Conformal Field Theory and Gravity
EPFL doctoral course

Marco Meineri, Joao Penedones

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Abstract

This course is an introduction to Conformal Field Theory (CFT) and its application to Quantum Gravity. The course starts with a review of scaling and renormalization in statistical physics leading to the CFT description of continuous phase transitions. We then proceed to the study of CFT on its own as a continuum quantum field theory. This leads us to the conformal bootstrap program as a practical tool to map out the space of CFTs. In the second part of the course, we introduce the Anti-de Sitter (AdS) spacetime and describe particle dynamics and QFT in this background. Finally, we introduce the AdS/CFT correspondence as a natural extension of QFT in a fixed AdS background. We also discuss some applications of the bulk geometric intuition to strongly coupled QFT.

Contents

Introduction

Conformal Field Theory (CFT) is a central subject in modern theoretical physics. It provides a field theory description of continuous phase transitions in statistical mechanics and quantum critical points in condensed matter theory. It describes the short distance (UV) and long distance (IR) behaviour of quantum field theories (QFT). It provides a non-perturbative definition of quantum gravity via the AdS/CFT correspondence.

These notes are divided in 4 main parts. The first part is dedicated to scaling and renormalization and discuss the emergence of CFT in statistical mechanics. This part follows closely the book [1]. The second part is focused on the study of CFT per se. The lecture notes [2] are a very useful complement to this section. [1] Section 3 deals with Anti-de Sitter (AdS) spacetime. The first goal here is to gain intuition about particle dynamics in AdS and QFT in a fixed AdS background. From this point-of-view, we will see that a gravitational theory with AdS boundary conditions naturally defines a CFT living on its boundary. In section 4, we discuss the AdS/CFT correspondence in more detail and emphasize its importance for quantum gravity. We also consider what kind of CFTs have simple AdS duals and the role of string theory. Furthermore, we discuss several applications of the gauge/gravity duality as a tool to geometrize QFT effects. Finally, in section 5, we introduce the Mellin representation of CFT correlation functions. We explain the analytic properties of Mellin amplitudes and their particular simplicity in the case of holographic CFTs.

The AdS/CFT correspondence [5, 6, 7] is a well established general approach to quantum gravity. However, it is often perceived as a particular construction specific to string theory. In these lectures I will argue that the AdS/CFT correspondence is the most conservative approach to quantum gravity. The quick argument goes as follows:

- *System in a box* - we work with Anti-de Sitter (AdS) boundary conditions because AdS is the most symmetric box with a boundary. This is useful to control large IR effects, even without dynamical gravity.

- *QFT in the box* - Quantum Field Theory (no gravity) in a fixed AdS background leads to the construction of boundary operators that enjoy an associative and

[1] See also the lecture notes [3, 4].

convergent Operator Product Expansion (OPE). The AdS isometries act on the boundary operators like the conformal group in one lower dimension.

- *Boundary stress-tensor from gravitons* - perturbative metric fluctuations around AdS lead to a boundary stress tensor (weakly coupled to the other boundary operators).

Starting from these 3 facts it is entirely natural to define quantum gravity with AdS boundary conditions as Conformal Field Theory (CFT) in one lower dimension. Of course not all CFTs look like gravity in our universe. That requires the size of the box to be much larger than the Planck length and all higher spin particles to be very heavy (relative to the size of the box). As we shall see, these physical requirements imply that the CFT is strongly coupled and therefore hard to find or construct. The major role of string theory is to provide explicit examples of such CFTs like maximally supersymmetric Yang-Mills (SYM) theory.

There are many benefits that follow from accepting the AdS/CFT perspective. Firstly, it makes the holographic nature of gravity manifest. For example, one can immediately match the scaling of the CFT entropy density with the Bekenstein-Hawking entropy of (large) black holes in AdS. Notice that this is a consequence because it was not used as an argument for AdS/CFT in the previous paragraph. More generally, the AdS/CFT perspective let us translate questions about quantum gravity into mathematically well posed questions about CFT. [2] Another benefit of the gauge/gravity duality is that it gives us a geometric description of QFT phenomena, which can be very useful to gain physical intuition and to create phenomenological models.

There are many reviews of AdS/CFT available in the literature. Most of them are complementary to these lecture notes because they discuss in greater detail concrete examples of AdS/CFT realized in string theory. I leave here an incomplete list [9, 10, 11, 12, 13, 14, 15, 16, 17] that can be useful to the readers interested in knowing more about AdS/CFT. The lecture notes [18] by Jared Kaplan discuss in greater detail many of the ideas presented here.

[2] It might not be possible to formulate all quantum gravity questions in CFT language. For example, it is unclear if the experience of an observer falling into a black hole in AdS is a CFT observable [8].

Chapter 1

Scaling and Renormalization

The traditional application of conformal field theory is the description of continuous phase transitions in Statistical Mechanics. This chapter is a brief review of how this arises. The book [1] is recommended as complementary reading to this chapter.

1.1 Phase transitions

The macroscopic properties (like density or magnetization) of a system depend on the external conditions (like temperature, pressure or applied magnetic field).

A phase transition is an abrupt change in macroscopic properties under a small variation of the external conditions. More precisely, the free energy F has a non-analyticity as a function of the external conditions. For example, $F(T)$ is not analytic at the critical temperature $T = T_c$.

A phase transition can be continuous or discontinuous. In a discontinuous or first order phase transition the first derivative of the free energy is discontinuous. At the transition, there are two phases that coexist and the correlation length is finite. First order phase transitions involve latent heat and hysteresis. The prototypical example is the liquid-vapour transition. Hysteresis means that it is possible to have a superheated liquid for $T > T_c$ and a supercooled gas for $T < T_c$.

In a continuous phase transition the first derivative of the free energy is continuous and the second (or higher) derivative is discontinuous. There is a unique phase and infinite correlation length.

The correlation length ξ can be defined using the two-point correlation function,

$$G(x-y) = \langle s(x)s(y)\rangle - \langle s(x)\rangle\langle s(y)\rangle\,, \tag{1.1}$$

where $s(x)$ is a local variable (like a spin) and we assumed translational and rotational invariance for simplicity. The correlation length can be defined by

$$\xi^2 = \frac{\sum_r r^2 G(r)}{\sum_r G(r)}\,. \tag{1.2}$$

Exercise 1.1.1 *Consider the canonical ensemble for a lattice spin system* [1]

$$\langle s(y) \rangle = \frac{1}{Z} \sum_{\{s\}} s(y) e^{-H[\{s\}]} \,, \qquad Z = \sum_{\{s\}} e^{-H[\{s\}]} \,. \tag{1.3}$$

Show that an infinitesimal local source h at the point x,

$$e^{-H[\{s\}]} \to e^{-H[\{s\}] + h s(x)} \,, \tag{1.4}$$

produces the response

$$\langle s(y) \rangle_h = \langle s(y) \rangle_0 + h\, G(x,y) + O(h^2) \tag{1.5}$$

where $G(x,y)$ is the connected two-point function in the absence of the source h.

In the context of QFT, the correlation length is inversely proportional to the mass of the lightest particle.

Exercise 1.1.2 *Consider the propagator of a massive scalar field in Euclidean space,*

$$G(x) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\,k\cdot x}}{k^2 + m^2} \,. \tag{1.6}$$

Determine its large-distance behaviour (for $x \gg 1/m$). Suggestion: *Use the identity*

$$\frac{1}{A} = \int_0^\infty dt\, e^{-tA} \,, \tag{1.7}$$

to do the Fourier transform to position space and evaluate the t integral using the saddle point approximation.

Estimate the behaviour of the propagator in the opposite limit $x \ll 1/m$. How does it compare with the propagator of a massless field?

A possible definition of correlation length is

$$\xi^2 = \frac{\int d^d x\, x^2 G(x)}{\int d^d x\, G(x)} \tag{1.8}$$

Show that this gives

$$\xi^2 = -\frac{1}{\hat{G}(0)} \frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k^\mu} \hat{G}(k) \bigg|_{k=0} = \frac{2d}{m^2} \tag{1.9}$$

where $\hat{G}(k)$ is the propagator in momentum space and in the last step we used the form of the propagator of a free massive scalar field.

[1]We write the Boltzmann weight $e^{-\beta H}$ as e^{-H} by absorbing the inverse temperature into the hamiltonian H. In this notation, the inverse temperature is just a parameter inside the dimensionless lattice Hamiltonian H.

1.1.1 Critical exponents

Uniaxial ferromagnets have spontaneous magnetization for temperatures $T < T_c$, where T_c denotes the Curie temperature. The magnetization is the order parameter for the phase transition between ferromagnet and paramagnet. The phase diagram has a critical point at $T = T_c$ and $H = 0$. In the vicinity of the critical point, it is natural to use the reduced temperature and magnetic field,

$$t = \frac{T - T_c}{T_c}\,, \qquad h = \frac{H}{T_c}\,. \tag{1.10}$$

Several quantities have singular behaviour near the critical point.

- Specific heat at $h = 0$
$$C \sim |t|^{-\alpha}\,. \tag{1.11}$$
- Spontaneous magnetization
$$\lim_{h\to 0^+} M \sim (-t)^\beta\,, \qquad t < 0\,. \tag{1.12}$$
- Susceptibility
$$\chi \equiv \left.\frac{\partial M}{\partial H}\right|_{H=0} \sim |t|^{-\gamma}\,. \tag{1.13}$$
- Magnetization at $T = T_c$
$$M \sim |h|^{1/\delta}\,. \tag{1.14}$$
- Correlation length
$$\xi \sim |t|^{-\nu}\,. \tag{1.15}$$
- Two-point function at the critical point ($T = T_c$ and $H = 0$)
$$G(r) \sim \frac{1}{r^{d-2+\eta}}\,. \tag{1.16}$$

The parameters $\alpha, \beta, \gamma, \delta, \nu, \eta$ are called critical exponents and d is the dimension of space.

For uniaxial ferromagnets we have

$$\alpha \approx 0.11\,, \qquad \beta \approx 0.33\,, \qquad \gamma \approx 1.24\,, \tag{1.17}$$
$$\delta \approx 4.79\,, \qquad \nu \approx 0.63\,, \qquad \eta \approx 0.036\,. \tag{1.18}$$

Remarkably, many different systems have the same critical exponents. For example, the critical point at the end of the line of first-order liquid-vapour transition has the same critical exponents as uniaxial ferromagnets. The same exponents also appear in binary mixtures, Coulombic and micellar systems [19]. This leads to the notion of *universality*. Systems with the same critical exponents belong to the same universality class. The examples above belong to the Ising universality class.

The intuitive explanation of universality is that the microscopic details of the system become irrelevant when the correlation length diverges. We shall see that critical points are described by scale invariant continuum quantum field theories. These are generically isolated and characterized by discrete data like its symmetries and space dimensionality.

The Ising model is the simplest microscopic model that belongs to the Ising universality class. It consists of a cubic lattice with a spin $s = \pm 1$ at each lattice site and the following hamiltonian

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i \,, \tag{1.19}$$

where the first sum runs over nearest neighbour pairs.

1.2 Renormalization

The basic idea of renormalization is to compute the large scale properties of the system by coarse graining the microscopic dynamics. This process is most intuitive in real space.

1.2.1 Block spin transformation

Consider a block spin transformation (BST) where we replace a block of spins by a single spin. For example, we can use the majority rule. We divide the square lattice in squares with $3 \times 3 = 9$ spins and replace each set of 9 spins by a single spin pointing in the direction of the majority of the original 9. If we start with a configuration above the critical temperature (paramagnetic phase) and we perform this operation many times we will reach a configuration where all spins are uncorrelated. This fixed point of the BST corresponds to $T = \infty$. If we start below the critical temperature, performing the block spin transformation many times will takes to another fixed point where all spins are aligned. This corresponds to $T = 0$. Both fixed points have $\xi = 0$. The critical point $T = T_c$ is also a fixed point of the BST but it has $\xi = \infty$.

