Homotopical algebra
Exercise Set 6

26.03.2018

Exercise 2 is to be handed in on 09.04.2018.

1. Let $L \dashv R$ be a pair of adjoint functors, where $L : \mathsf{C} \to \mathsf{D}$ and $R : \mathsf{D} \to \mathsf{C}$. Let $\eta : \mathrm{Id}_{\mathsf{C}} \to RL$ and $\varepsilon : LR \to \mathrm{Id}_{\mathsf{D}}$ be the unit and counit of the adjunctions. Prove that η and ε satisfy the *triangle identities*, i.e.,

   (a) $\mathrm{Id}_{L(C)} = \varepsilon_{L(C)} \circ L(\eta_C)$ for all $C \in \mathrm{Ob}\,\mathsf{C}$, and

   (b) $\mathrm{Id}_{R(D)} = R(\varepsilon_D) \circ \eta_{R(C)}$ for all $D \in \mathrm{Ob}\,\mathsf{D}$.

2. Let C be a cocomplete category.

   (a) Let J be any small category. Prove that C^{J} is cocomplete and that, in particular, *colimits are created objectwise*, i.e., for all small categories I and all I-shaped diagrams $\Phi : \mathsf{I} \to \mathsf{C}^{\mathsf{J}}$ the functor $\mathrm{colim}_{\mathsf{I}}\,\Phi : \mathsf{J} \to \mathsf{C}$ satisfies $\big(\mathrm{colim}_{\mathsf{I}}\,\Phi\big)(j) = \mathrm{colim}_{\mathsf{I}}\,\big(\Phi(-)(j)\big)$.

   (b) Let $F : \mathsf{J} \to \mathsf{K}$ be a functor between small categories. Prove that the "precomposition with F" functor $F^* : \mathsf{C}^{\mathsf{K}} \to \mathsf{C}^{\mathsf{J}}$ admits a left adjoint. (Hint: Consider special cases first, such as when J is the category with one object and one morphism or the category with two objects and one non-identity morphism. The general formula can be expressed in terms of colimits of functors with domains that have as objects morphisms in K of the form $F(j) \to k$ for a fixed $k \in \mathrm{Ob}\,\mathsf{K}$.)

   (c) Let $F_! : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}^{\mathsf{K}}$ denote the left adjoint to F^*. For any $\Phi \in \mathrm{Ob}\,\mathsf{C}^{\mathsf{J}}$, the functor $F_!(\Phi) : \mathsf{K} \to \mathsf{C}$ is called the *left Kan extension of* Φ *along* F. Show that the diagram of functors

$$\begin{tikzcd} \mathsf{J} \arrow[r, "\Phi"] \arrow[d, "F"'] & \mathsf{C} \\ \mathsf{K} \arrow[ur, "F_!(\Phi)"'] & \end{tikzcd}$$

   "commutes up to a natural transformation", i.e., there is a natural transformation $\eta : \Phi \to F_!(\Phi) \circ F$.

(d) Let N denote the category with $\mathrm{Ob}\,\mathsf{N} = \mathbb{N}$, the set of natural numbers, and only identity morphisms. Let $S : \mathsf{N} \times \mathsf{N} \to \mathsf{N}$ denote functor specified by $S(m, n) = m+n$. A functor $\Phi : \mathsf{N}\times\mathsf{N} \to \mathsf{Ab}$ corresponds to a *bigraded abelian group*, i.e., a set $\{A_{m,n,} \mid m, n \in \mathbb{N}\} \subset \mathrm{Ob}\,\mathsf{Ab}$. Compute $S_!(\Phi)$.

3. Dualize the exercise above in the case where C is complete, leading to a right adjoint $F_* : \mathsf{C}^\mathsf{J} \to \mathsf{C}^\mathsf{K}$ to F^* and then to the definition of a *right Kan extension.*

4. Let C be a bicomplete category, and let J be any small category. Recall the functors $\mathrm{colim}_\mathsf{J} : \mathsf{C}^\mathsf{J} \to \mathsf{C}$ and $\lim_\mathsf{J} : \mathsf{C}^\mathsf{J} \to \mathsf{C}$ of Exercise 6 in Exercise set 5.

