathcal{M}_{\mathbb{K}}$ admits a a model category structure with

$$\mathrm{WE}_{\mathcal{M}_{\mathbb{K}}} = \mathcal{U}_{\mathbb{K}}^{-1}(\mathrm{WE}_{\mathcal{M}}),\ \mathrm{Cof}_{\mathcal{M}_{\mathbb{K}}} = \mathcal{U}_{\mathbb{K}}^{-1}(\mathrm{Cof}_{\mathcal{M}}) \text{ and}$$

$$\mathrm{Fib}_{\mathcal{M}_{\mathbb{K}}} \cap \mathrm{WE}_{\mathcal{M}_{\mathbb{K}}} = \widehat{\mathrm{Post}}_{F_{\mathbb{K}}(\mathcal{Z})}.$$

In particular, $\mathcal{U}_{\mathbb{K}} : \mathcal{M}_{\mathbb{K}} \overset{\perp}{\rightleftarrows} \mathcal{M} : F_{\mathbb{K}}$ is a Quillen pair.

Remark on the proof: Relies heavily on a very recent result of Makkai and Rosicky, on the existence of left-induced weak factorization systems.

Corollary: Let $\mathcal{M}$ be a combinatorial model category with sets $\mathcal{I}$ and $\mathcal{J}$ of generating cofibrations. Let $\mathbb{T}$ be a monad on $\mathcal{M}$ such that T preserves filtered colimits and monomorphisms or countable inverse limits. Let $\mathcal{Z} \subseteq \mathrm{Mor}\,\mathcal{M}^{\mathbb{T}}$ satisfy $(\mathcal{U}^{\mathbb{T}})^{-1}(\mathrm{Fib}_{\mathcal{M}} \cap \mathrm{WE}_{\mathcal{M}}) = \widehat{\mathrm{Post}}_{\mathcal{Z}}$.
If:

i) $F^{\mathbb{T}}(\mathcal{I}), F^{\mathbb{T}}(\mathcal{J})$ small wrt $F^{\mathbb{T}}(\mathcal{I})$-cell, $F^{\mathbb{T}}(\mathcal{J})$-cell, resp.;

ii) $\mathcal{U}^{\mathbb{T}}(F^{\mathbb{T}}(\mathcal{J})\text{-cell}) \subseteq \mathrm{WE}_{\mathcal{M}}$, and

iii) $\mathcal{U}^{\mathbb{T}}\mathcal{U}_{\mathbb{K}^{\mathbb{T}}}(\mathrm{Post}_{F_{\mathbb{K}^{\mathbb{T}}}(\mathcal{Z})}) \subseteq \mathrm{WE}_{\mathcal{M}}$,

then $\exists$ Quillen pair $\mathcal{U}_{\mathbb{K}^{\mathbb{T}}} : \mathcal{D}(\mathbb{T}) \overset{\perp}{\rightleftarrows} \mathcal{M}^{\mathbb{T}} : F_{\mathbb{K}^{\mathbb{T}}}$.

(Similar result can be formulated for codescent.)

__Remarks__: 1) Both $\mathcal{D}(\mathbb{T})$ and $\mathcal{M}^{\mathbb{T}}$ combinatorial, with

$$WE_{\mathcal{D}(\mathbb{T})} = (\mathcal{U}_{\mathbb{K}^{\mathbb{T}}})^{-1}(\mathcal{U}^{\mathbb{T}})^{-1}(WE_{\mathcal{M}})$$

and

$$WE_{\mathcal{M}^{\mathbb{T}}} = (\mathcal{U}^{\mathbb{T}})^{-1}(WE_{\mathcal{M}}).$$

2) [H-Shipley] Different conditions for existence of model category structure on $\mathcal{M}_{\mathbb{K}}$, relying on a sort of "stability" in $\mathcal{M}$ and a filtration of $WE_{\mathbb{K}}$ that is appropriately compatible with $\mathbb{K}$.

A sort of flatness condition.

__Example__: Let $\varphi: A \rightarrow B$ be a morphism of non-negatively graded, dg $\Bbbk$-algebras such that $-\otimes_A B$ preserves monomorphisms (i.e., degreewise injective maps), e.g., B is A-semifree as a left A-module. Since colimits in $\underline{\mathrm{Mod}}_A$ and $\underline{\mathrm{Mod}}_B$ are created in $Ch_{\Bbbk}^{\geq 0}$, $\mathbb{T}_\varphi = \varphi^*(-\otimes_A B)$ preserves all colimits. Moreover, conditions i) and ii) can easily be seen to hold (cf. [Schwede-Shipley]). It remains thus only to determine conditions under which condition iii) also holds.

For example, if B is A-semifree on a free graded $\Bbbk$-module of finite type, then all limits in $\mathcal{D}(\mathbb{T}_\varphi) = \mathcal{D}(\varphi)$ are created in $\underline{\mathrm{Mod}}_B$, so an argument based on the Mittag-Leffler condition shows easily that $\exists$ Quillen pair

$$\mathcal{U}: \mathcal{D}(\varphi) \underset{\perp}{\rightleftarrows} \underline{\mathrm{Mod}}_B .$$

2) Homotopic descent

__Henceforth__: $\mathcal{M}$ a model category, $\mathbb{T}$ a monad on $\mathcal{M}$ such that $\exists$ diagram of Quillen pairs

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dl, shift right] \arrow[dl, phantom, "\perp"] \arrow[dr, shift left] \arrow[dr, phantom, "\perp"] & \\
\mathcal{D}(\mathbb{T}) \arrow[ur, shift right] \arrow[rr, shift right] \arrow[rr, phantom, "\top"] & & \mathcal{M}^{\mathbb{T}} \arrow[ul, shift left] \arrow[ll, shift right]
\end{tikzcd}$$

with weak equivalences created in $\mathcal{M}$.

Recall/Defn: [Dwyer-Kan] Let $\mathcal{M}$ be a model category. The simplicial localization of $\mathcal{M}$ is a simplicial enrichment of $\mathcal{M}$, denoted $\mathrm{Map}_{\mathcal{M}}(-,-)$, such that

$$\left.\begin{array}{l} X' \xrightarrow{\sim} X \\ Y \xrightarrow{\sim} Y' \end{array}\right\} \Rightarrow \mathrm{Map}_{\mathcal{M}}(X,Y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{M}}(X',Y').$$

and $\pi_0 \mathrm{Map}_{\mathcal{M}}(X,Y) \cong \mathrm{Ho}(\mathcal{M})(X,Y) \cong [X^c, Y^f]$.

$\mathrm{Map}_{\mathcal{M}}(X,Y)$ is the derived mapping space.