Mathematically, a BST is implemented by a projection operator

$$T_b(s'; s_i) = \Theta\left(s' \sum s_i\right) \,, \tag{1.20}$$

where s_i are the spins inside the block b and the Heaviside Θ-function is given by $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ if $x < 0$. The hamiltonian of the blocked system is given by

$$e^{-H'[\{s'\}]} = \sum_{\{s\}} e^{-H[\{s\}]} \prod_b T_b(s'; s_i) \,. \tag{1.21}$$

One can check that the partition function remains the same,

$$Z' = \sum_{\{s'\}} e^{-H'[\{s'\}]} = \sum_{\{s'\}} \sum_{\{s\}} e^{-H[\{s\}]} \prod_b T_b(s'; s_i) = \sum_{\{s\}} e^{-H[\{s\}]} = Z \,. \tag{1.22}$$

If we parametrize the Hamiltonian by a set of couplings $\{k\} = \{k_1, k_2, \dots\}$ then the BST generates a map

$$\{k\} \to \{k'\} = \mathcal{R}[\{k\}] \,. \tag{1.23}$$

Exercise 1.2.1 *The BST can be implement exactly in the 1D Ising model. Divide the lattice in blocks of 2 spins and erase the first spin in each block. This operation is known as decimation,*

$$T(s'; s_1, s_2) = \Theta(s' s_2) \,. \tag{1.24}$$

Show that if the hamiltonian is given by

$$H = -k \sum_i s_i s_{i+1} - \sum_i c \tag{1.25}$$

then decimation leads to the map

$$c' = 2c + \frac{1}{2} \log(4 \cosh 2k) \tag{1.26}$$

$$k' = \frac{1}{2} \log(\cosh 2k) \tag{1.27}$$

Use this to derive that the correlation length (in lattice units) is given by

$$\xi = \frac{const}{\log \tanh k} \,, \tag{1.28}$$

which diverges at low temperatures $k \to \infty$.

There are many possibilities for BSTs. The BST will automatically preserve the partition function as long as $\sum_{s'} T_b(s'; s_i) = 1$. In addition, a useful BST should preserve the local structure of the Hamiltonian.

In higher dimensions ($d > 1$), BSTs generate new couplings from the ones originally present in the Hamiltonian. This means that after a larger number of steps the number of couplings that one has to keep track becomes very large. In practice, one has to resort to approximations and truncations.

1.2.2 Scaling variables

A fixed point $\{k^*\}$ in the space of couplings is defined by $\{k^*\} = \mathcal{R}[\{k^*\}]$. It is useful to linearize the map $\mathcal{R}$ around the fixed point

$$k'_a - k^*_a \approx \sum_b T_{ab} \left(k_b - k^*_b\right) \,, \qquad T_{ab} = \left.\frac{\partial k'_a}{\partial k_b}\right|_{k=k^*} \,. \tag{1.29}$$

Using the left eigenvectors e^i_a of the matrix T_{ab},

$$\sum_a e^i_a T_{ab} = \lambda^i e^i_b \,, \tag{1.30}$$

we can define the *scaling variables*

$$u_i \equiv \sum_a e^i_a \left(k_a - k^*_a\right) \,, \tag{1.31}$$

that transform multiplicatively $u'_i = \lambda^i u_i$ (no sum). The associated *renormalization group eigenvalues* y_i are defined by $\lambda^i = b^{y_i}$ where b is the size of the block (in lattice units) used in the BST. Scaling variables are classified accordingly to

- $y_i > 0 \Rightarrow u_i$ is **relevant** because it grows under a BST.
- $y_i < 0 \Rightarrow u_i$ is **irrelevant** because it decreases under a BST.
- $y_i = 0 \Rightarrow u_i$ is **marginal**. The linear approximation is not sufficient to know its behaviour under a BST.

In general there is an infinite number of irrelevant scaling variables and a finite number of relevant ones. For example, the critical point of the Ising model has two relevant scaling variables: the reduced temperature ($\mathbb{Z}_2$ even) and the magnetic field ($\mathbb{Z}_2$ odd). This is consistent with the idea of universality because most parameters in the hamiltonian are irrelevant for the long distance properties of the system close to the critical point. The number of relevant scaling variables is the number of parameters that must be tuned to achieve criticality. Equivalently, it is the co-dimension of the critical surface in the space of couplings.

1.2.3 Free energy

Let $f = -\frac{1}{N}\log Z$ be the (dimensionless) free energy per spin. In general, there is a constant term in f which does not flow to a fixed value at the critical point (see exercise 1.2.1). It is convenient to write

$$f = c + \hat{f}[\{k\}] \tag{1.32}$$

where we exclude the constant c appearing in the hamiltonian from the list of couplings $\{k\}$. Under a BST we have

$$f' = b^d f \qquad \Rightarrow \qquad \hat{f}[\{k\}] = g[\{k\}] + b^{-d}\hat{f}[\{k'\}]\,, \tag{1.33}$$

where b is the linear size of the block and d is the space dimension. In general, the function g is regular at the fixed point $\{k^*\}$ because it is generated by integrating out the short distance degrees of freedom in each block. [2] Therefore, the singular part of the free energy scales as

$$f_s[\{k\}] = b^{-d} f_s[\{k'\}]\,. \tag{1.34}$$

Using scaling variables, we can write

$$f_s(u_t, u_h, u_I, \dots) = b^{-d} f_s\left(b^{y_t}u_t, b^{y_h}u_h, b^{y_I}u_I, \dots\right) \tag{1.35}$$

$$= b^{-nd} f_s\left(b^{ny_t}u_t, b^{ny_h}u_h, b^{ny_I}u_I, \dots\right) \tag{1.36}$$

where u_I represents an irrelevant scaling variable and we have used the Ising critical point variables for concreteness. Suppose we start very close to the fixed point $u_t = u_h = 0$ and perform n BSTs until $|b^{ny_t}u_t| = 1$. [3] This gives

$$f_s(u_t, u_h, u_I, \dots) = |u_t|^{\frac{d}{y_t}} f_s\left(\pm 1, u_h|u_t|^{-\frac{y_h}{y_t}}, 0, \dots\right)\,, \tag{1.37}$$

[2] The free energy can be written as $f = c + \sum_{n=0}^{\infty} b^{-dn} g[\{k\}_n]$. The non-analyticity at the fixed point emerges from the infinite sum. However, the sum can not diverge because the free energy is bounded if the hamiltonian is bounded from below and there is a finite number of degrees of freedom per site. The non-analyticity can be seen as a divergence of some derivatives of the free energy (with respect to couplings or temperature) at the fixed point.

[3] More precisely, we should set $|b^{ny_t}u_t| = u_{t_0}$ where u_{t_0} is a small but fixed number.

or equivalently

$$f_s(t,h) = \left|\frac{t}{t_0}\right|^{\frac{d}{y_t}} \Phi_\pm \left(\frac{h}{h_0} \left|\frac{t}{t_0}\right|^{-\frac{y_h}{y_t}} \right) , \tag{1.38}$$

where we have used $u_t = t/t_0$ and $u_h = h/h_0$. The functions $\Phi_\pm$ are universal scaling functions and the parameters t_0 and h_0 are model dependent. $\Phi_\pm$ describes the paramagnetic/ferromagnetic phases close to the critical point, *i.e.* $t > 0$ and $t < 0$.

Exercise 1.2.2 *Show that the scaling form* (1.38) *of the singular part of the free energy, predicts the following expressions for the critical exponents introduced in section 1.1.1,*

$$\alpha = 2 - \frac{d}{y_t} , \qquad \beta = \frac{d - y_h}{y_t} , \qquad \gamma = \frac{2y_h - d}{y_t} , \qquad \delta = \frac{y_h}{d - y_h} . \tag{1.39}$$

Check that these imply the following scaling relations,

$$\alpha + 2\beta + \gamma = 2 , \qquad \alpha + \beta + \beta\delta = 2 . \tag{1.40}$$

In general, the scaling relations can be more complicated if there are more relevant operators.

Exercise 1.2.3 *Argue that the correlation length* ξ *satisfies*

$$\xi(u_t, u_h, u_I, \dots) = b\, \xi \left(b^{y_t} u_t, b^{y_h} u_h, b^{y_I} u_I, \dots\right) . \tag{1.41}$$

Use this to derive the following critical behaviour of ξ *at zero magnetic field,*

$$\xi \sim |t|^{-\frac{1}{y_t}} \qquad \Rightarrow \qquad \nu = \frac{1}{y_t} . \tag{1.42}$$

Compare this behaviour with what you found in problem 1.2.1.

1.2.4 Scaling operators

In the vicinity of the fixed point the hamiltonian takes the form

$$H = H^* + \sum_a (k_a - k_a^*) \sum_x S_a(x) \tag{1.43}$$

where $S_a(x)$ are local operators associated with the couplings k_a. Moving to the scaling variables defined in (1.31) we find

$$H = H^* + \sum_i u_i \sum_x \mathcal{O}_i(x) \tag{1.44}$$

where $\mathcal{O}_i(x)$ are scaling operators which are related to the original local operators by $S_a(x) = \sum_i e_a^i \mathcal{O}_i(x)$. It is convenient to consider scaling variables that are space dependent,

$$H = H^* + \sum_i \sum_x u_i(x) \mathcal{O}_i(x) , \tag{1.45}$$

so that connected correlation functions can be computed as derivatives of the free energy. In particular, the two-point function is given by

$$G_i(x,y) \equiv \langle \mathcal{O}_i(x)\mathcal{O}_i(y)\rangle - \langle \mathcal{O}_i(x)\rangle\langle \mathcal{O}_i(y)\rangle = \frac{\partial^2 \log Z}{\partial u_i(x)\partial u_i(y)} \,. \tag{1.46}$$

If the scaling variables $u_i(x)$ are slowly varying fields and the hamiltonian is dominated by local interactions then we expect that the BST has the same effect as for constant scaling variables. Namely $u_i'(x') = b^{y_i} u_i(x)$, where x' labels the block of size b and x labels all sites contained in that block prior to the BST. Therefore,

$$G_i'(x',y') = \frac{\partial^2 \log Z'}{\partial u_i'(x')\partial u_i'(y')} = b^{-2y_i} \sum_{\substack{x\in x'\\ y\in y'}} \frac{\partial^2 \log Z}{\partial u_i(x)\partial u_i(y)} = b^{2(d-y_i)} G_i(x,y) \,, \tag{1.47}$$

where we assumed that the correlation function varies slowly for $|x-y| \gg b$ so that the sum over sites in each block only produces a volume factor b^d.

At the critical point, (1.47) reduces to

$$G_i'(x',y') \to G_i^*(x'-y') = G_i^*\left(\frac{x-y}{b}\right) = b^{2(d-y_i)} G_i^*(x-y) \,, \tag{1.48}$$

which implies that

$$G_i^*(x-y) = \frac{const}{|x-y|^{2\Delta_i}} \,, \qquad\qquad \Delta_i \equiv d - y_i \,, \tag{1.49}$$

where we introduce the scaling dimensions Δ_i.

Exercise 1.2.4 *Apply the previous result to the spin two-point function and show that*

$$\eta = d + 2 - 2y_h \,. \tag{1.50}$$

Exercise 1.2.5 *Generalize the previous argument to n-point functions at criticality to derive*

$$\left\langle \mathcal{O}_1\left(\frac{x_1}{b}\right) \dots \mathcal{O}_n\left(\frac{x_n}{b}\right)\right\rangle = b^{\Delta_1} \dots b^{\Delta_n} \langle \mathcal{O}_1(x_1)\dots \mathcal{O}_n(x_n)\rangle \,. \tag{1.51}$$

Conclude that the one-point functions $\langle \mathcal{O}_i(x)\rangle = 0$ if $\Delta_i \neq 0$.