   (a) Show that $\mathrm{colim}_\mathsf{J}$ and $\lim_\mathsf{J}$ are left and right adjoints to the functor $\Delta : \mathsf{C} \to \mathsf{C}^\mathsf{J}$ that takes an object C to the constant diagram Δ_C and a morphism $f : C \to D$ to the "constant at f" natural transformation $\Delta_f : \Delta_C \to \Delta_D$.

   (b) Let I be another small category.

      i. Prove that are isomorphisms of categories $(\mathsf{C}^\mathsf{I})^\mathsf{J} \xleftarrow{\Phi} \mathsf{C}^{\mathsf{I}\times\mathsf{J}} \xrightarrow{\Psi} (\mathsf{C}^\mathsf{J})^\mathsf{I}$.

      ii. Prove *Fubini's Theorem* (categorical version): For all $F \in \mathrm{Ob}\,\mathsf{C}^{\mathsf{I}\times\mathsf{J}}$,

      $$\mathrm{colim}_\mathsf{I}\big(\mathrm{colim}_\mathsf{J}\,\Phi(F)\big) \cong \mathrm{colim}_\mathsf{J}\big(\mathrm{colim}_\mathsf{I}\,\Psi(F)\big) \cong \mathrm{colim}_{\mathsf{I}\times\mathsf{J}}\,F$$

      and

      $$\lim{}_\mathsf{I}\big(\lim{}_\mathsf{J}\Phi(F)\big) \cong \lim{}_\mathsf{J}\big(\lim{}_\mathsf{I}\Psi(F)\big) \cong \lim{}_{\mathsf{I}\times\mathsf{J}}F.$$

   (c) Prove that if $f : Z \to X$ and $g : Z \to Y$ are morphisms in $G\mathsf{Set}$, then

   $$(X \coprod_Z Y)_G \cong (X_G) \coprod_{Z_G} (Y_G),$$

   where the pushout is computed on the left in $G\mathsf{Set}$ and on the right in Set, and $(-)_G$ denotes the G-orbits functor. Similarly, prove that if $f : X \to Z$ and $g : Y \to Z$ are morphisms in $G\mathsf{Set}$, then

   $$(X \times_Z Y)^G \cong (X^G) \times_{Z^G} (Y^G),$$

   where the pullback is computed on the left in $G\mathsf{Set}$ and on the right in Set, and $(-)^G$ denotes the G-fixed points functor.

5. Let J be a small category, and let C be any category. The functor category $\mathsf{C}^{\mathsf{J}^{op}}$ is the category of C-*valued presheaves on* J. (Set-valued presheaves are often simply called *presheaves.*)

   (a) Show that there is a functor $Y : \mathsf{J} \to \mathsf{Set}^{\mathsf{J}^{op}}$ defined on objects by $Y(j) = \mathsf{J}(-, j) : \mathsf{J}^{op} \to \mathsf{Set}$. The presheaf $Y(j)$ is said to be *representable*, represented by $j \in \mathrm{Ob}\,\mathsf{J}$.

(b) Prove the *Yoneda Lemma*: For every $j \in \mathrm{Ob}\,\mathsf{J}$ and every presheaf $\Phi \in \mathrm{Ob}\,\mathsf{Set}^{\mathsf{J}^{op}}$, there is an isomorphism

$$\Phi(j) \cong \mathsf{Set}^{\mathsf{J}^{op}}\big(Y(j), \Phi\big)$$

that is natural in j, More formally, there is a natural isomorphism between Φ and the functor $\mathsf{Set}^{\mathsf{J}^{op}}\big(Y(-), \Phi\big) : \mathsf{J}^{op} \to \mathsf{Set}$.

(c) Explain why the Yoneda Lemma implies that the functor Y is *full* (i.e., every morphism $Y(j) \to Y(j')$ is of the form $Y(f)$ for some $f \in \mathsf{J}(j, j')$) and *faithful* (i.e., if $f, g \in \mathrm{Mor}\,\mathsf{J}$ and $Y(f) = Y(g)$, then $f = g$.), so that

$$\mathsf{Set}^{\mathsf{J}}\big(Y(j), Y(j')\big) \cong \mathsf{J}(j, j').$$

(d) Let $j, j' \in \mathrm{Ob}\,\mathsf{J}$. Prove that if $Y(j) \cong Y(j')$, then $j \cong j'$.