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Homotopy Exponents for Large H-Spaces

Wojciech Chachólski[1], Wolfgang Pitsch[2], Jérôme Scherer[2], and Don Stanley[3]

[1]The Royal Institute of Technology, Matematik, S-10044 Stockholm, Sweden, [2]Universitat Autònoma de Barcelona, Departament de Matemàtiques, E-08193 Bellaterra, Spain, and [3]Department of Mathematics and Statistics, University of Regina, College West 307.14, Regina, Saskatchewan S4S 0A2, Canada

Correspondence to be sent to: pitsch@mat.uab.cat

We show that H-spaces with finitely generated cohomology, as an algebra or as an algebra over the Steenrod algebra, have homotopy exponents at all primes. This provides a positive answer to a question of Stanley.

Introduction

Moore's conjecture, see for example [9], predicts that elliptic complexes have an exponent at any prime p, meaning that there is a bound on the p-torsion in the graded group of all homotopy groups. Relying on results by James [6] and Toda [11] about the homotopy groups of spheres, the fourth author (re)proved in [10] Long's result that finite H-spaces have an exponent at any prime [7]. He proved in fact the result for H-spaces for which the mod p cohomology is finite and also asked whether this would hold for finitely generated cohomology rings. The aim of this note is to give a positive answer to this question and to provide a way larger class of H-spaces that have homotopy exponents.

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Theorem 1.2. Let X be a connected and p-complete H-space such that $H^*(X; \mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then X has an exponent at p. $\square$

This class of H-spaces is optimal in the sense that H-spaces with a larger mod p cohomology, such as an infinite product of Eilenberg–Mac Lane spaces $K(\mathbb{Z}/p^n, n)$, will not have in general an exponent at p. The theorem should be compared with the computations done by Clément and the third author of homological exponents, [4]. Whereas such H-spaces always have homotopy exponents, they almost never have homological exponents. The only H-spaces for which the 2-torsion in $H_*(X; \mathbb{Z})$ has a bound are products of copies of the circle, classifying spaces of cyclic groups, the infinite complex projective space and $K(\mathbb{Z}, 3)$. As a corollary, we obtain the desired result. In fact, we obtain the following global theorem.

Theorem 1.4. Let X be a connected H-space such that $H^*(X; \mathbb{Z})$ is finitely generated as an algebra. Then X has an exponent at each prime p. $\square$

The methods we use are based on the deconstruction techniques of Castellana, Crespo, and Scherer [3].

1 Homotopy Exponents

Our starting point is the fact that mod p finite H-spaces always have homotopy exponents. The following is a variant of Stanley [10, Corollary 2.9]. Whereas he focused on spaces localized at a prime, we will stick to p-completion in the sense of Bousfield and Kan [2].

Proposition 1.1 Stanley**.** Let p be a prime and X be a p-complete and a connected H-space such that $H^*(X; \mathbb{F}_p)$ is finite. Then X has an exponent at p. $\square$

We will not repeat the proof, but let us sketch the main steps. Let us consider a decomposition of X by p-complete cells, i.e. X is obtained by attaching cones along maps from $(S^n)^\wedge_p$. The natural map $X \to \Omega\Sigma X$ factors then through the loop spaces on a wedge W of a finite number of such p-completed spheres, up to multiplying by some integer N: the composite $X \to \Omega\Sigma X \xrightarrow{N} \Omega\Sigma X$ is homotopic to $X \to \Omega W \to \Omega\Sigma X$. The proof goes by induction on the number of p-complete cells and the key ingredient here is Hilton's description of the loop space on a wedge of spheres [5]. Note that the suspension of a map between spheres is torsion except for the multiples of the identity. This idea to "split off" all the cells of X up to multiplication by some integer is dual to Arlettaz' way to split off Eilenberg–Mac Lane spaces in H-spaces with finite-order k-invariants, [1, Section 7]. The final step relies on the classical results by James [6] and Toda [11], that spheres do have homotopy exponents at all primes.