[Dwyer-H]: $L : \mathcal{M} \underset{\leftarrow}{\overset{\perp}{\rightarrow}} \mathcal{M}' : R$ Quillen pair

$\Rightarrow \mathrm{Map}^h_{\mathcal{M}}(X, R(Y^f)) \simeq \mathrm{Map}^h_{\mathcal{M}'}(L(X^c), Y)$

Remark: The simplicial localization of a model category always exists, and there are many weakly equivalent ways of constructing the derived mapping spaces, e.g., by [Dugger]:

$$(\star\star)\quad \mathrm{Map}^h_{\mathcal{M}}(X,Y) \simeq \mathrm{Nerve}\left(\begin{tikzcd} & \bullet \arrow[dd, dotted, "\sim"] & \\ X \arrow[ur] \arrow[dr] & & Y \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & \bullet & \end{tikzcd}\right), \text{ if } X \text{ cofibrant.}$$

(Need to be a little careful about smallness...)

Defn: The monad $\mathbb{T}$ satisfies homotopic descent if

$$\mathrm{Map}^h_{\mathcal{M}}(X,Y) \xrightarrow{\sim} \mathrm{Map}^h_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}^{\mathbb{T}}X, \mathrm{Can}^{\mathbb{T}}Y)$$

$\forall X, Y$ bifibrant.

Remark: The simplicial map of derived mapping spaces above arises from the functor

$$(X \longrightarrow W \xleftarrow{\sim} Y) \longmapsto (\mathrm{Can}^{\mathbb{T}}X \longrightarrow \mathrm{Can}^{\mathbb{T}}W \xleftarrow{\sim} \mathrm{Can}^{\mathbb{T}}Y)$$

where we use that $\mathrm{Can}^{\mathbb{T}}$ is a left Quillen functor to deduce that $\mathrm{Can}^{\mathbb{T}}X$ is cofibrant (so that $(\star\star)$ applies) and that the map from $\mathrm{Can}^{\mathbb{T}}Y$ to $\mathrm{Can}^{\mathbb{T}}W$ is an acyclic cofibration.

Defn: The monad $\mathbb{T}$ satisfies effective homotopic descent if $\mathrm{Can}^{\mathbb{T}} : \mathcal{M} \longrightarrow \mathcal{D}(\mathbb{T})$ is a Quillen equivalence.

Remark: If $\mathbb{T}$ satisfies effective homotopic descent, then it also satisfies homotopic descent. Indeed,

$\mathrm{Can}^{\mathbb{T}}$ Quillen equivalence $\Rightarrow \tilde{\eta}_X : X \xrightarrow{\sim} \mathrm{Prim}^{\mathbb{T}}(\mathrm{Can}^{\mathbb{T}}X)^f$, $\forall X$ cofib.

so $\exists$ diagram in sSet $\forall X, Y$ bifibrant in $\mathcal{M}$

$$\begin{tikzcd}
\mathrm{Map}^h_{\mathcal{M}}(X,Y) \arrow[r, "\sim"] \arrow[d, "\therefore\ \sim"'] & \mathrm{Map}^h_{\mathcal{M}}(X, \mathrm{Prim}^{\pi}(\mathrm{Can}^{\pi}Y)^f) \\
\mathrm{Map}^h_{\mathcal{D}(\pi)}(\mathrm{Can}^{\pi}X, \mathrm{Can}^{\pi}Y) \arrow[r, "\sim"] & \mathrm{Map}^h_{\mathcal{D}(\pi)}(\mathrm{Can}^{\pi}X, (\mathrm{Can}^{\pi}Y)^f). \arrow[u, "\sim\ \text{[Dwyer-H.]}"']
\end{tikzcd}$$

Questions: How to determine when (effective) homotopic descent satisfied? What does it mean?

Basic observations: π satisfies effective homotopic descent

(Using known characterization of Quillen equivalences.)

$\Leftrightarrow$
- $\tilde{\eta}_X : X \xrightarrow{\sim} \underline{\mathrm{Prim}^{\pi}(\mathrm{Can}^{\pi}X)^f}$ $\forall X$ cofibrant

π-completion, in a sense to be specified

and
- $(A,m,\delta), (A',m',\delta')$ fibrant in $\mathcal{D}(\pi) \Rightarrow$:

  $f: (A,m,\delta) \longrightarrow (A',m',\delta') \in WE_{\mathcal{D}(\pi)} \Leftrightarrow \mathrm{Prim}^{\pi}f \in WE_{\mathcal{M}}$

Realizability!

$\Leftrightarrow$
- $\varepsilon_{(A,m,\delta)} : \mathrm{Can}^{\pi}(\mathrm{Prim}^{\pi}(A,m,\delta))^c \xrightarrow{\sim} (A,m,\delta)$ $\forall (A,m,\delta)$ fibrant

and
- X, X' cofibrant in $\mathcal{M} \Rightarrow$

  $\Leftarrow$ = rigidity problems!

  $f: X \longrightarrow X' \in WE_{\mathcal{M}} \Leftrightarrow \mathrm{Can}^{\pi}f \in WE_{\mathcal{D}(\pi)}$,

  i.e., π preserves and reflects weak equivalences between cofibrant objects: homotopic faithful flatness.

3) The descent spectral sequence: tool for studying homotopic descent, interpolating backwards along $\mathrm{Map}^h_{\mathcal{M}}(X,Y) \longrightarrow \mathrm{Map}^h_{\mathcal{D}(\pi)}(\mathrm{Can}^{\pi}X, \mathrm{Can}^{\pi}Y)$

[Blumberg-Riehl], [Noël-Johnson], [Noël]: closely related work

Need more structure on $\mathcal{M}$ to do computations...

Def^n: A simplicial model category consists of an sSet-enriched category $\mathcal{M}$ that is tensored and cotensored over sSet such that the underlying (ordinary) category $\mathcal{M}_0$ is endowed with a model category structure, compatible with the enrichment in the sense that

(Recall tensoring, cotensoring, etc.)

(SM7): $i: A \rightarrowtail X,\ p: E \twoheadrightarrow B \Rightarrow (i^*, p_*): \mathrm{Map}(X,E) \twoheadrightarrow \mathrm{Map}(A,E) \times_{\mathrm{Map}(A,B} \mathrm{Map}(X,B)$

Rmk: If $\mathcal{M}$ is a simplicial model category, then

$$\mathrm{Map}^h_{\mathcal{M}}(X,Y) \simeq \mathrm{Map}(X^c, Y^f) \quad \forall X,Y \in \mathrm{Ob}\,\mathcal{C}.$$

Def^n: Let $\mathcal{M}$ be a simplicial model category. The totalization functor

$$\mathrm{Tot}: \mathcal{M}^{\Delta} \longrightarrow \mathcal{M}$$

(Need only cotensoring for def^n) is defined on objects by

$$\mathrm{Tot}(X^\bullet) = \mathrm{equal}\Big(\prod_{n\geq 0} (X^n)^{\Delta[n]} \rightrightarrows \prod_{\substack{\Delta(n,k)\\ k,n\geq 0}} (X^k)^{\Delta[n]} \Big).$$

Key properties:
1) $X^\bullet$ Reedy fibrant $\Rightarrow$ $\mathrm{Tot}\, X^\bullet$ fibrant in $\mathcal{M}$.
2) $\mathrm{Tot}(cc^\bullet X) \cong X$ $\forall X \in \mathrm{Ob}\,\mathcal{M}$. (Exercise!)
3) $cc^\bullet X \underset{r^\bullet}{\overset{i^\bullet}{\rightleftarrows}} Y^\bullet \,\&\, h$ "external" SDR in $\mathcal{M}^{\Delta}$ $\Rightarrow$ $X \simeq \mathrm{Tot}\, Y^\bullet$.