Exercise 1.2.6 *Consider the hamiltonian*

$$H = -\frac{1}{2}\sum_{x,y} \hat{J}(x-y)s(x)s(y) - \mathcal{H}\sum_x s(x) + \lambda \sum_x \left(s(x)^2 - 1\right)^2 \tag{1.52}$$

where x and y label sites in a d-dimensional hypercubic lattice and $s(x) \in \mathbb{R}$. The coupling $\hat{J}(x-y)$ is equal to J if x and y are nearest neighbours and it vanishes otherwise. Argue that in the limit $\lambda \to \infty$ this hamiltonian reduces to the usual Ising model where $s(x) = \pm 1$. We shall assume that the hamiltonian with finite λ can also describe the Ising critical point. This is motivated by universality and the intuition from BSTs.

Show that in the continuum limit where the lattice spacing $a \to 0$, the hamiltonian can be written as

$$H = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + t\, a^{-2}\phi^2(x) + u\, a^{d-4}\phi^4(x) + h\, a^{-1-\frac{d}{2}}\phi(x) + \dots \right], \tag{1.53}$$

where the field $\phi(x) = \sqrt{J} a^{\frac{2-d}{2}} s(x)$ and the dimensionless couplings are

$$t = -\frac{2\lambda}{J} - d\,, \qquad u = \frac{\lambda}{J^2}\,, \qquad h = -\frac{\mathcal{H}}{\sqrt{J}}\,. \tag{1.54}$$

The ... in (1.53) *stand for terms with more than two derivatives. Use dimensional analysis to conclude that a generic term with p derivatives and n fields,*

$$g\, a^{-y_g} \int d^d x \phi \partial^p \phi^{n-1}\,, \tag{1.55}$$

has $y_g = d - p - n\frac{d-2}{2}$. The fixed point $t = u = h = 0$ is called the gaussian fixed point. Check that for $d > 4$, t and h are the only relevant couplings of the gaussian fixed point. In fact the operator $\phi^3(x)$ is relevant. However, it is a ***redundant*** *operator because it can be removed from the hamiltonian by a field redefinition $\phi(x) \to \phi(x) + const$. In general, there is an infinite number of redundant operators that can be removed by local field redefinitions $\phi(x) \to \phi(x) + c_1\phi^3(x) + c_2\partial^2\phi(x) + \dots$.*

The continuum limit of the partition function

$$Z = \int \prod_x d\phi(x) e^{-H[\phi]} \tag{1.56}$$

is equivalent to the path integral formulation of (euclidean) QFT, with the hamiltonian playing the role of the euclidean action.

1.2.5 RG flows

We are free to choose the rescaling factor b of the BST. On a lattice we usually have some geometric constraints but using QFT we have more freedom. It is convenient to choose

$$b = e^{\delta\ell} \approx 1 + \delta\ell\,, \qquad \delta\ell \ll 1\,. \tag{1.57}$$

By performing a sequence of infinitesimal BSTs we can define a renormalization group (RG) flow in the space of couplings. It is natural to define a vector field tangent to these RG trajectories

$$\vec{\beta}(\vec{k}) = -\lim_{\delta\ell \to 0} \frac{\vec{k}' - \vec{k}}{\delta\ell}\,, \tag{1.58}$$

such that fixed points are characterized by

$$\vec{\beta}(\vec{k}^*) = 0\,. \tag{1.59}$$

Notice that

$$\vec{k}' = \vec{\mathcal{R}}_b(\vec{k}) = \vec{k} + \delta\ell \left.\frac{d\vec{\mathcal{R}}_b(\vec{k})}{db}\right|_{b=1} + O(\delta\ell^2) \tag{1.60}$$

which implies that

$$\vec{\beta}(\vec{k}) = -\left.\frac{d\vec{\mathcal{R}}_b(\vec{k})}{db}\right|_{b=1} = \frac{d\vec{k}}{d\log\Lambda}\,, \tag{1.61}$$

where in the last expression we introduced a momentum cutoff $\Lambda \sim 1/a$.

Recall that the matrix

$$T_{cd} = \left.\frac{\partial k'_c}{\partial k_d}\right|_{k=k^*} \tag{1.62}$$

had left eigenvalues b^{y_i}. For $b = 1 + \delta\ell$, we find $b^{y_i} = 1 + y_i\delta\ell + O(\delta\ell^2)$ and

$$T_{cd} = \delta_{cd} + \delta\ell M_{ac} + O(\delta\ell^2)\,, \tag{1.63}$$

where

$$M_{cd} = -\left.\frac{\partial\beta_c}{\partial k_d}\right|_{k=k^*}\,. \tag{1.64}$$

Therefore the renormalization group eigenvalues y_i are the eigenvalues of the matrix M_{cd}.

Exercise 1.2.7 *Consider an invertible redefinition $\tilde{g}(g)$ of a coupling constant g. Assume that $g = 0$ is a fixed point of the RG flow and that $\tilde{g}(0) = 0$. Write the beta function*

$$\tilde{\beta}(\tilde{g}) = \frac{d\tilde{g}}{d\log\Lambda} \tag{1.65}$$

as a perturbative expansion in the coupling $\tilde{g}$ using the perturbative expansions

$$\beta(g) = \frac{dg}{d\log\Lambda} = b_1 g + b_2 g^2 + \dots \tag{1.66}$$

and

$$\tilde{g} = a_1 g + a_2 g^2 + \dots \tag{1.67}$$

If $b_1 \neq 0$ then show that one can choose a_i's such that $\tilde{\beta}$ is linear to all orders in $\tilde{g}$. Is the result physically reasonable?

It is instructive to consider a simplified model with only one coupling. In this case

$$\beta(g) = \frac{dg}{d\log\Lambda}\,, \tag{1.68}$$

which means that if $\beta > 0$ then g decreases towards the IR. This is the case of $g\phi^4$ in $d > 4$ or QED in $d = 4$. In both examples the theory becomes weakly coupled at low energies. If $\beta < 0$ then g increases towards the IR. This is the case of $g\phi^4$ in $d < 4$ or QCD in $d = 4$. In both examples the theory becomes weakly coupled at high energies.

So far we have only talked about theories with a UV cutoff. In fact, we have defined the RG flow by the change in couplings necessary to compensate a change of UV cutoff so that the physical observables remain invariant. However, some QFTs are **UV complete** in the sense that we can compute correlation functions at arbitrarily small distances. [4]

[4] In the QFT textbooks UV complete theories are often called renormalizable theories.

Clearly, this is only possible in theories where the correlation length is infinite in units of the lattice spacing. More precisely, one can define QFT correlations functions by the limit of the lattice correlators

$$G_{1\dots n}(x_1,\dots,x_n;\{u_i\}) \equiv \langle \mathcal{O}_1(x_1)\dots\mathcal{O}(x_n)\rangle_{\{u_i\}} \tag{1.69}$$

when the correlation length $\xi \to \infty$ keeping the ratios $|x_i - x_j|/\xi$ fixed. As shown in exercise 1.2.3, the correlation length close to the critical point can be written as

$$\xi = |u_1|^{-\frac{1}{y_1}} F(U_1,\dots,U_r) \tag{1.70}$$

where we assumed that u_1 was a relevant coupling ($y_1 > 0$) and $U_i \equiv u_i |u_1|^{-\frac{y_i}{y_1}}$. Notice that when we approach the critical point at $u_i = 0$, the U's associated to irrelevant operators tend to zero. On the other hand, the U's associated to the r relevant operators can be kept fixed as we approach the fixed point $u_i = 0$. The variables $U_1,\dots,U_r$ encode the direction in parameter space in which we are approaching the fixed point. Under p consecutive BSTs, the correlator of scaling operators transforms as

$$G_{1\dots n}\left(\frac{x_1}{b^p},\dots,\frac{x_n}{b^p};\{b^{py_i}u_i\}\right) = G_{1\dots n}(x_1,\dots,x_n;\{u_i\}) \prod_{i=1}^{n} b^{p\Delta_i}\,. \tag{1.71}$$

We can then choose $b^p = \xi$ and take the limit $\xi \to \infty$ by approaching the fixed point. This gives

$$\lim_{\xi\to\infty} G_{1\dots n}(x_1,\dots,x_n;\{u_i\}) \prod_{i=1}^{n} \xi^{\Delta_i} = G_{1\dots n}\left(\frac{x_1}{\xi},\dots,\frac{x_n}{\xi};\{U_i F\}\right)\,, \tag{1.72}$$

which shows that in the continuum limit the correlation functions only depend on the ratios $(u_i)^{\frac{1}{y_i}}/(u_j)^{\frac{1}{y_j}}$ involving relevant couplings. Therefore, UV complete QFTs are scale invariant theories (fixed points) and relevant deformations of these,

$$S = S^* + \sum_i g_i \mu^{y_i} \int d^d x \mathcal{O}_i(x) \tag{1.73}$$

with $y_i > 0$ and μ a renormalization scale. These QFTs describe the scaling region of the fixed point. In particular, they contain all the universal observables like critical exponents, the scaling functions $\Phi_\pm$ in (1.38) and the universal correlators (1.72).

If S^* is a free QFT and the deformation $\mathcal{O}_i$ introducing interactions is weakly relevant (i.e. $0 < y_i \ll 1$), then a perturbative analysis can be useful. For example, consider the action for a scalar field

$$S = \int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} t\mu^2\phi^2 + \frac{1}{4!}\lambda\mu^{d-4}\phi^4\right]\,. \tag{1.74}$$

We can identify S^* with the first term and the other terms as relevant deformations, assuming $2 \le d < 4$. One can compute the $\beta-$function at small coupling [5]

$$\beta(\lambda) = -(4-d)\lambda + c\lambda^2 + O(\lambda^3) \tag{1.75}$$

[5] There is also a β–function for the dimensionless coupling t. We are assuming that we can tune t so that this β–function also vanishes.

where c is a constant that depends on the renormalization scheme. This $\beta-$function has two zeros: $\lambda = 0$ and $\lambda = \lambda^* \equiv (4-d)/c$. The first zero corresponds to the gaussian fixed point where the operator ϕ^4 is relevant because its associated RG eigenvalue is $(4-d)$. The zero at $\lambda = \lambda^*$ corresponds to an interacting fixed point where the operator ϕ^4 is irrelevant because its RG eigenvalue is $-(4-d)$. When $\epsilon \equiv 4-d \ll 1$ the interacting fixed point is weakly coupled and can be reliable described with perturbative methods. This leads to the famous ϵ–expansion proposed by Wilson and Fisher [**?**].

1.3 Problems

Exercise 1.3.1 ***Fluid model***

Consider a simple lattice model of a fluid, where there can be at most one molecule per lattice site and there is an energy U associated to the attractive interaction between two molecules in neighbouring sites. Show that this model in the grand canonical ensemble is equivalent to the Ising model in a magnetic field and relate the physical parameters of the two models.

Exercise 1.3.2 ***Decimation in 1D Ising with magnetic field***

Consider the Ising model in one dimension, with partition function

$$Z = \sum_{\{s_i=\pm 1\}} e^{k\sum_i s_i s_{i+1} + h\sum_i s_i + \sum_i c} \tag{1.76}$$

where k is the nearest neighbour coupling, h is the magnetic field and c is a constant that we introduced for convenience. Notice that the standard free-energy per spin is given by

$$f(k,h) = -\frac{1}{N}\log Z + c \tag{1.77}$$

where N is the number of spins and we consider periodic boundary conditions.