Theorem 1.2. Let X be a connected and p-complete H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then X has an exponent at p. □

Proof. A connected H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra can always be seen as the total space of an H-fibration $F \to X \to Y$, where Y is an H-space with finite mod p cohomology and F is a p-torsion Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$ [3, Theorem 7.3]. This is a fibration of H-spaces and H-maps, so that we obtain another fibration $F^\wedge_p \to X \to Y^\wedge_p$ by p-completing it. The base space $Y^\wedge_p$ now satisfies the assumptions of Proposition 1.1. It therefore has an exponent at p. The homotopy groups of the fiber $F^\wedge_p$ are finite direct sums of cyclic groups $\mathbb{Z}/p^n$ and copies of the p-adic integers $\mathbb{Z}^\wedge_p$. Thus, $F^\wedge_p$ has an exponent at p as well. The homotopy long exact sequence of the fibration allows us to conclude. ■

We see here how the p-completeness assumption plays an important role. The space $K(\mathbb{Z}_{p^\infty}, 1)$, for example, has obviously no exponent at p, but its p-completion is $K(\mathbb{Z}^\wedge_p, 2) = (\mathbb{C}P^\infty)^\wedge_p$, which is a torsion-free space. The following corollary is the answer to Stanley's question.

Corollary 1.3. Let X be a connected and p-complete H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra. Then X has an exponent at p. □

In fact, when the mod p cohomology is finitely generated, the fiber F in the fibration described in the proof of Theorem 1.2 is a single Eilenberg–Mac Lane space $K(P, 1)$. Thus, the typical example of an H-space with finitely generated mod p cohomology is the 3-connected cover of a simply connected finite H-space (P is $\mathbb{Z}_{p^\infty}$ in this case). Likewise, the typical example in Theorem 1.2 are highly connected covers of finite H-spaces. This explains why such spaces have homotopy exponents!

If one does not wish to work at one prime at a time and prefers to find a global condition which permits to conclude that a certain class of spaces have exponents at all primes, one must replace mod p cohomology by integral cohomology.

Theorem 1.4. Let X be a connected H-space such that $H^*(X;\mathbb{Z})$ is finitely generated as an algebra. Then X has an exponent at each prime p. □

Proof. Since the integral cohomology groups are finitely generated, it follows from the universal coefficient exact sequence (see [8]) that the integral homology groups are also finitely generated. Since X is an H-space, we may use a standard Serre class argument to conclude that so are the homotopy groups. Therefore, the p-completion map $X \to X^\wedge_p$ induces an isomorphim on the p-torsion at the level of homotopy groups. The theorem is now a direct consequence of the next lemma. ■

Lemma 1.5. Let X be a connected space. If $H^*(X;\mathbb{Z})$ is finitely generated as an algebra, then so is $H^*(X;\mathbb{F}_p)$. □

Proof. Let $u_1, \ldots, u_r$ generate $H^*(X;\mathbb{Z})$ as an algebra. Consider the universal coefficients short exact sequences

$$0 \to H^n(X;\mathbb{Z}) \otimes \mathbb{Z}/p \longrightarrow H^n(X;\mathbb{F}_p) \xrightarrow{\partial} \mathrm{Tor}(H^{n+1}(X;\mathbb{Z});\mathbb{Z}/p) \to 0\,.$$

Since $H^*(X;\mathbb{Z})$ is finitely generated as an algebra, it is degreewise finitely generated as a group, and therefore $\mathrm{Tor}(H^*(X;\mathbb{Z});\mathbb{Z}/p)$ can be identified with the ideal of elements of order p in $H^*(X;\mathbb{Z})$. This ideal must be finitely generated, since $H^*(X;\mathbb{Z})$ is Noetherian. Choose generators $a_1, \ldots, a_s$. Each a_i corresponds to a pair $\alpha_i, \beta\alpha_i$ in $H^*(X;\mathbb{F}_p)$, where β denotes the Bockstein.

We claim that the elements $\alpha_1, \ldots, \alpha_s$ together with the mod p reduction of the algebra generators, denoted by $\bar{u}_1, \ldots, \bar{u}_r$, generate $H^*(X;\mathbb{F}_p)$ as an algebra. Let $x \in H^*(X;\mathbb{F}_p)$ and write its image $\partial(x) = \sum \lambda_j a_j$ with $\lambda_j = \lambda_j(u)$ a polynomial in the u_i's. Define now $\bar{\lambda}_j = \lambda_j(\bar{u}) \in H^*(X;\mathbb{F}_p)$ to be the corresponding polynomial in the $\bar{u}_i$'s. As the action of $H^*(X;\mathbb{Z})$ on the ideal $\mathrm{Tor}(H^*(X;\mathbb{Z});\mathbb{Z}/p)$ factors through the mod p reduction map $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{F}_p)$, the element $x - \sum \bar{\lambda}_j \alpha_j$ belongs to the kernel of ∂, i.e. it lives in the image of the mod p reduction. It can be written therefore as a polynomial $\bar{\mu}$ in the $\bar{u}_i$'s. Thus $x = \bar{\mu} + \sum \bar{\lambda}_j \alpha_j$. ■

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