Proof by dualization of classical result by J-P Meyer. $\rightarrow$

The descent SS associated to a monad $\mathbb{T}$ is a special case of the following type of SS.

Def^n: [Bousfield-Kan] Let $Y^\bullet$ be a pointed Reedy fibrant cosimplicial simplicial object. The extended homotopy SS (BKSS) associated to $Y^\bullet$ has

$$E_2^{s,t} = \pi^s \pi_t Y^\bullet \quad \forall t \geq s \geq 0$$

(in particular: $t \geq 2 \Rightarrow E_2^{s,t} = H^s(N_*(\pi_t Y^\bullet))$) cosimplicial abelian group

and abuts to (subtle convergence issues!)

$$\pi_*(\mathrm{Tot}\, Y^\bullet).$$

The cosimplicial simplicial set to which we apply the BKSS construction is defined as follows, for any monad on a simplicial model category $\mathcal{M}$.

Rmk: $\mathcal{M}$ simplicial model cat $\not\Rightarrow$ $\mathcal{D}(\mathbb{T})$ simplicial in general (cotensoring issues).

Def^n: Let $Y \in \mathrm{Ob}\,\mathcal{M}$. The $\mathbb{T}$-cobar construction on Y, $\Omega^\bullet_{\mathbb{T}} Y \in \mathcal{M}^\Delta$, is defined by

$$\begin{tikzcd}
\Omega^\bullet_{\mathbb{T}} Y = \Big(\mathbb{T}Y \arrow[r, shift left=6, "\eta_{\mathbb{T}Y}"] \arrow[r, shift right=6, "\mathbb{T}\eta_Y"'] & \mathbb{T}^2 Y \arrow[l, "\mu_Y"'] \arrow[r, shift left=10, "\eta_{\mathbb{T}^2 Y}"] \arrow[r, "\mathbb{T}\eta_Y"] \arrow[r, shift right=10, "\mathbb{T}^2\eta_Y"'] & \mathbb{T}^3 Y \arrow[l, shift right=5, "\mu_{\mathbb{T}Y}"'] \arrow[l, shift left=5, "\mathbb{T}\mu_Y"] & \cdots \Big)
\end{tikzcd}$$

(Exercise: Check that the cosimplicial identities hold.)

Remark: $\exists \quad \eta^\bullet : cc^\bullet Y \longrightarrow \Omega^\bullet_{\mathbb{T}} Y \quad \forall Y \in \mathcal{M}$

If $Y = \mathcal{U}^{\mathbb{T}}(Y, m)$, then $\eta^\bullet$ fits into an external SDR

$$\begin{tikzcd}
cc^\bullet Y \arrow[r, shift left, "\eta^\bullet"] & \Omega^\bullet_{\mathbb{T}} Y \arrow[l, shift left, "m^\bullet"] \arrow[loop right, "h"]
\end{tikzcd}$$

and so $Y \simeq \mathrm{Tot}\,\Omega^\bullet_{\mathbb{T}} Y$.

Also, $\exists$ external SDR in $(\mathcal{M}^{\mathbb{T}})^\Delta$

$$cc^\bullet F^{\mathbb{T}} Y \rightleftarrows (F^{\mathbb{T}})^\Delta \Omega^\bullet_{\mathbb{T}} Y \quad \forall Y,$$

so $F^{\mathbb{T}} Y \simeq \mathrm{Tot}(F^{\mathbb{T}})^\Delta \Omega^\bullet_{\mathbb{T}} Y$ if $\mathbb{T}$ simpl.

Input data for the $\mathbb{T}$-descent SS

- $f : X \longrightarrow Y \in \mathcal{M}$, where X cofibrant
- $j : \Omega^\bullet_{\mathbb{T}} Y \xrightarrow{\sim} \hat{Y}^\bullet$, where $\hat{Y}^\bullet$ is Reedy fibrant and levelwise $\mathbb{T}$-injective.

Always levelwise $\mathbb{T}$-injective.

Lemma: If $\mathcal{U}^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ preserves acyclic cofibrations, then $\hat{Y}^\bullet$ exists $\forall Y \in \mathrm{Ob}\,\mathcal{M}$.

Def^n: The $\mathbb{T}$-descent spectral sequence at f is the BKSS of the cosimplicial simplicial set $\mathrm{Map}_{\mathcal{M}}(X, \hat{Y}^\bullet)$, pointed at $j\eta^\bullet \circ cc^\bullet f$.

Notation: $E^{\mathbb{T}}_f$

Interpretation:

since $\hat{Y}^n$ $\mathbb{T}$-injective $\forall n$

- $(E^{\mathbb{T}}_f)_2$: $\mathrm{Map}_{\mathcal{M}}(X, \hat{Y}^\bullet) \cong \mathrm{Map}_{\mathcal{M}}(X, (\mathrm{Prim}^{\mathbb{T}} \mathrm{Can}^{\mathbb{T}})^\Delta \hat{Y}^\bullet)$

Using result of [Dwyer-H],
OK since X cofib, $\hat{Y}^n$ fib $\forall n$

$$\simeq \mathrm{Map}^h_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}^{\mathbb{T}} X, (\mathrm{Can}^{\mathbb{T}})^\Delta \hat{Y}^\bullet)$$

So: $(E^{\mathbb{T}}_f)^{s,t}_2 \cong \pi^s \pi_t \mathrm{Map}^h_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}^{\mathbb{T}} X, (\mathrm{Can}^{\mathbb{T}})^\Delta \hat{Y}^\bullet)$.

Let's analyze the target component, assuming $\mathbb{T}$ is simplicial.

$$cc^\bullet : \mathrm{Can}^{\mathbb{T}}Y \longrightarrow (\mathrm{Can}^{\mathbb{T}})^{\Delta}\,\Omega^\bullet_{\mathbb{T}}Y \longrightarrow (\mathrm{Can}^{\mathbb{T}})^{\Delta}\,\hat{Y}^\bullet \quad \text{in } \mathcal{D}(\mathbb{T})^{\Delta}$$

is a cosimplicial injective resolution, i.e.,

- $(\mathrm{Can}^{\mathbb{T}})^{\Delta}\hat{Y}^\bullet$ is Reedy fibrant and levelwise injective wrt to the monad associated to $\mathcal{U}_{\mathbb{K}^{\mathbb{T}}} : \mathcal{D}(\mathbb{T}) \rightleftarrows \mathcal{M}^{\mathbb{T}} : F_{\mathbb{K}^{\mathbb{T}}}$, with underlying functor is
$$(A, m, \delta) \longmapsto (\mathbb{T}A, \mu_A, \mathbb{T}\eta_A);$$
- $\mathcal{U}_{\mathbb{K}^{\mathbb{T}}}\mathrm{Can}^{\mathbb{T}}Y = F^{\mathbb{T}}Y \simeq \mathrm{Tot}\,(F^{\mathbb{T}})^{\Delta}\,\Omega^\bullet_{\mathbb{T}}Y = \mathrm{Tot}\,(\mathcal{U}_{\mathbb{K}^{\mathbb{T}}}\mathrm{Can}^{\mathbb{T}})^{\Delta}\,\Omega^\bullet_{\mathbb{T}}Y$
and under reasonable hypotheses ← e.g., Y fibrant and $\mathbb{T}$ preserves fibrant objects and w.eq. between fibrants
$(\mathcal{U}_{\mathbb{K}}^{\mathbb{T}}\mathrm{Can}^{\mathbb{T}})^{\Delta}\,\Omega^\bullet_{\mathbb{T}}Y \xrightarrow{\simeq} (\mathcal{U}_{\mathbb{K}^{\mathbb{T}}}\mathrm{Can}^{\mathbb{T}})^{\Delta}\,\hat{Y}^\bullet$