Perform a block spin transformation (BST) that erases half of the spins. More precisely, sum over the spins s_i with odd index i and show that the partition function can be written as

$$Z = Z' = \sum_{\{s'_i=\pm 1\}} e^{k'\sum_i s'_i s'_{i+1} + h'\sum_i s'_i + \sum_i c'} \tag{1.78}$$

where $s'_i = s_{2i}$ are the even-numbered spins and

$$e^{2h'} = e^{2h}\frac{\cosh(2k+h)}{\cosh(2k-h)} \tag{1.79}$$

$$e^{4k'} = \frac{\cosh(2k+h)\cosh(2k-h)}{\cosh^2 h} \tag{1.80}$$

$$e^{4c'} = 16e^{8c}\cosh(2k+h)\cosh(2k-h)\cosh^2 h \tag{1.81}$$

What are the fixed points of this BST in the plane $(x = e^{-4k}, y = e^{-2h})$?

Can you use this BST to determine the exact free energy per spin $f(k,h)$ in the thermodynamic limit?

Exercise 1.3.3 ***Approximate Block Spin Transformation***

Consider the (generalized) 2D Ising model

$$H[\{s\}] = -\sum_{<x,y>} J_{xy}\, s(x)s(y) \tag{1.82}$$

on an infinite square lattice. The local spins can take values $s(x) = \pm 1$. The sum in the hamiltonian runs over all pairs of sites in the lattice (not necessarily nearest neighbours). Our goal is to perform a block spin transformation that removes the dotted sites in figure 1.1. Let us denote the spins s on dotted sites by $\dot{s}$ and the remaining ones by s'.

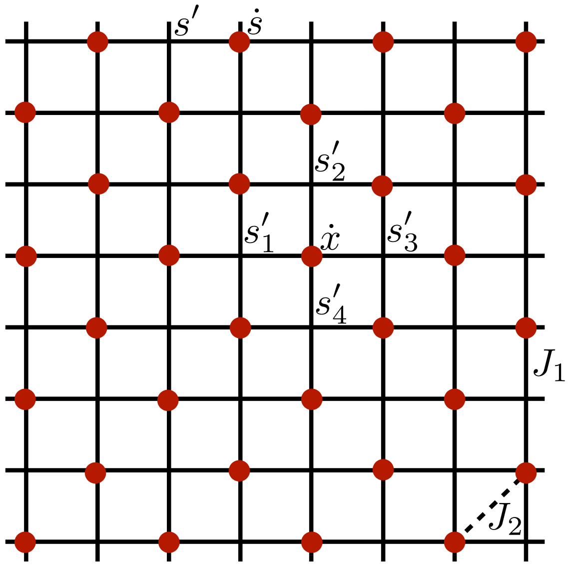

Figure 1.1 A two dimensional square lattice divided into two sub-lattices labelled by s' and $\dot{s}$. The four nearest neighbours s'_i of a dotted site x' are shown. In question **d.** we consider nearest neighbour interactions J_1 and next-to-nearest neighbour interactions J_2.

a. *Show that the partition function can be written as*

$$Z = \sum_{\{s\}} e^{-H[\{s\}]} = \left[\prod_{<x,y>} \cosh J_{xy}\right] \sum_{\{s\}} \prod_{<x,y>} \left[1 + s(x)s(y)\, \tanh J_{xy}\right] . \tag{1.83}$$

Recall the identity $e^{\pm x} = \cosh x\,(1 \pm \tanh x)$.

b. *Show that the hamiltonian H' generated by the block spin transformation that removes the dotted spins is given by*

$$H'[\{s'\}] = -\sum_{<x,y>} \log\left(\cosh J_{xy}\right) - \log\left(\sum_{\{\dot{s}\}} \prod_{<x,y>} \left[1 + s(x)s(y)\, \tanh J_{xy}\right]\right) \tag{1.84}$$

c. *Consider first the simplest case where* $J_{xy} = J$ *when* x *and* y *are nearest neighbours and* $J_{xy} = 0$ *otherwise. In this case, show that*

$$\begin{aligned} H'[\{s'\}] = &- \sum_{\dot{x}} \log\left(2\cosh^4 J\right) \\ &- \sum_{\dot{x}} \log\left[1 + (s'_1 s'_2 + s'_1 s'_3 + s'_1 s'_4 + s'_2 s'_3 + s'_2 s'_4 + s'_3 s'_4)\tanh^2 J + s'_1 s'_2 s'_3 s'_4 \tanh^4 J\right] \end{aligned} \tag{1.85}$$

where the sum is over the dotted sites $\dot{x}$ *and* s'_i *are its four nearest neighbours. Argue that this can be written as*

$$H'[\{s'\}] = -\sum_{\dot{x}} \left[K_0 + (s'_1 s'_2 + s'_1 s'_3 + s'_1 s'_4 + s'_2 s'_3 + s'_2 s'_4 + s'_3 s'_4) K_1 + s'_1 s'_2 s'_3 s'_4 K_2\right] \tag{1.86}$$

where

$$K_0 = \log 2 + 2J^2 + O(J^4) \tag{1.87}$$

$$K_1 = J^2 + O(J^4) \tag{1.88}$$

$$K_2 = -2J^4 + O(J^6) \tag{1.89}$$

Notice that applying a block spin transformation to an hamiltonian with only nearest neighbour interactions we generated a more complicated hamiltonian containing 4 spin interactions. In order to proceed we must do some approximation.

d. *Consider now the case where* $J_{xy} = J_1$ *when* x *and* y *are nearest neighbours,* $J_{xy} = J_2$ *when* x *and* y *are next-to-nearest neighbours, and* $J_{xy} = 0$ *otherwise. In this case, show that*

$$H'[\{s'\}] = C - \sum_{<x,y>} J'_{xy} s'(x) s'(y) + \ldots \tag{1.90}$$

where C *is a constant, the dots stand for interaction terms with 4 or more spins and the nearest and next-to-nearest neighbour couplings in the* s' *lattice are given by*

$$J'_1 = J_2 + 2J_1^2 + O(J_2^3, J_1^4, J_2 J_1^2) \tag{1.91}$$

$$J'_2 = J_1^2 + O(J_2^2, J_1^4, J_2 J_1^2) \tag{1.92}$$

e. *Let us approximate the exact block spin transformation above by truncating to*

$$J'_1 = J_2 + 2J_1^2\ , \qquad\qquad J'_2 = J_1^2\ . \tag{1.93}$$

Determine the fixed points of this transformation. Determine also the renormalization group eigenvalue y *associated with the relevant scaling variable* u *at the non-trivial fixed point (with* $J_1 \neq 0$*). Recall the definition of scaling variable* $u' = b^y u$ *with* b *being the scaling factor of the block spin transformation.*

Exercise 1.3.4 *Mean Field Approximation*

Prove Feynman's inequality

$$Tr\, e^{-H} \geq Tr\, e^{-H' - \langle H - H' \rangle_{H'}}\ , \tag{1.94}$$

where $[H, H'] = 0$ *and*

$$\langle \mathcal{O} \rangle_{H'} = \frac{Tr\, \mathcal{O}\, e^{-H'}}{Tr\, e^{-H'}} \ . \tag{1.95}$$

Choosing the hamiltonian H' *to maximize the right hand side of (1.94) is a systematic way to implement a mean-field approximation. Use the hamiltonian*

$$H' = -h' \sum_x s(x) \tag{1.96}$$

to study the Ising model hamiltonian on a hyper-cubic lattice,

$$H = -J \sum_{<x,y>} s(x)s(y) - h \sum_x s(x) \tag{1.97}$$

in the mean-field approximation. Show that the free energy per spin in this approximation

$$f_{MF} = -\frac{1}{N} \log \max_{h'} Tr\, e^{-H' - \langle H - H' \rangle_{H'}} \ , \tag{1.98}$$

can be written as

$$f_{MF} = \min_M \left[-\log 2 - hM + \frac{1 - Jz}{2} M^2 + \frac{1}{12} M^4 + \mathcal{O}(M^6) \right] \ , \tag{1.99}$$

where z *denotes the number of nearest neighbours of each spin. What is the critical temperature? Plot the phase diagram and compute the thermodynamical critical exponents for the Ising model in* d *dimensions using this approximate free energy.*

To determine the spin two-point correlation function we need to allow for space dependent magnetic fields $h(x)$ *and* $h'(x)$*. By moving to Fourier space, determine the spin two-point function at zero magnetic field and read off the* ν *and* η *critical exponents in the mean-field approximation.*

Exercise 1.3.5 ***Dangerously irrelevant operators***

Consider the (singular part of the) free energy of the Ising model in the mean field approximation

$$f(h, t, \lambda) = \min_M \left[-hM + \frac{t}{2} M^2 + \frac{\lambda}{12} M^4 \right] \ , \tag{1.100}$$

and verify that it satisfies the scaling law

$$f(h, t, \lambda) = b^{-d} f\left(b^{y_h} h, b^{y_t} t, b^{y_\lambda} \lambda\right) \tag{1.101}$$

with renormalization group eigenvalues

$$y_h = \frac{d}{2} + 1 \ , \qquad y_t = 2 \ , \qquad y_\lambda = 4 - d \ . \tag{1.102}$$

Recall that these are the RG eigenvalues of the Gaussian fixed point. Using the standard formulas derived in the lectures, one would conclude that the critical exponents are given by the second column in next table. However, these are different from the actual critical exponents associated with Mean Field Theory free energy (1.100), which are given in the third column.

Exponent	*Gaussian*	*MFT*
α	$2-\frac{d}{2}$	*0*
β	$\frac{d-2}{4}$	$\frac{1}{2}$
γ	*1*	*1*
δ	$\frac{d+2}{d-2}$	*3*
ν	$\frac{1}{2}$	$\frac{1}{2}$
η	*0*	*0*

Go through the general argument presented in the lectures and find the assumption made that is not valid in this case. Hint: study the free energy when $\lambda \to 0$ *and* $t < 0$*.*

Exercise 1.3.6 ***The Transfer Matrix***

The partition function of one dimensional lattice models with local interactions can be computed using the transfer matrix method. One starts by writing the hamiltonian as a sum of terms associated to each pair of neighbouring sites

$$H = \sum_{i=1}^{N} H(s_i, s_{i+1}) \tag{1.103}$$

where we assumed periodic boundary conditions $s_{N+1} = s_1$*. The partition function can then be written as*

$$Z = \sum_{\{s_i\}} e^{-H} = \sum_{\{s_i\}} e^{-H(s_1,s_2)} e^{-H(s_2,s_3)} \dots e^{-H(s_N,s_1)} = Tr\, T^N \tag{1.104}$$

where T *is a square matrix whose elements are*

$$[T]_{s_1,s_2} = e^{-H(s_1,s_2)} \ . \tag{1.105}$$

Notice that the size of the matrix is given by the number of degrees of freedom per site. Correlation functions of local observables can be written as

$$\langle f_1(s_{i_1}) \dots f_n(s_{i_n}) \rangle = \frac{1}{Z} Tr\, F_1 T^{i_2-i_1} F_2 T^{i_3-i_2} F_3 \dots F_n T^{N-i_n+i_1} \tag{1.106}$$

where F_i *is a diagonal matrix associated to the local observable* $f_i(s)$*,*

$$[F]_{s_1,s_2} = \delta_{s_1,s_2} f(s_1) \ . \tag{1.107}$$

Show that the correlation length is given by

$$\frac{1}{\xi} = -\log \frac{\lambda_1}{\lambda_0} \tag{1.108}$$

where $\lambda_0, \lambda_1, \lambda_2, \dots$ *are the eigenvalues of the transfer matrix* T *ordered by decreasing size (modulus).*

Use this method to show that the free energy per site of the one-dimensional Ising model (1.110), is given by

$$f = -\lim_{N\to\infty} \frac{1}{N} \log Tr\, e^{-H} = -\log \lambda_0 = -J - \log\left(\cosh h + \sqrt{\sinh^2 h + e^{-4J}}\right) \ . \quad (1.109)$$

Study the thermodynamic quantities and the spin-spin two-point function near the critical temperature. Compare your results with the mean-field approximation.

Exercise 1.3.7 ***Finite Size Scaling***

Consider the 2D Ising model

$$H = -J \sum_{<x,y>} s(x)s(y) - h \sum_x s(x) \quad (1.110)$$

on a square lattice with cylinder topology. Let N be the number of sites along the periodic direction of the cylinder. Argue that the specific heat per spin (at zero magnetic field) has the following scaling behavior (for large N and small t)

$$c \sim N^{\alpha/\nu} \psi\left(tN^{1/\nu}\right) \quad (1.111)$$

where α and ν are critical exponents of the two dimensional system and $t \sim J - J_c$ with $J_c = \frac{1}{2}\log(1+\sqrt{2})$. The theory of finite size scaling is explained in section 4.4 of [1]. In fact, the divergence in the specific heat of the two-dimensional Ising model is only logarithmic ($\alpha = 0$). Thus, we expect

$$c \sim \psi\left(tN^{1/\nu}\right) \log N \quad (1.112)$$

Define a transfer matrix $2^N \times 2^N$ for this system. Plot the specific heat per spin of the system as a function of J (at zero magnetic field) for several values of $N = 1, 2, 3, \ldots$ (it is possible to go up to $N = 7$ in a reasonable time in a personal laptop). Compare your results with the scaling (1.112). More precisely, test the logarithmic growth of the maximum $c_{max}(N)$ of the specific heat and fit the value of J that maximizes c for each N with

$$J_{max}(N) \approx J_c + B\, N^{-1/\nu} \ . \quad (1.113)$$

What values do you obtain for the fitting parameters J_c, B and ν?

A better way to estimate the value of ν is to plot $c(J,N)/c_{max}(N)$ as a function of $(J - J_{max}(N))N^{1/\nu}$ for each value $N = 3, 4, 5, 6, 7$ and for several values of ν. You can use the function `Manipulate` *of* `MATHEMATICA` *to vary ν until the curves for different values of N collapse on top of each other (approximately). Estimate ν using this technique.*

Suggestion 1: A possible way to numerically determine the largest (in modulus) eigenvalue λ_0 of a matrix T is to use

$$T|v_0\rangle = \lambda_0|v_0\rangle \ , \qquad |v_0\rangle = \lim_{n\to\infty} |n\rangle \ , \qquad |n+1\rangle = \frac{T|n\rangle}{||\,T|n\rangle\,||} \ , \quad (1.114)$$

assuming that the initial vector $|0\rangle$ of the iteration is not orthogonal to $|v_0\rangle$.

Suggestion 2: Another possibility is to consider a lengthy but finite cylinder. Then, you can use

$$Z = Tr\, T^{2^M} = Tr\,(((T^2)^2)^2 \dots)^2 \tag{1.115}$$

where you just need to square a matrix M times. A cylinder of length $64 = 2^6$ is already good.

Exercise 1.3.8 ***High and low temperature expansions***

Derive the high-temperature series expansion for the partition function of the 2D Ising model at zero magnetic field,

$$Z = 2^N\,(\cosh J)^{2N}\left(1 + \sum_n a_n (\tanh J)^n\right) \tag{1.116}$$

where N is the total number of spins in the system and a_n is the number of ways to draw closed loops of total length n in the square lattice (each link can only be used once). Derive the low temperature expansion of the same model,

$$Z = e^{2JN}\left(2 + \sum_n b_n e^{-2Jn}\right) \tag{1.117}$$

where b_n is the number of configurations with exactly n anti-aligned nearest neighbour spin pairs. Show that $b_n = 2a_n$ (Kramers-Wannier duality) and use that to determine the critical temperature of the 2D Ising model. Compare the result with the prediction from mean field theory.

Exercise 1.3.9 ***Worm algorithm for Monte-Carlo simulation***

The high temperature expansion of the Ising model discussed in the previous problem can be easily generalized to arbitrary dimension,

$$Z = 2^N\,(\cosh J)^{dN}\left(1 + \sum_n a_n (\tanh J)^n\right)\ , \tag{1.118}$$

where N is the total number of spins in the system and a_n is the number of ways to draw closed loops of total length n in the hyper-cubic lattice. Show that the high temperature expansion for the spin two point function takes the form

$$\langle s(x)s(y)\rangle = \frac{1}{Z} 2^N\,(\cosh J)^{dN} \sum_n (\tanh J)^n a_n(x,y) = \frac{\sum_n (\tanh J)^n a_n(x,y)}{1 + \sum_n (\tanh J)^n a_n}\ , \tag{1.119}$$

where $a_n(x,y)$ is the number of ways to draw a path from x to y and closed loops of total length n in the hyper-cubic lattice.

The basic idea of the worm algorithm is to think of the space of paths on the hyper-cubic lattice as the configuration space. This means that one configuration is uniquely characterized by the list of links crossed by the paths. Then we put a probability distribution on this configuration space such that the probability of each configuration is proportional to

$(\tanh J)^n$ *where* n *is the total length of the paths. Finally, we compute averages* $\langle \ldots \rangle_{worm}$ *with this distribution. This gives*

$$\langle s(x)s(y) \rangle = \frac{\langle L_1(x,y) \rangle_{worm}}{\langle L_0 \rangle_{worm}} \,, \tag{1.120}$$

where $L_1(x,y) = 1$ *if the configuration contains only closed loops and one open path from* x *to* y*, otherwise* $L_1(x,y) = 0$*. On the other hand,* $L_0 = 1$ *if the configuration contains only closed loops and* $L_0 = 0$ *otherwise.*

To make this idea useful we need to invent a method to generate path configurations with the correct probability distribution. Show that the following Markov chain of path configurations produces the correct probability distribution at late times:

- *Choose a random point in the lattice and set the two endpoints* m *and* i *to this site.*
- *Choose with equal probability which end point you will try to move. Let us say the result was* m*.*
- *Try to move* m *randomly to one of the* $2d$ *neighbouring sites and activate the link from* i *to* m *with probability* $\tanh J$*. You will need to keep a list of active links in the lattice.*
- *Try to move one end point randomly to its neighbouring sites. In each move, you can either activate a new link or remove a link from the list of active links. If the proposal is to activate a new link, it should only be accepted with probability* $\tanh J$*. If the proposal is to deactivate a link then it is always accepted.*
- *Repeat the previous step.*

Notice that this algorithm explores the full configuration space consisting of closed loops together with at most one open path (with endpoints m *and* i*). This is sufficient to calculate the spin-spin two-point function using (1.120). The idea is simply to compute the average* $\langle \ldots \rangle_{worm}$ *in the ensemble of configurations generated by the Markov chain above. Implement this algorithm in the computer and plot the two-point function as a function of the distance* $|x-y|$ *for temperatures above, below and at the critical temperature of the 2D Ising model. Start by computing the magnetic susceptibility*

$$\chi = \sum_y \langle s(x)s(y) \rangle = \frac{1}{\langle L_0 \rangle_{worm}} \tag{1.121}$$

as function of the coupling J*.*

Exercise 1.3.10 ***RG Flow with Cubic Anisotropy***

Consider the following Euclidean action

$$S = \int d^d x \left[\frac{1}{2} \sum_{a=1}^{n} \partial_\mu \varphi^a \partial^\mu \varphi^a + \frac{1}{2} t \sum_{a=1}^{n} \varphi^a \varphi^a + u \left(\sum_{a=1}^{n} \varphi^a \varphi^a \right)^2 + v \sum_{a=1}^{n} (\varphi^a)^4 \right] \tag{1.122}$$

for n real scalar fields φ^a. Notice that the action with $v = 0$ is invariant under $O(n)$ rotations of the fields, while for $v \neq 0$ this symmetry is broken to a finite group: the symmetry group of an n-dimensional cube.

***a**) For constant field of fixed magnitude $\bar{\varphi}^2 = \sum_{a=1}^{n} \varphi^a \varphi^a$, what directions in field space minimize the action for $v > 0$ and $v < 0$?*

***b**) What are the allowed values of u and v such that the action is bounded from below? Draw the allowed region in the (u, v) plane.*

***c**) Studying the system in $d = 4 - \epsilon$ dimensions, derive the following (one-loop) renormalization group equations*

$$\frac{du}{dl} = -\beta_u = \epsilon u - c\left[(n+8)u^2 + 6uv\right] \tag{1.123}$$

$$\frac{dv}{dl} = -\beta_v = \epsilon v - c(12uv + 9v^2) \tag{1.124}$$

where c is a numerical constant that can be set to 1 by rescaling the couplings u and v.

***d**) Find the fixed points in the (u, v) plane. Determine the scaling variables and respective RG eigenvalues at each fixed point.*

***e**) Sketch the RG flow lines in the (u, v) plane for $n > 4$ and for $n < 4$. What is the symmetry of the most stable RG fixed point in each case?*

Exercise 1.3.11 ***Quantum Ising in transverse field***

Consider the hamiltonian

$$\hat{H} = -J \sum_i \hat{S}_i^z \hat{S}_{i+1}^z - h \sum_i \hat{S}_i^x \tag{1.125}$$

for a one-dimensional lattice with a spin-$\frac{1}{2}$ degree of freedom per site. Assume periodic boundary conditions in a chain of length N, $\hat{S}_{N+1} = \hat{S}_1$. Recall that the spin operators can be represented using the Pauli matrices

$$\hat{S}^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.126}$$

***a**) What is the ground state of the system when $J > 0$ and $h = 0$? What is the ground state of the system when $h > 0$ and $J = 0$? Comment on the qualitative difference between these two cases.*

***b**) The partition function of the system can be written as*

$$Z = Tr\, e^{-\beta \hat{H}} = \lim_{\Delta\tau \to 0} Tr \left(1 - \Delta\tau \hat{H}\right)^{\frac{\beta}{\Delta\tau}}. \tag{1.127}$$

This has the form of the partition function of a system defined on a two-dimensional square lattice with $N \times \frac{\beta}{\Delta\tau}$ sites (with periodic boundary conditions) and transfer matrix $1 - \Delta\tau\hat{H}$ connecting consecutive rings of length N. Show that

$$\langle \sigma_1', \ldots, \sigma_N' | (1 - \Delta\tau \hat{H}) | \sigma_1, \ldots, \sigma_N \rangle = e^{K_1 \sum_i \sigma_i \sigma_{i+1} + K_2 \sum_i (\sigma_i \sigma_i' - 1)} + O(\Delta\tau^2) \tag{1.128}$$

where $|\sigma_1, \ldots, \sigma_N\rangle$*, with* $\sigma_i = \pm 1$*, denotes a state of the spin chain in the eigenbasis of* $\hat{S}_i^z$ *(the eigenvalues being* $\sigma_i/2$*) and*

$$K_1 = \frac{J\Delta\tau}{4}\,, \qquad e^{-2K_2} = \frac{h\Delta\tau}{2}\,. \tag{1.129}$$

c) *Notice that the transfer matrix above corresponds to an anisotropic two-dimensional Ising model, with hamiltonian*

$$H = -K_1 \sum_{i,j} \sigma_{i,j}\sigma_{i+1,j} - K_2 \sum_{i,j} \sigma_{i,j}\sigma_{i,j+1} \tag{1.130}$$

where we dropped an irrelevant constant and labelled the local Ising spins $\sigma_{i,j}$ *by the row and column numbers* i, j*. It can be shown that this model is critical for* $\sinh 2K_1 \sinh 2K_2 = 1$ *where it is described by the two-dimensional Ising universality class with renormalization group eigenvalues* $y_h = 15/8$ *and* $y_t = 1$*. With this in mind what can you say about the large* r *behaviour of the correlation function*

$$G(r) = \langle \hat{S}_i^z \hat{S}_{i+r}^z \rangle - \langle \hat{S}_i^z \rangle \langle \hat{S}_{i+r}^z \rangle\,, \tag{1.131}$$

at zero temperature and for $J = 2h$*?*

Exercise 1.3.12 ***Lattice Gauge Theory***

Consider a 4-dimensional hyper-cubic lattice and associate a $N \times N$ *unitary matrix* U_{ij} *to the link (or edge) of the lattice that connects the neighbouring sites* i *and* j*. We suppress the indices associated with the fact that* U_{ij} *is a matrix in the group* $U(N)$*. Reversing orientation corresponds to hermitian conjugation* $U_{ij} = U_{ji}^\dagger$*. Wilson proposed the following action*

$$S = \beta \sum_{\square} \left[1 - \frac{1}{N} Re\, Tr\, U_{ij}U_{jk}U_{kl}U_{ki}\right]\,, \tag{1.132}$$

where the sum runs over all elementary squares or plaquettes of the lattice. The labels i, j, k, l *are the vertices of these plaquettes.*

Show that the action is invariant under the lattice gauge transformation

$$U_{ij} \to \Omega_i U_{ij} \Omega_j^\dagger\,, \tag{1.133}$$

with $\Omega_i \in U(N)$ *depending on the site* i*. The partition function*

$$Z = \int \prod_{\langle ij \rangle} dU_{ij} e^{-S[U]}\,, \tag{1.134}$$

is obtained by integrating each link variable over the manifold of $N \times N$ *unitary matrices. The natural measure in this space (Haar measure) is invariant under the transformation* (1.133).