So under good conditions

$$F^{\mathbb{T}}Y \xrightarrow{\simeq} \mathrm{Tot}\,(F^{\mathbb{T}})^{\Delta}\,\hat{Y}^\bullet,$$

justifying

$$(E_f^{\mathbb{T}})_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{D}(\mathbb{T})}(\mathrm{Can}^{\mathbb{T}}Y, \mathrm{Can}^{\mathbb{T}}Y).$$

Independent of choice of $\hat{Y}^\bullet$ [HKRS]: Under reasonable conditions...

<u>$(E_f^{\mathbb{T}})_\infty$</u>: $\mathrm{Tot}\,\mathrm{Map}_{\mathcal{M}}(X, \hat{Y}^\bullet) \simeq \mathrm{Map}_{\mathcal{M}}(X, \mathrm{Tot}\,\hat{Y}^\bullet) \simeq \mathrm{Map}_{\mathcal{M}}(X, \mathrm{Prim}^{\mathbb{T}}(\mathrm{Can}^{\mathbb{T}}Y)^f).$

$\mathrm{Tot}\,\hat{Y}^\bullet$ can be seen as a $\mathbb{T}$-completion of Y in the following sense, which doesn't require the full strength of simplicial model categories.

- A morphism $f: X \to Y$ in $\mathcal{M}$ is a $\mathbb{T}$-equivalence if
$$\mathrm{Map}_{\mathcal{M}}(Y, \mathcal{U}^{\mathbb{T}}(A,m)) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{M}}(X, \mathcal{U}^{\mathbb{T}}(A,m))$$
for all $(A,m) \in \mathcal{M}^{\mathbb{T}}$ such that (A,m) fibrant, i.e., A fibrant in $\mathcal{M}$.

<u>Exercise</u>: Let $f: X \to Y \in \mathcal{M}$, where X, Y cofibrant. Prove that $\mathbb{T}f \in WE_{\mathcal{M}} \Rightarrow f$ a $\mathbb{T}$-equivalence and that the converse holds if $\mathcal{M}$ is a simplicial model category.

- An object Z in $\mathcal{M}$ is Π-complete if
$$f: X \longrightarrow Y \text{ } \Pi\text{-equivalence} \Rightarrow f^*: \mathrm{Map}^h(Y,Z) \xrightarrow{\sim} \mathrm{Map}^h(X,Z).$$

Exercise: Show that:

i) $(A,m) \in \mathcal{M}^\Pi \Rightarrow A$ Π-complete

ii) W retract of Z, Z Π-complete $\Rightarrow W$ Π-complete

iii) i) + ii) $\Rightarrow$ every Π-injective object is Π-complete

iv) Z, Z' weakly equivalent $\Rightarrow$
Z Π-complete $\Leftrightarrow$ Z' Π-complete.

Lemma: $Z^\bullet \in \mathcal{M}^\Delta$ Reedy fibrant such that Z^n Π-complete $\forall n$
$\Rightarrow \mathrm{Tot}\, Z^\bullet$ is Π-complete

Here we assume that $\mathcal{M}$ is a simplicial model category.

Thus, under our hypotheses, $\mathrm{Tot}\, \hat{Y}^\bullet$ is indeed Π-complete.

Notation: $Y^\wedge_\Pi = \mathrm{Tot}\, \hat{Y}^\bullet$.

Conclusion: The Π-descent SS at f abuts to

under reasonable conditions
$$\pi_* \mathrm{Map}_{\mathcal{M}}(X, Y^\wedge_\Pi) \simeq \mathrm{Map}_{\mathcal{M}}(X, \mathrm{Prim}^\Pi(\mathrm{Can}^\Pi Y)^f).$$

So the Π-descent SS can be seen as "interpolating backwards" from the target of $\mathrm{Map}^h_{\mathcal{M}}(X,Y) \longrightarrow \mathrm{Map}^h_{\mathcal{D}(\Pi)}(\mathrm{Can}^\Pi X, \mathrm{Can}^\Pi Y)$ to its source. It's therefore a tool for measuring deviation from satisfying homotopic descent.

Example: Let $\varphi: A \longrightarrow B$ be a monoid morphism in a monoidal model category such that $\mathcal{D}(\varphi)$ admits a model category structure of the sort studied above.

Then $\Omega^\bullet_{\Pi_\varphi} M$ is the well-known Amitsur complex.

If $\Omega^\bullet_{\Pi_\varphi} M$ is Reedy fibrant, so that we may choose completion of M along $\mathbb{T}$ (cf. e.g., [Carlsson]).

The $\mathbb{T}_\varphi$-descent SS looks like:

$$\mathrm{Ext}^{s,t}_{\mathfrak{D}(\varphi)}\big((M\otimes_A B, \rho_M), (N\otimes_A B, \rho_N)\big)$$
$$\Longrightarrow \pi_{t-s}\mathrm{Map}_{\underline{\mathrm{Mod}}_A}(M, N^\wedge_\varphi).$$

Special cases: a) $\eta: I \longrightarrow B \Rightarrow$ Adams-type SS

b) $$\begin{tikzcd} A \arrow[rr, "\varepsilon"] \arrow[dr, tail, "\tilde{\varepsilon}"'] & & I \Rightarrow \text{Quillen-homology SS} \\ & Q \arrow[ur, "\sim"'] & \end{tikzcd}$$

$$\mathrm{Ext}^{s,t}_{Q\otimes_A Q}(M\otimes_A Q, N\otimes_A Q) \Rightarrow \pi_{t-s}\mathrm{Map}_{\underline{\mathrm{Mod}}_A}(M, N^\wedge_{\tilde{\varepsilon}}).$$

htpy indecomposables

Remarks: 1) One should be able extract an "obstruction theory à la Bousfield", for measuring obstructions to realizing a $\mathbb{T}$-alg as a free $\mathbb{T}$-algebra and to realizing a morphism between free $\mathbb{T}$-algebras as an element of $\mathrm{Im}\, F^{\mathbb{T}}$, from the $\mathbb{T}$-descent SS.

2) $\exists$ dual theory of homotopic codescent and an associated SS. Applied to the Ganea comonad, one should be able to use this machinery to study approximations to LS-category.