When β *is large, the partition function is dominated by unitary matrices close to the identity. It is then convenient to write*

$$U_{ij} = e^{iaA_\mu} \tag{1.135}$$

where a is the lattice spacing and A_μ is a $N \times N$ hermitian matrix associated with the link $\langle ij \rangle$ which is along the direction μ. Take the (naive) continuum limit $a \to 0$, assuming smoothness of the field A_μ so that you can use Taylor expansions. You should find the Yang-Mills action

$$S = \frac{1}{4g^2} \int d^4x \, Tr F_{\mu\nu} F^{\mu\nu} \,, \tag{1.136}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the field strength. What is the relation between the Yang-Mills coupling g and the lattice coupling β?

The simplest lattice gauge theory has gauge group $\mathbb{Z}_2$. In this case, the link variables U_{ij} can only take the values ± 1. Show that the partition function of a $\mathbb{Z}_2$ lattice gauge theory in a three dimensional cubic lattice is equal to the partition function of the Ising model on the dual lattice, which is defined as follows. The vertices of the dual lattice are at the center of the elementary cubes of the original lattice. A pedagogical reference is Duality in field theory and statistical systems *by R. Savit. You can google it or find it at the following link: https://journals.aps.org/rmp/pdf/10.1103/RevModPhys.52.453*

Chapter 2

Conformal Field Theory

In the previous chapter, we saw that continuous phase transitions correspond to scale invariant systems with infinite correlation length. Generically, these systems are also invariant under local scale transformations, *i.e.* conformal transformations.[1] Conformal symmetry has far reaching consequences.

This chapter describes the concepts necessary to formulate a non-perturbative definition of CFT. In the last part, we explain the embedding space formalism for CFT and 't Hooft's large N expansion, which will be very useful in the context of the AdS/CFT correspondence described in the following chapters.

2.1 Conformal Transformations

For simplicity, in most formulas, we will consider Euclidean signature. We start by discussing conformal transformations of $\mathbb{R}^d$ in Cartesian coordinates,

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu \ . \tag{2.1}$$

A conformal transformation (CT) is a coordinate transformation $x^\mu \to \tilde{x}^\mu$ that preserves the form of the metric tensor up to a scale factor,

$$\delta_{\mu\nu} \frac{d\tilde{x}^\mu}{dx^\alpha} \frac{d\tilde{x}^\nu}{dx^\beta} = \Omega^2(x) \delta_{\alpha\beta} \ . \tag{2.2}$$

In other words, a CT is a local dilatation. Therefore, CTs preserve the intersection angle between two curves. Poincaré transformations are a particular case of CTs with $\Omega(x) = 1$.

It is useful to consider an infinitesimal CT

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x) \, . \tag{2.3}$$

In this case, (2.2) reduces to the conformal Killing vector equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\alpha \epsilon^\alpha \delta_{\mu\nu} \, . \tag{2.4}$$

[1] There is long history to the question: "Does scale imply conformal invariance?". Under reasonable assumptions, we think the answer is yes but this has only been proven in 2 [**?**] and 4 [**?**] dimensions. See exercises ?? and ?? for more comments about this question.

CTs can also be defined in general spacetimes by replacing the flat metric $\delta_{\mu\nu}$ by a general metric $g_{\mu\nu}(x)$ in equation (2.2). Notice that this implies that two spacetimes with metrics proportional to each other, $g^{(1)}_{\mu\nu}(x) = \Lambda^2(x) g^{(2)}_{\mu\nu}$, have the same conformal transformations. Such spacetimes are said to be conformally related.

Exercise 2.1.1 *Consider infinitesimal CTs in a general spacetime with metric $g_{\mu\nu}(x)$ and show that this leads to the following conformal killing vector equation*

$$\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \frac{2}{d} \nabla_\alpha \epsilon^\alpha g_{\mu\nu} \,, \tag{2.5}$$

where ∇_μ is the covariant derivative (Levi-Civita connection).

Returning to flat Euclidean space, let us know discuss the solutions to equation (2.4). In 1 dimension, any coordinate transformation is a conformal transformation because there are no angles. In 2 dimensions, the conformal group is infinite dimensional. This makes 2D CFT an extremely powerful tool. However, in this course we will not delve into this topic and instead focus on CFTs in $d > 2$ dimensions.

Exercise 2.1.2 *Consider the coordinate transformation*

$$x \to \tilde{x} = f(x+iy) + f(x-iy) + i\,[g(x+iy) - g(x-iy)] \tag{2.6}$$

$$y \to \tilde{y} = g(x+iy) + g(x-iy) - i\,[f(x+iy) - f(x-iy)] \,, \tag{2.7}$$

where f and g are arbitrary real analytic functions, i.e $f(z^*) = [f(z)]^*$. *Check that this is a CT of the Euclidean plane and obtain*

$$ds^2 = d\tilde{x}^2 + d\tilde{y}^2 = 4|f'(x+iy) + ig'(x+iy)|^2 (dx^2 + dy^2) \,. \tag{2.8}$$

It is convenient to use a complex coordinate $z = x + iy$. Then a coordinate transformation is any holomorphic map $z \to \tilde{z} = F(z)$,

$$ds^2 = d\tilde{z} d\bar{\tilde{z}} = |F'(z)|^2 dz d\bar{z} \,. \tag{2.9}$$

Exercise 2.1.3 *The infinitesimal transformation*

$$x^\mu \to x^\mu + \epsilon^\mu(x) \tag{2.10}$$

is conformal if and only if ϵ^μ is a conformal Killing vector, i.e.

$$\nabla_\nu \epsilon_\mu + \nabla_\mu \epsilon_\nu = f(x) g_{\mu\nu} \tag{2.11}$$

for some function $f(x)$.

Take $g_{\mu\nu}$ to be the Euclidean metric in cartesian coordinates in $d > 2$ dimensions. Start by deriving the identity

$$2\nabla_\mu \nabla_\nu \epsilon_\rho = g_{\mu\rho} \nabla_\nu f + g_{\nu\rho} \nabla_\mu f - g_{\mu\nu} \nabla_\rho f \ . \tag{2.12}$$

By combining the last two equations appropriately, show that

$$(2-d)\nabla_\mu \nabla_\nu f = g_{\mu\nu}\nabla^2 f \tag{2.13}$$

and conclude that the function f must take the form $f(x) = c + 4b_\mu x^\mu$. Finally, show that the most general conformal Killing vector is given by

$$\epsilon^\mu = a^\mu + c\, x^\mu + m^{\mu\nu} x_\nu + x^2 b^\mu - 2(x\cdot b)x^\mu \ , \tag{2.14}$$

where a^μ corresponds to a translation, c to dilatation, $m^{\mu\nu} = -m^{\nu\mu}$ to a rotation and b^μ to a special conformal transformation. How many independent conformal transformations exist in d dimensions?

The finite versions of the infinitesimal transformations (2.14) are:

- Translation
$$x^\mu \to x^\mu + a^\mu \tag{2.15}$$
- Dilatation
$$x^\mu \to \lambda x^\mu \tag{2.16}$$
- Rotation
$$x^\mu \to M^\mu_{\ \mu} x^\nu \tag{2.17}$$
where $M^\mu_{\ \mu}$ is an orthogonal matrix, $M^\mu_{\ \alpha} M^\nu_{\ \beta}\delta_{\mu\nu} = \delta_{\alpha\beta}$.
- Special Conformal Transformation (SCT)
$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} = \frac{\frac{x^\mu}{x^2} - b^\mu}{\left(\frac{x^\mu}{x^2} - b^\mu\right)^2} \tag{2.18}$$

In spacetime dimension $d > 2$, conformal transformations form a group of dimension $\frac{1}{2}(d+2)(d+1)$. We shall see that this group is isomorphic to $SO(d+1,1)$.

The generators P_μ and $M_{\mu\nu}$ correspond to translation and rotations and they are present in any relativistic invariant QFT. In addition, we have the generators of dilatations D and special conformal transformations K_μ. It is convenient to think of the special conformal transformations as the composition of an inversion followed by a translation followed by another inversion. Inversion is the conformal transformation[2]

$$x^\mu \to \frac{x^\mu}{x^2} \ . \tag{2.19}$$

[2] Inversion is outside the component of the conformal group connected to the identity. Thus, it is possible to have CFTs that are not invariant under inversion. In fact, CFTs that break parity also break inversion.

Exercise 2.1.4 *Show that inversion*

$$x^\mu \to x'^\mu = \frac{x^\mu}{x^2} \tag{2.20}$$

is a conformal transformation. By applying an inversion, after a translation, after another inversion, we obtain a special conformal transformation

$$x^\mu \to x'^\mu = \frac{\frac{x^\mu}{x^2} - b^\mu}{\left(\frac{x^\mu}{x^2} - b^\mu\right)^2} = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \ . \tag{2.21}$$

Contrary to inversion, this conformal transformation is continuously connected to the identity. Show that

$$dx'^\mu dx'_\mu = \frac{dx^\mu dx_\mu}{(1 - 2b\cdot x + b^2x^2)^2} \ . \tag{2.22}$$

Exercise 2.1.5 *Verify that the action*

$$S[\varphi] = \int d^dx \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \tag{2.23}$$

describing the Gaussian fixed point is conformal invariant. In other words, show that

$$S[\varphi'] = S[\varphi] \ , \qquad\qquad \varphi'(x) = \left|\frac{\partial x'}{\partial x}\right|^{\Delta/d} \varphi(x') \tag{2.24}$$

where $x \to x'$ is a conformal transformation and $\Delta = \frac{d-2}{2}$. Hint: the only non-trivial transformation to check is the special conformal transformation.

2.2 Conformal Algebra

The form of the generators of the conformal algebra acting on functions can be obtained from

$$\phi\left(x^\mu + \epsilon^\mu(x)\right) = \left[1 + i\,a^\mu P_\mu - cD + \frac{i}{2}m^{\mu\nu}M_{\mu\nu} + i\,b^\mu K_\mu\right]\phi\left(x^\mu\right) \ , \tag{2.25}$$

which leads to [3]

$$P_\mu = -i\partial_\mu \ , \qquad\qquad D = -x^\mu\partial_\mu \ , \tag{2.26}$$

$$M_{\mu\nu} = -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \ , \qquad\qquad K_\mu = 2ix_\mu x^\nu\partial_\nu - i\,x^2\partial_\mu \ . \tag{2.27}$$

Exercise 2.2.1 *Show that the generators obey the following commutation relations*

$$\begin{gathered}
[D, P_\mu] = P_\mu \ , \qquad [D, K_\mu] = -K_\mu \ , \qquad [K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2i\,M_{\mu\nu} \ , \\
[M_{\mu\nu}, P_\alpha] = i\left(\delta_{\mu\alpha}P_\nu - \delta_{\nu\alpha}P_\mu\right) \ , \qquad [M_{\mu\nu}, K_\alpha] = i\left(\delta_{\mu\alpha}K_\nu - \delta_{\nu\alpha}K_\mu\right) \ , \\
[M_{\alpha\beta}, M_{\mu\nu}] = i\left(\delta_{\alpha\mu}M_{\beta\nu} + \delta_{\beta\nu}M_{\alpha\mu} - \delta_{\beta\mu}M_{\alpha\nu} - \delta_{\alpha\nu}M_{\beta\mu}\right) \ .
\end{gathered} \tag{2.28}$$

[3] We define the dilatation generator D in a non-standard fashion so that it has real eigenvalues in radial quantization of unitary CFTs.

The third line of (2.28) is the usual $SO(d)$ (or Lorentz) algebra. The second line means that P_μ and K_μ transform as vectors under rotations. The first line is more interesting. It says that P_μ and K_μ act as raising and lowering operators for D.

2.3 Local operators

For a conformal transformation g ($x^\mu \to \tilde{x}^\mu = g^\mu(x)$), the correlation functions of local operators in a CFT should be invariant,

$$\langle \mathcal{O}_1^{a_1}(x_1) \dots \mathcal{O}_n^{a_n}(x_n) \rangle = T_{b_1}{}^{a_1}(x_1, g) \dots T_{b_n}{}^{a_n}(x_n, g) \left\langle \mathcal{O}_1^{b_1}(g(x_1)) \dots \mathcal{O}_n^{b_n}(g(x_n)) \right\rangle . \tag{2.29}$$

The question we would like to answer is what can the matrices $T_b{}^a(x, g)$ be? We shall start by considering the little group of conformal transformations that preserves the origin $x = 0$. Then, the matrices $T_b{}^a(0, g)$ must for a representation of the little group generated by dilatations, rotations and SCTs. The action of the little group generators can then be written as

$$\begin{aligned} D\mathcal{O}^a(0) &= [\Delta]_b{}^a \, \mathcal{O}^b(0) \\ M_{\mu\nu}\mathcal{O}^a(0) &= [S_{\mu\nu}]_b{}^a \, \mathcal{O}^b(0) \\ K_\mu \mathcal{O}^a(0) &= [q_\mu]_b{}^a \, \mathcal{O}^b(0) \end{aligned} \tag{2.30}$$

where the matrices Δ, $S_{\mu\nu}$ and q_μ form a representation of the little algebra. Given this representation, we can easily derive the action of the conformal generators at any point x. We start by defining

$$\mathcal{O}^a(x) \equiv e^{iP\cdot x}\mathcal{O}^a(0) \tag{2.31}$$

and then use the commutation relations and the action (2.30) to define the action of the generators on $\mathcal{O}^a(x)$. This leads to

$$\begin{aligned} P_\mu \mathcal{O}^a(x) &= -i\partial_\mu \mathcal{O}^a(x) \\ D\mathcal{O}^a(x) &= \left([\Delta]_b{}^a - x\cdot\partial\,\delta_b^a\right)\mathcal{O}^b(x) \\ M_{\mu\nu}\mathcal{O}^a(x) &= \left([S_{\mu\nu}]_b{}^a - i(x_\mu\partial_\nu - x_\nu\partial_\mu)\,\delta_b^a\right)\mathcal{O}^b(x) \\ K_\mu\mathcal{O}^a(x) &= \left([q_\mu]_b{}^a - 2x_\mu\,[\Delta]_b{}^a + 2i\,[S_{\mu\nu}]_b{}^a\,x^\nu + i(2x_\mu x\cdot\partial - x^2\partial_\mu)\,\delta_b^a\right)\mathcal{O}^b(x) \end{aligned} \tag{2.32}$$

2.4 Primary operators and their correlation functions

We have seen that scaling operators transform simply under global scale transformations (see equation (1.51)). The natural generalization for conformal transformations $x \to \tilde{x}$ is

$$\langle \mathcal{O}_1(\tilde{x}_1)\dots\mathcal{O}_n(\tilde{x}_n)\rangle = \left|\frac{\partial\tilde{x}}{\partial x}\right|_{x_1}^{-\frac{\Delta_1}{d}} \dots \left|\frac{\partial\tilde{x}}{\partial x}\right|_{x_n}^{-\frac{\Delta_n}{d}} \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle . \tag{2.33}$$

If the operators have spin, *i.e.* if they transform under rotations then the rule is

$$\left\langle \Omega(\tilde{x}_1)^{\Delta_1} D(R(\tilde{x}_1))_b{}^a \mathcal{O}_1^b(\tilde{x}_1)\dots \right\rangle = \langle \mathcal{O}_1^a(x_1)\dots\rangle . \tag{2.34}$$

where [4]

$$\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \Omega(\tilde{x}) R^\mu{}_\nu(\tilde{x}) \,, \qquad\qquad R^\mu{}_\nu(\tilde{x}) \in SO(d) \,. \tag{2.35}$$

The matrix $D(R)^a{}_b$ implements the action of the $SO(d)$ rotation R in the representation of $\mathcal{O}_1$. Operators that satisfy this transformation rule are called *primary.*

Exercise 2.4.1 *Check that the rule* (2.34) *is compatible with the composition of two conformal transformations. In other words, verify that it gives the same result for* $x \to \tilde{x} = g_1(g_2(x))$ *and for* $x \to y = g_2(x)$ *followed by* $y \to \tilde{x} = g_1(y)$.

As explained above, if the action of Poincaré transformations and inversion are correctly realized then all conformal transformations will act correctly. In practice, this means that imposing conformal symmetry of a correlator amounts to imposing Poincaré covariance and

$$\left\langle \left(x_1^2\right)^{-\Delta_1} D(I(x_1))_b{}^a \mathcal{O}_1^b\left(\frac{x_1}{x_1^2}\right) \dots \right\rangle = \left\langle \mathcal{O}_1^a(x_1) \dots \right\rangle \,, \tag{2.36}$$

where [5]

$$I^\mu{}_\nu(x) = \delta^\mu_\nu - \frac{2 x_\mu x^\nu}{x^2} \,. \tag{2.37}$$

This implies that vacuum one-point functions $\langle \mathcal{O}(x) \rangle$ vanish except for the identity operator (which is the unique operator with $\Delta = 0$). It also fixes the form of the two and three point functions,

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle = \frac{\delta_{ij}}{(x-y)^{2\Delta_i}} \,, \tag{2.38}$$

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} \, |x_{13}|^{\Delta_1+\Delta_3-\Delta_2} \, |x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \,,$$

where we have normalized the operators to have unit two-point function. Notice that after normalizing the operators using the two-point function, the normalization of the three-point function is physically meaningful.

Exercise 2.4.2 *Derive expressions* (2.38).

The four-point function is not fixed by conformal symmetry because with four points one can construct two independent conformal invariant cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \,. \tag{2.39}$$

The physical meaning of the cross-ratios is clear in the *conformal frame.* In the conformal frame, we use the symmetries to place the operators at special points. Notice

[4]Here we are considering conformal transformations connected to the identity. If the CFT is parity invariant then $R^\mu{}_\nu$ can also be a reflection.

[5]Notice that $I^\mu{}_\nu \in O(d)$ is a reflection and not a rotation. This is related to the fact that inversion is not in the component of the conformal group connected to the identity.

that we can always reconstruct the general case by acting with the inverse symmetries. We send x_4 to ∞ by performing a special conformal transformation. Then we bring x_1 to the origin with a translation. Next, we do a dilatation and a rotation to bring x_3 to $(1, 0, \ldots, 0)$. Finally, we perform a rotation (that leaves x_3 fixed) to bring x_2 to the plane $(x, y, 0, \ldots, 0)$. If we evaluate the cross-ratios in this configuration we obtain

$$u = z\bar{z} \,, \qquad v = (1-z)(1-\bar{z}) \,, \tag{2.40}$$

where $z = x + iy$ and $\bar{z} = x - iy$.

The general form of the four point function is

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_4) \rangle = \frac{\mathcal{A}(u,v)}{\left(x_{12}^2 x_{34}^2\right)^{\Delta}} \,. \tag{2.41}$$

Exercise 2.4.3 *Generalize* (2.41) *for 4 different operators. Generalize it also for the case of a n-point function. How many independent cross-ratios are there in this case?*

Exercise 2.4.4 *The correlator* (2.41) *is invariant under permutations of the points x_i. Show that this implies*

$$\mathcal{A}(u,v) = \mathcal{A}(u/v, 1/v) \,, \qquad \mathcal{A}(u,v) = \left(\frac{u}{v}\right)^{\Delta} \mathcal{A}(v,u) \,. \tag{2.42}$$

Exercise 2.4.5 ***Tensor primary fields - two point function***

A tensor primary field of scaling dimension Δ and spin J transforms as follows

$$T'_{\mu_1 \ldots \mu_J}(x) = \left| \frac{\partial x'}{\partial x} \right|^{\frac{\Delta - J}{d}} \frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\nu_J}}{\partial x^{\mu_J}} T_{\nu_1 \ldots \nu_J}(x') \,. \tag{2.43}$$

Show that the two-point function of a vector primary operator in a CFT is given by

$$\langle j_\mu(x) j_\nu(y) \rangle = \frac{C_j}{|x-y|^{2\Delta}} I_{\mu\nu}(x-y) \tag{2.44}$$

where

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \,. \tag{2.45}$$

If j_μ is a conserved current, what is its scaling dimension Δ?

Show that the two-point function of a spin 2 (symmetric and traceless) tensor primary operator in a CFT is given by

$$\langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle = \frac{C_T}{|x-y|^{2\Delta}} \left[\frac{1}{2} \left(I_{\mu\alpha}(x-y) I_{\nu\beta}(x-y) + I_{\mu\beta}(x-y) I_{\nu\alpha}(x-y) \right) - \frac{1}{d} \eta_{\mu\nu} \eta_{\alpha\beta} \right] .$$

Check that if $T_{\mu\nu}$ is the stress-energy tensor, then its conservation requires $\Delta = d$.

Compute the constant C_T for a real massless scalar field in flat space. Use the (normal ordered version of the) stress-energy tensor

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{4(d-1)} \left((d-2) \partial_\mu \partial_\nu + \eta_{\mu\nu} \partial^2 \right) \varphi^2 \,, \tag{2.46}$$

which can be derived from the action

$$S[\varphi] = \int d^dx\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\mu\varphi + \frac{d-2}{8(d-1)}R\,\varphi^2\right) \ , \tag{2.47}$$

as explained in exercise 2.5.1.