Example: Let $p: P_*B \twoheadrightarrow B$ be a Kan fibration in $\underline{\mathrm{sSet}}$, where $P_*B \simeq *$. Consider $p_*: \underline{\mathrm{sSet}}/P_*B \underset{\leftarrow}{\overset{\perp}{\rightarrow}} \underline{\mathrm{sSet}}/B : p^*$. The canonical codescent datum associated to $X \xrightarrow{f} B$ is ${}^{GB}\circlearrowright \mathrm{hfib}(f) \to P_*B$, where GB = the Kan loop gp on B. The associated comonad $\mathbb{K}_p$ satisfies homotopic codescent: $\mathrm{Map}_{\underline{\mathrm{sSet}}/B}(f, q) \xrightarrow{\sim} \mathrm{Map}_{\mathfrak{D}^{co}(\mathbb{K}_p)}(\mathrm{hfib}(f), \mathrm{hfib}(q))$ $\forall\, q$ Kan fib.

II. Homotopic Hopf-Galois extensions

monoid morphisms satisfying homotopic Grothendieck descent.

A. Classical Hopf-Galois extensions [Chase-Sweedler], [Kreimer-Takeuchi]

Hopf-Galois data Dualizing the notion of a Galois extension . . .

Notation: $\varphi : A \longrightarrow B^{\circlearrowleft H}$

- $\Bbbk$ commutative ring
- H - a $\Bbbk$-bialgebra
- A - a $\Bbbk$-algebra with trivial H-coaction
- B - an H-comodule algebra with coaction $\rho : B \longrightarrow B \otimes H$
- $\varphi : A \longrightarrow B$ - a morphism of H-comodule algebras

Associated homomorphisms

- The Galois map

$$\begin{tikzcd}
B \otimes_A B \arrow[r, "B \otimes_A \rho"] \arrow[rr, bend right, "\beta_\varphi"'] & B \otimes_A B \otimes H \arrow[r, "\bar{\mu} \otimes H"] & B \otimes H
\end{tikzcd}$$

- The corestriction map

$$A \xrightarrow{i_\varphi} B^{coH} = B \mathbin{\square}_H \Bbbk = \{ b \in B \mid \rho(b) = b \otimes 1 \}.$$

Defn: $\varphi : A \longrightarrow B^{\circlearrowleft H}$ is a Hopf-Galois extension if β_φ and i_φ are both isomorphisms.

Examples: 1) Let G be a finite group, $G < \mathrm{Aut}_{\Bbbk}(E)$ for some field extension $\Bbbk \subset E$. Let $F = E^G$.
[Montgomery]

Then: $F \hookrightarrow E$ is a G-Galois extension

$\Longleftrightarrow$ $F \hookrightarrow E$ is a $\Bbbk^G$-Hopf-Galois extension,

where $\Bbbk^G = \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk)$.

2) Let $r: X \times G \to X$ be an action of a finite group on a finite set. Let $Y = X_G$ = the set of G-orbits, and let $q: X \to Y$ denote the quotient map.

Consider:

$$\begin{tikzcd}
X \times G \arrow[r, "\Delta \times G"] \arrow[dr] \arrow[drr, bend right, "\alpha"'] & X \times X \times G \arrow[r, "X \times r"] & X \times X \\
 & X \times_Y X \times G \arrow[u, hook] \arrow[r, "X \times r"] & X \times_Y X \arrow[u, hook]
\end{tikzcd}$$

Let $\Bbbk$ be a field. Let $\Bbbk^G$ be defined as above, while $\Bbbk^X = \underline{\mathrm{Set}}(X, \Bbbk)$, $\Bbbk^Y = \mathrm{Set}(Y, \Bbbk)$, endowed with pointwise $+$ and $\cdot$.

$\Rightarrow$ Hopf-Galois data $\Bbbk^Y \xrightarrow{q^*} \Bbbk^X \circlearrowleft \Bbbk^G$

<u>Exercise</u>: q^* is a $\Bbbk^G$-Hopf-Galois extension
$\iff$ α is a bijection
$\iff$ the action r is free.
(Show that $\beta_{q^*} = \alpha^*$.)

3) Let H be a $\Bbbk$-bialgebra, seen as an H-comodule algebra over itself. Consider $\varphi = \eta: \Bbbk \to H$.

Then:
- i_η clearly an iso
- $$\begin{tikzcd}
H \otimes H \arrow[r, "\Delta \otimes H"] \arrow[rr, bend right, "\beta_\eta"'] & H \otimes H \otimes H \arrow[r, "H \otimes \mu"] & H \otimes H
\end{tikzcd}$$
 – iso iff H is a Hopf algebra

More generally, if H is a Hopf algebra, then
$$A \otimes \eta_H : A \hookrightarrow A \otimes H$$
is an H-Hopf-Galois extension, of normal basis type.

Why algebraists care about HG-extensions

- Generalization of Galois theory
- Faithfully flat HG-extensions over coordinate ring of an affine group scheme correspond to G-principal fiber bundles
- Can study Hopf algebras via their associated HG-extensions.

Why homotopy theorists might care

somewhat ad hoc defn

[Rognes] The unit map $\eta : S \to MU$ is a homotopic HG-ext over $S[BU]$, which is NOT a homotopic Galois extension for any group G.

B. Homotopic HG-extensions

1) Co-rings and their comodules

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category.
If (A, μ, η) is a monoid in $\mathcal{V}$, then $({}_A\underline{\mathrm{Mod}}_A, \otimes_A, A)$ is also a monoidal category.

Defn: An A-co-ring is a comonoid in ${}_A\underline{\mathrm{Mod}}_A$, i.e., (V, δ, ε) where:
- V is an A-bimodule,
- $\delta : V \to V \otimes_A V$ + coassociativity and
- $\varepsilon : V \to A$ counitality.

Defn: Let (V, δ, ε) be a V-co-ring. A (right) V-comodule consists of (M, ρ) where $M \in \underline{\mathrm{Mod}}_A$ and

$$\rho : M \to M \otimes_A V \quad \in \underline{\mathrm{Mod}}_A$$

such that $(M \otimes_A \delta)\rho = (\rho \otimes_A V)\rho$, $(M \otimes_A \varepsilon)\rho = \mathrm{Id}_M$.

Notation: $\mathcal{V}_A^V$ = category of right V-comodules in $\underline{\mathrm{Mod}}_A$

Remark: • $A = \mathbb{I} \Rightarrow {}_A\mathrm{Mod}_A = \mathcal{V}$ and an A-co-ring is just a comonoid in $\mathcal{V}$

• $V = A \Rightarrow \mathcal{V}_A^A \cong \mathrm{Mod}_A$.

Remark: If (V,δ,ε) is an A-co-ring, then $(\overset{\overset{\mathbb{K}_V}{\|}}{-\otimes_A V}, -\otimes_A \delta, -\otimes_A \varepsilon)$ is a comonad on Mod_A, so can apply [BHKKRS] thm to get model cat structure.

Important adjunctions

- The forgetful/cofree - adjunction

$$\begin{tikzcd} \mathcal{V}_A^V \arrow[r, shift left, "U_{\mathbb{K}_V}"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_A \arrow[l, shift left, "-\otimes_A V = F_{\mathbb{K}_V}"] \end{tikzcd}$$

(Special case of the adjunction below, for $g=\varepsilon$.)

Rmk: Henceforth consider only co-rings V st $-\otimes_A V$ preserves monos, so that $\mathcal{V}_A^V$ complete.

- For every $g:(V,\delta,\varepsilon) \longrightarrow (V',\delta',\varepsilon')$ morphism of co-rings, ∃ "restriction of coefficients" adjunction

$$\begin{tikzcd} \mathcal{V}_A^V \arrow[r, shift left, "g_*"] & \mathcal{V}_A^W, \arrow[l, shift left, "-\square_{V'} V"] \end{tikzcd}$$

where $\forall (M,\rho) \in \mathrm{Ob}\, \mathcal{V}_A^W$,

$$M \square_{V'} V = \mathrm{equal}\Big(\begin{tikzcd} M\otimes_A V \arrow[r, shift left, "\rho\otimes_A V"] \arrow[r, shift right, "M\otimes_A (g\otimes_A V)\delta"'] & M\otimes_A V'\otimes_A V \end{tikzcd} \Big)$$

– computed in $\mathcal{V}_A^V$.

E.g., coaugmented co-ring $\eta: A \longrightarrow V$

$$\Rightarrow \begin{tikzcd} \mathrm{Mod}_A = \mathcal{V}_A^A \arrow[r, shift left, "\eta_*"] & \mathcal{V}_A^V \arrow[l, shift left, "-\square_V A"] \end{tikzcd}$$

– the coinvariants functor

Homotopy theory

Suppose now that $(\mathcal{M}, \otimes, \mathbb{I})$ is a monoidal model category, which is combinatorial and such that every object in $\mathcal{M}$ is small relative to Mor $\mathcal{M}$.

Let A be a monoid in $\mathcal{M}$, and let V be an co-ring such that $-\otimes_A V$ preserves monos.

Let $\mathcal{J}$ denote the set of generating acyclic cofibrations of $\mathcal{M}$.

If $(\mathcal{J}\otimes A)\text{-cell} \underset{(*)}{\subseteq} WE$, then by the Schwede-Shipley theorem

$\exists$ Quillen pair
$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "-\otimes A"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Mod}}_A \arrow[l, shift left, "\mathcal{U}"] \end{tikzcd}$$

with $\mathrm{Fib}_{\underline{\mathrm{Mod}}_A} = \mathcal{U}^{-1}(\mathrm{Fib}_{\mathcal{M}})$, $WE_{\underline{\mathrm{Mod}}_A} = \mathcal{U}^{-1}(WE_{\mathcal{M}})$

$\mathrm{Cof}_{\underline{\mathrm{Mod}}_A} = \widehat{(\mathcal{I}\otimes A)\text{-cell}}$.

often simply all acyclic cofib.

If $\exists\, \mathcal{Z} \subseteq \mathrm{Fib}_{\underline{\mathrm{Mod}}_A} \cap WE_{\underline{\mathrm{Mod}}_A}$ st $\mathrm{Fib}_{\underline{\mathrm{Mod}}_A} \cap WE_{\underline{\mathrm{Mod}}_A} = \widehat{\mathrm{Post}_{\mathcal{Z}}}$, then by the HKKRS theorem:

$\mathcal{U}(\mathrm{Post}_{\mathcal{Z}\otimes_A V}) \underset{(**)}{\subseteq} WE \Rightarrow \exists$ Quillen pair $\mathcal{U} : \mathcal{M}_A^V \rightleftarrows \underline{\mathrm{Mod}}_A : -\otimes_A V$

with $WE_{\mathcal{M}_A^V} = \mathcal{U}^{-1}(WE_{\underline{\mathrm{Mod}}_A})$

$\mathrm{Cof}_{\mathcal{M}_A^V} = \mathcal{U}^{-1}(\widehat{\mathrm{Cof}_{\underline{\mathrm{Mod}}_A}})$

$\mathrm{Fib}_{\mathcal{M}_A^V} \cap WE_{\mathcal{M}_A^V} = \widehat{\mathrm{Post}_{\mathcal{Z}\otimes_A V}}$.

Remark: Henceforth we always assume $(*)$ holds for A, and we consider only co-rings for which $(**)$ holds.

Note that if $(**)$ holds for V, W, then any morphism of co-rings $g: V \to W$ gives rise to a Quillen pair
$$\begin{tikzcd} \mathcal{M}_A^V \arrow[r, shift left, "g_*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_A^W \arrow[l, shift left, "- \square_W V"] \end{tikzcd}.$$

In particular, $\exists$ Quillen pair
$$\begin{tikzcd} \underline{\mathrm{Mod}}_A \arrow[r, shift left, "\eta_* = \mathrm{Triv}_V(-)"] & \mathcal{M}_A^V \arrow[l, shift left, "(\ )^{coV}"] \end{tikzcd}$$
$\forall$ coaugmented co-rings V.

Example: $\mathcal{M} = Ch_{\Bbbk}^{\geq 0}$, A any dg $\Bbbk$-algebra, V an A-coring that's semi-free as a left A-module on a generating graded $\Bbbk$-module of finite type

$\Rightarrow$ have the desired model cat structure on $\mathcal{M}_A^V$.

2) Homotopify the HG-framework

The data $\varphi: A \longrightarrow B^{hH}$ that satisfies the monoid axiom.

- H is a bimonoid with comultiplication $\Delta: H \longrightarrow H\otimes H$
- A is a monoid, seen as an H-comodule with trivial coaction $A \xrightarrow{A\otimes\eta} A\otimes H$
- B is an H-comodule monoid with coaction $\rho: B \to B\otimes H$, which is a monoid morphism
- $\varphi: A \longrightarrow B$ is a morphism of H-comodule monoids.

Example: Continuation of the previous example...

Recall: $\underset{M\otimes_{\mathcal{C}}\Omega C\otimes_{\mathcal{C}}N}{\overset{\wr\parallel}{\Omega(M;C;N)}}$ where C dg coalg, M right C-comod, N left C-comodule.

Let H be a dg Hopf algebra, and let E be a right H-comodule algebra.

Proposition: [H-Levi] The multiplication on E extends naturally to $\Omega(E;H;\Bbbk)$ and to $\Omega(E;H;H)$ so that the the inclusion $\underset{E\tilde{\otimes}\Omega H}{\overset{\wr\parallel}{\Omega(E;H;\Bbbk)}} \hookrightarrow \underset{E\tilde{\otimes}\Omega H\tilde{\otimes}H}{\overset{\wr\parallel}{\Omega(E;H;H)}}$ is HG-data.

– a homotopic normal basis extension

Associated co-rings

- The canonical (or descent) co-ring: $W_\varphi = (B\otimes_A B, \delta_\varphi, \varepsilon_\varphi)$
  - already seen in very first lecture
- The Hopf co-ring: $W_\rho = (B\otimes H, B\otimes\Delta, B\otimes\varepsilon_H)$, with B-actions

$$B\otimes W_\rho \xrightarrow{\mu_B\otimes H} B\otimes H$$

and

$$W_\rho\otimes B \xrightarrow{W_\rho\otimes\rho} W_\rho\otimes B\otimes H \xrightarrow{\cong} B^{\otimes 2}\otimes H^{\otimes 2} \xrightarrow{\mu_B\otimes\mu_H} B\otimes H.$$

- The Hopf-Galois map

$$\begin{tikzcd} B\otimes_A B \arrow[rr, "\beta_\varphi"] \arrow[dr, "B\otimes_A \rho"'] & & B\otimes H \\ & B\otimes_A B\otimes H \arrow[ur, "\bar{\mu}_B\otimes H"'] & \end{tikzcd}$$

is a morphism of B-co-rings. (Exercise!)

Introducing homotopy

Henceforth assume $\varphi: A\to B$, H are such that $\exists$ Quillen pairs

$$\begin{tikzcd} \mathcal{M}_B^{W_\varphi} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_B \arrow[l, shift left] \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \mathcal{M}_B^{W_\rho} \arrow[l, shift right] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Mon}^H \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Mon} \arrow[l, shift left] \end{tikzcd}$$

wrt to model category structures of the type discussed above.

Consequently, $\exists$ Quillen pair $\begin{tikzcd} \mathrm{Mon} \arrow[r, shift left, "\mathrm{Triv}_H"] \arrow[r, phantom, "\perp"] & \mathrm{Mon}^H \arrow[l, shift left, "(-)^{coH}"] \end{tikzcd}$.

Defn: A model for the homotopy coinvariants of a H-comod monoid (B,ρ) is $((B,\rho)^f)^{coH}$ for some fibrant replacement of (B,ρ) in Alg^H. Notn: $(B,\rho)^{hcoH}$

Example: (cont.) $\Omega(E;H;H)$ is a functorial fibrant replacement of E in Alg^H (cf. [H-Shipley]).

Consequently, $E^{hcoH} = \Omega(E;H;\Bbbk)$.

Defn: The homotopy corestriction map

$$\begin{tikzcd} A\cong (A)^{coH} \arrow[r, "\varphi^{coH}"] \arrow[rr, bend right, "i_\varphi"'] & B^{coH} \arrow[r] & (B^f)^{coH} \end{tikzcd}$$

Putting it all together…

Def^n: $\varphi: A \to B^{2H}$ is a homotopic H-Hopf-Galois extension if

$$\begin{tikzcd} \mathcal{M}_B^{W_\varphi} \arrow[r, shift left, "\beta_\varphi^*"] \arrow[r, phantom, "\perp"] & \mathcal{M}_B^{W_p} \arrow[l, shift left, "-\square_{W_p} W_\varphi"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \underline{\mathrm{Mod}}_A \arrow[r, shift left, "-\otimes_A B^{hcoH}"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Mod}}_{B^{hcoH}} \arrow[l, shift left, "i_\varphi^*"] \end{tikzcd}$$

are Quillen equivalences.

Example: (cont.) $\varphi: \Omega(E;H;\Bbbk) \to \Omega(E;H;H)$

- Since $\Omega(E;H;H)$ is fibrant in $\underline{\mathrm{Alg}}^H$, can take $\Omega(E;H;H)^{hcoH} = \Omega(E;H;H)^{coH} \cong \Omega(E;H;\Bbbk)$, i.e., i_φ is an isomorphism.
- $\Omega(E;H;H) \otimes_{\Omega(E;H;\Bbbk)} \Omega(E;H;H) \xrightarrow[\cong]{\beta_\varphi} \Omega(E;H;H) \otimes H$

So φ is even a strict H-HG-extension and thus a homotopic HG-extension, since $\Omega(E;H;H)$ is fibrant in $\underline{\mathrm{Alg}}^H$.

Example: Let H be a simplicial monoid, seen as a simplicial bimonoid via the diagonal map.
Let A be a simplicial monoid.
Let B be a fibrant H-comodule monoid, i.e., a simplicial monoid endowed with a simplicial hm $p: B \to H$ that is also a Kan fibration.
Let $\varphi: A \to B$ be a simplicial hm st $p\varphi(a) = e$ $\forall a \in A$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[rd, "*"'] & B \arrow[d, "p"] \\ & H \end{tikzcd}$$

(Model cat theory here much simpler, since working with cartesian monoidal cat.)

Thm: φ is a homotopic H-HG-extension iff $A \simeq \mathrm{hfib}(p)$, i.e., φ is a principal fibration.

Remark: If $\mathcal{M} = Ch_{\Bbbk}^{\geq 0}$, B is left A-semifree of finite type, and H is degreewise $\Bbbk$-free of finite type, then the desired model category structures exist on $\underline{\mathrm{Alg}}^H$, $\mathcal{M}_B^{W_\varphi}$ and $\mathcal{M}_B^{W_p}$, since W_φ, W_p are then both B-semifree of finite type.

Properties of homotopic HG-extensions [Karpova]

Framework: $\mathcal{M} = Ch_{\Bbbk}^{\geq 0}$

(Properties analogous to those proved by Rognes for htpic Galois extensions of ring spectra.)

Theorem: Let $\varphi: A \longrightarrow B^{\circlearrowleft H}$, $f: A \longrightarrow A'$ be morphisms of commutative dg $\Bbbk$-algebras such that B and A' are left A-semifree of finite type. In the pushout diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi"'] & A' \arrow[d, "\bar{\varphi}"] \\
B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A' \otimes_A B
\end{tikzcd}$$

if φ is a homotopic H-HG-extension, then so is $\bar{\varphi}$.

Conversely ...

Theorem: Under the same hypotheses as above, if A' is faithfully flat over A, then:

$\bar{\varphi}$ homotopic H-HG $\Rightarrow$ φ homotopic H-HG.

Currently in progress: one direction of a Hopf-Galois correspondence, i.e., $\varphi: A \to B^{\circlearrowleft H}$ htpic HG-extension

$\Rightarrow$ (quotients of $H \Rightarrow$ subextensions of φ).

Rmk: Under appropriate connectivity/nilpotency and finite-type conditions, any morphism of commutative dg algebras can be replaced up to quasi-isomorphism by a semifree extension of the type required in the theorems above. So, up to homotopy, the semifreeness hypothesis is not a real constraint.

III. From Grothendieck to Hopf and Galois via Koszul

Goal: To understand the BIG PICTURE relating homotopic Grothendieck descent to homotopic Hopf-Galois extensions, at least for $\mathcal{M} = Ch_{\mathbb{k}}^{\geq 0}$, though the results stated here almost certainly hold more generally.

First we need one more notion blending algebras and coalgebras.

A. Generalized Koszul duality

Motivation:

Let A and C be a dg $\mathbb{k}$-algebra and a dg $\mathbb{k}$-coalgebra, respectively. A **twisting cochain** from C to A is a $\mathbb{k}$-linear map

$$t : C_* \longrightarrow A_{*-1}$$

of degree -1 such that $d_A t + t d_C = \mu(t \otimes t)\Delta$.

Key property: $\mathrm{Alg}(\Omega C, A) \overset{1\text{-}1}{\longleftrightarrow} \mathrm{Tw}(C, A) \overset{1\text{-}1}{\longleftrightarrow} \mathrm{Coalg}(C, BA)$

$$\alpha_t \longleftrightarrow t \longleftrightarrow \gamma_t$$

Theorem: [Lefèvre] If $t : C \to A$ is a twisting cochain such that α_t (equiv. γ_t) is a quasi-isomorphism, then $\exists$ Quillen equivalence

$$-\otimes_\tau A : \mathcal{M}^C \underset{\perp}{\rightleftarrows} \mathcal{M}_A : -\otimes_\tau C. \quad (!)$$

In particular, if A is Koszul and C is its Koszul dual, then $\exists$ Quillen equivalence (!).

Example: Universal examples: $\mathcal{M}^C \rightleftarrows \mathcal{M}_{\Omega C}$, $\mathcal{M}^{BA} \rightleftarrows \mathcal{M}_A$.

Def^n: A coalgebra C is a **generalized Koszul dual** of A if $\exists$ Quillen equivalence $\mathcal{M}^C \underset{\perp}{\rightleftarrows} \mathcal{M}_A$.

B. Grothendieck vs Hopf-Galois

For any underlying monoidal (model) category, we have

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & B & \underline{\mathrm{Mon}}^{\to} \arrow[rrrdd, "\mathrm{Descent}"] & A \arrow[r, "\varphi"] & B \arrow[rd, mapsto] & \\
 & & & & & W_\varphi = (B \otimes_A B, \delta_\varphi, \varepsilon_\varphi) \\
A \arrow[r, "\varphi"] \arrow[u, mapsto] \arrow[d, mapsto] & B^{\partial H} & \underline{\mathrm{HopfGalois}} \arrow[u] \arrow[d] \arrow[rrr, shift left=2, ""{name=U, below}] \arrow[rrr, shift right=2, ""{name=D}] & & & \underline{\mathrm{CoRing}}. \\
 & & & & & W_p = (B \otimes H, B \otimes \Delta, B \otimes \varepsilon_H) \\
B^{\partial H} & & \underline{\mathrm{Hopf}} \arrow[rrruu, "\mathrm{Hopf}"'] & & & \\
 & & & B^{\partial H} \arrow[rruu, mapsto] & &
\arrow[Rightarrow, from=U, to=D, "\beta"]
\end{tikzcd}$$

The following two theorems describe the precise relationship between homotopic Grothendieck descent and HG-extensions, at least when $\mathcal{M} = Ch_{\Bbbk}^{\geq 0}$.

Theorem I: Let $A \xrightarrow{\varphi} B^{\partial H}$ be Hopf-Galois data in $Ch_{\Bbbk}^{\geq 0}$.
If $i_\varphi^* : \mathcal{M}_{B^{hcoH}} \longrightarrow \mathcal{M}_A$ is a Quillen equivalence, then:

φ homotopic HG-extension $\iff$ φ satisfies effective homotopic Grothendieck descent.

Theorem II: Let $\varphi : A \longrightarrow B^{\partial H}$ be Hopf-Galois data in $\mathcal{M} = Ch_{\Bbbk}^{\geq 0}$.
If $\eta_B : \Bbbk \xrightarrow{\simeq} B$, then

$i_\varphi^* : \mathcal{M}_{B^{hcoH}} \longrightarrow \mathcal{M}_A$ is a Quillen equivalence

$\iff$ H is a generalized Koszul dual of A.

Proof of the Theorems:

For any choice of Hopf-Galois data in $Ch_{\Bbbk}^{\geq 0}$

$$A \xrightarrow{\varphi} B^{\circlearrowleft H}$$

there is a commuting diagram of HG-data

$$\begin{tikzcd}
 & A \arrow[r, "\varphi"] \arrow[dd, "i_\varphi"'] & B^{\circlearrowleft H} \arrow[dd, "j"', "\simeq"] & \\
\text{algebra morphism} \arrow[r] & {} & {} & \text{morphism of } H\text{-comodule algebras} \arrow[l] \\
 & \underbrace{B^{hcoH} = \Omega(B;H;\Bbbk)}_{= A'} \arrow[r, "\varphi'"] & \underbrace{\Omega(B;H;H)}_{= B'} . &
\end{tikzcd}$$

The commuting diagram of HG-data above induces a commuting diagram of categories and functors: ($\mathcal{M} = Ch_{\Bbbk}^{\geq 0}$)

$$\begin{tikzcd}
\mathcal{M}_{A} \arrow[rr, shift left, "- \otimes_A A'"] \arrow[rr, phantom, "\perp"] \arrow[dd, shift right, "\mathrm{Can}_\varphi"'] \arrow[dd, phantom, "\dashv"] & & \mathcal{M}_{A'} \arrow[ll, shift left, "i_\varphi^*"] \arrow[dd, shift left, "\mathrm{Can}_{\varphi'}"] \arrow[dd, phantom, "\vdash"] & \\
 & & {} & \text{equivalence of categories} \arrow[l] \\
\mathcal{M}_B^{W_\varphi} \arrow[uu, shift right] \arrow[rr, shift left, "(**)"] \arrow[rr, phantom, "\perp"] \arrow[dd, shift right] \arrow[dd, phantom, "\dashv"] & & \mathcal{M}_{B'}^{W_{\varphi'}} \arrow[uu, shift left] \arrow[ll, shift left] \arrow[dd, shift left] \arrow[dd, phantom, "\vdash"] & \\
 & & {} & \text{isomorphism of categories} \arrow[l] \\
\mathcal{M}_B^{W_\rho} \arrow[uu, shift right] \arrow[rr, shift left, "(*)"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left] \arrow[dr, phantom, "\top"] & & \mathcal{M}_{B'}^{W_{\rho'}} \arrow[uu, shift left] \arrow[ll, shift left] \arrow[dl, shift right] \arrow[dl, phantom, "\top"'] & \\
 & \mathcal{M}^H \arrow[ul, shift left, "(***)"] \arrow[ur, shift right, "(***)"'] & &
\end{tikzcd}$$

(*) Quillen equivalence, since $B \otimes H \xrightarrow{\simeq} B' \otimes H$ as B-corings.

(**) No reason to expect $B \otimes_A B \longrightarrow B' \otimes_{A'} B'$ to be a quasi-isomorphism in general.

(***) If $\Bbbk \xrightarrow{\simeq} B$, then $\Bbbk \xrightarrow{\simeq} B'$, and these are Quillen equivalences as in (*).

and II easily. The key to the proof is the particularly nice behavior of the "normal basis extension" $\varphi' : A' \hookrightarrow B'$. //

Scholium: Given HG-data, $A \xrightarrow{\varphi} B^{\partial H}$, if $A \xrightarrow{\simeq} B^{hcoH}$, then $B \otimes_A B$ can be viewed as "generalized Koszul dual" of A.