Exercise 2.4.6 ***Tensor primary fields - three point function***
A tensor primary field of scaling dimension Δ and spin J transforms as follows

$$T'_{\mu_1\dots\mu_J}(x) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta-J}{d}} \frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\nu_J}}{\partial x^{\mu_J}} T_{\nu_1\dots\nu_J}(x') \ . \tag{2.48}$$

a. *Verify that*

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)j^\mu(x_3)\rangle = C_{12j}\frac{V^\mu(x_1,x_2,x_3)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta+1}|x_{13}|^{\Delta_1+\Delta-\Delta_2-1}|x_{23}|^{\Delta_2+\Delta-\Delta_1-1}} \tag{2.49}$$

has the correct transformation properties of a three point function of a vector and two scalar primary operators in a CFT. In this expression, Δ is the dimension of the vector operator j^μ, Δ_i is the dimension of the scalar operator $\mathcal{O}_i$, C_{12j} is a constant and

$$V^\mu(x_1,x_2,x_3) = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2} \ . \tag{2.50}$$

Suggestion: start by showing that under inversion $x'^\mu = x^\mu/x^2$, we have

$$(x'_{ij})^2 = \frac{x_{ij}^2}{x_i^2 x_j^2} \ , \qquad V_\mu(x_1,x_2,x_3) = \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_{x=x_3} V_\nu(x'_1,x'_2,x'_3) \ . \tag{2.51}$$

b. *Similarly, verify that*

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)T^{\mu\nu}(x_3)\rangle = C_{12T}\frac{H^{\mu\nu}(x_1,x_2,x_3)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta+2}|x_{13}|^{\Delta_1+\Delta-\Delta_2-2}|x_{23}|^{\Delta_2+\Delta-\Delta_1-2}} \tag{2.52}$$

transforms appropriately under conformal transformations with $T^{\mu\nu}$ a primary field of dimension Δ and spin 2 (symmetric traceless tensor). Here, the numerator is

$$H^{\mu\nu} = V^\mu V^\nu - \frac{1}{d}V_\alpha V^\alpha \delta^{\mu\nu} \ , \tag{2.53}$$

and you can use the identities (2.51) without proof.

c. *Consider a free massless scalar field with Euclidean action*

$$S[\varphi] = \int d^dx \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \ . \tag{2.54}$$

Show that the two-point function is given by (assume $d > 2$)

$$\langle\varphi(x)\varphi(y)\rangle = \frac{\mathcal{N}}{|x-y|^{d-2}} \ , \qquad \mathcal{N} = \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}} \ . \tag{2.55}$$

Recall that the Γ*-function is defined by*

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \ , \qquad \Re z > 0 \ . \tag{2.56}$$

d*. In the same theory, compute the three point function*

$$\langle \varphi(x_1)\varphi(x_2)T^{\mu\nu}(x_3)\rangle \ , \tag{2.57}$$

where

$$T_{\mu\nu} =: \partial_\mu\varphi\partial_\nu\varphi : -\frac{1}{4(d-1)}\left((d-2)\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial^2\right) : \varphi^2 : \tag{2.58}$$

is the stress-energy tensor. Compare your result with (2.52) and determine the dimension Δ *of the stress-energy tensor, the dimension* Δ_φ *and the constant* $C_{\varphi\varphi T}$*.*

2.5 Stress-energy tensor

To define the stress-energy tensor it is convenient to consider the theory in a general background metric $g_{\mu\nu}$. Formally, we can write

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g = \frac{1}{Z[g]}\int [d\phi] e^{-S[\phi,g]}\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n) \ , \tag{2.59}$$

where $Z[g] = \int [d\phi]e^{-S[\phi,g]}$ is the partition function for the background metric $g_{\mu\nu}$. Recalling the classical definition

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)} \ , \tag{2.60}$$

it is natural to define the quantum stress-energy tensor operator via the equation

$$\frac{Z[g+\delta g]}{Z[g]} = 1 + \frac{1}{2}\int dx\sqrt{g}\delta g_{\mu\nu}(x)\left\langle T^{\mu\nu}(x)\right\rangle_g + O(\delta g^2) \ , \tag{2.61}$$

and

$$\begin{aligned} &\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_{g+\delta g} - \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g \\ =&\frac{1}{2}\int dx\sqrt{g}\delta g_{\mu\nu}(x)\Big[\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g \\ &\qquad - \langle T^{\mu\nu}(x)\rangle_g\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g\Big] + O(\delta g^2) \ . \end{aligned} \tag{2.62}$$

Exercise 2.5.1 *Consider the action for a free scalar field in a curved background*

$$S[g_{\mu\nu},\varphi] = \int d^dx\sqrt{g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\mu\varphi + \frac{1}{2}\xi R\varphi^2\right) \tag{2.63}$$

where R *is the Ricci scalar of the metric* $g^{\mu\nu}$ *and* ξ *is a dimensionless coupling constant. Show that the action is Weyl invariant*

$$S[\Omega^2 g_{\mu\nu},\Omega^{-\Delta}\varphi] = S[g_{\mu\nu},\varphi] \tag{2.64}$$

for $\Delta = \frac{d-2}{2}$ *and* $\xi = \frac{d-2}{4(d-1)}$. *Hint: you will need to use the following formula*

$$\tilde{R} = \Omega^{-2}\left[R - 2(d-1)g^{\mu\nu}\nabla_\mu\nabla_\nu \log\Omega - (d-2)(d-1)g^{\mu\nu}(\nabla_\mu \log\Omega)(\nabla_\nu \log\Omega)\right] \quad (2.65)$$

where $\tilde{R}$ *is the Ricci scalar for the metric* $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. *You can find the derivation of this formula in appendix D of Wald's book on General Relativity.*

Obtain the stress-energy tensor for this theory and check that it is traceless and conserved on-shell (i.e., if φ *solves the classical equations of motion).*

Exercise 2.5.2 *Show that the action for electrodynamics*

$$S[A_\mu] = \int d^4x \frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \quad (2.66)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, *is invariant under conformal transformations,*

$$S[A'_\mu] = S[A_\mu]\ , \qquad A'_\mu(x) = \frac{\partial x'^\alpha}{\partial x^\mu}A_\alpha(x') \quad (2.67)$$

Rederive the same result by considering electrodynamics in a general curved background

$$S[g_{\mu\nu}, A_\mu] = \int d^4x\sqrt{g}\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} \quad (2.68)$$

and showing that

$$S[\Omega^2 g_{\mu\nu}, A_\mu] = S[g_{\mu\nu}, A_\mu] \quad (2.69)$$

Obtain the electromagnetic stress-energy tensor $T_{\mu\nu}$.

Returning to flat Minkowski space (for simplicity), show that the following integrals over all space at a fixed time t,

$$D(t) = \int_{\Sigma(t)} d^3x\, T^{0\mu}x_\mu\ , \qquad K_\alpha(t) = \int_{\Sigma(t)} d^3x\, T^{0\mu}(\eta_{\mu\alpha}x^2 - 2x_\mu x_\alpha) \quad (2.70)$$

are conserved quantities for an electromagnetic wave propagating in vacuum.

2.6 Ward identities

Under an infinitesimal coordinate transformation $\tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$, the metric tensor changes $\tilde{g}_{\mu\nu} = g_{\mu\nu} - \nabla_\mu\epsilon_\nu - \nabla_\nu\epsilon_\mu$ but the physics should remain invariant. In particular, the partition function $Z[g] = Z[\tilde{g}]$ and the correlation functions [6]

$$\langle\mathcal{O}_1(\tilde{x}_1)\dots\mathcal{O}_n(\tilde{x}_n)\rangle_{\tilde{g}} = \langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g\ , \quad (2.71)$$

do not change. This leads to the conservation equation $\langle\nabla_\mu T^{\mu\nu}(x)\rangle_g$ and

$$\begin{aligned}&\sum_{i=1}^n \epsilon^\mu(x_i)\frac{\partial}{\partial x_i^\mu}\langle\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g \\ &= -\int dx\sqrt{g}\epsilon_\nu(x)\langle\nabla_\mu T^{\mu\nu}(x)\mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle_g\end{aligned} \quad (2.72)$$

[6] If the operators are not scalars (e.g. if they are vector operators) then one also needs to take into account the rotation of their indices.

for all $\epsilon^\mu(x)$ that decays sufficiently fast at infinity. Thus $\nabla_\mu T^{\mu\nu} = 0$ up to contact terms.

Correlation functions of primary operators transform homogeneously under Weyl transformations of the metric [7]

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{\Omega^2 g} = \frac{\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g}{[\Omega(x_1)]^{\Delta_1} \dots [\Omega(x_n)]^{\Delta_n}} \ . \tag{2.73}$$

Exercise 2.6.1 *Show that this transformation rule under local rescalings of the metric (together with coordinate invariance) implies (2.33) under conformal transformations.*

Consider now an infinitesimal Weyl transformation $\Omega = 1 + \omega$, which corresponds to a metric variation $\delta g_{\mu\nu} = 2\omega g_{\mu\nu}$. From (2.62) and (2.73) we conclude that

$$\begin{aligned} &\sum_{i=1}^{n} \Delta_i \, \omega(x_i) \, \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g \\ &= - \int dx \sqrt{g} \, \omega(x) g_{\mu\nu} \Big[\langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g \\ &\qquad\qquad - \langle T^{\mu\nu}(x) \rangle_g \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g \Big] \ . \end{aligned} \tag{2.74}$$

Consider the following codimension 1 integral over the boundary of a region B, [8]

$$\begin{aligned} I = \int_{\partial B} dS_\mu \epsilon_\nu(x) \Big[&\langle T^{\mu\nu}(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g \\ &- \langle T^{\mu\nu}(x) \rangle_g \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g \Big] \ . \end{aligned} \tag{2.75}$$

One can think of this as the total flux of the current $\epsilon_\nu T^{\mu\nu}$, where $\epsilon_\nu(x)$ is an infinitesimal conformal transformation. Gauss law tells us that this flux should be equal to the integral of the divergence of the current

$$\nabla_\mu \left(\epsilon_\nu T^{\mu\nu} \right) = \epsilon_\nu \nabla_\mu T^{\mu\nu} + \nabla_\mu \epsilon_\nu T^{\mu\nu} = \epsilon_\nu \nabla_\mu T^{\mu\nu} + \frac{1}{d} \nabla_\alpha \epsilon^\alpha g_{\mu\nu} T^{\mu\nu} \ , \tag{2.76}$$

where we used the symmetry of the stress-energy tensor $T^{\mu\nu} = T^{\nu\mu}$ and the definition of an infinitesimal conformal transformation $\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \frac{2}{d} \nabla_\alpha \epsilon^\alpha g_{\mu\nu}$. Using Gauss law and (2.72) and (2.74) we conclude that

$$I = - \sum_{x_i \in B} \left[\epsilon^\mu(x_i) \frac{\partial}{\partial x_i^\mu} + \frac{\Delta_i}{d} \nabla_\alpha \epsilon^\alpha(x_i) \right] \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g \ . \tag{2.77}$$

The equality of (2.75) and (2.77) for any infinitesimal conformal transformation (2.14) is the most useful form of the conformal Ward identities.

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[7] In general, the partition function is not invariant in even dimensions. This is the Weyl anomaly $Z[\Omega^2 g] = Z[g] e^{-S_{Weyl}[\Omega, g]}$.

[8] In the notation of the [2] this is the topological operator $Q_\epsilon[\partial B]$ inserted in the correlation function $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_g$.

Exercise 2.6.2 *Conformal symmetry fixes the three-point function of a spin 2 primary operator and two scalars up to an overall constant,* [9]

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)T^{\mu\nu}(x_3)\rangle = C_{12T}\frac{H^{\mu\nu}(x_1,x_2,x_3)}{|x_{12}|^{2\Delta-d+2}|x_{13}|^{d-2}|x_{23}|^{d-2}}\ , \tag{2.78}$$

where

$$H^{\mu\nu} = V^\mu V^\nu - \frac{1}{d}V_\alpha V^\alpha \delta^{\mu\nu}\ , \qquad V^\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}\ . \tag{2.79}$$

Write the conformal Ward identity (2.75)=(2.77) for the three point function $\langle T^{\mu\nu}(x)\mathcal{O}(0)\mathcal{O}(y)\rangle$ for the case of an infinitesimal dilation $\epsilon^\mu(x) = \lambda x^\mu$ and with the surface ∂B being a sphere centred at the origin and with radius smaller than $|y|$. Use this form of the conformal Ward identity in the limit of an infinitesimally small sphere ∂B and formula (2.78) for the three point function to derive

$$C_{\mathcal{O}\mathcal{O}T} = -\frac{d\Delta}{d-1}\frac{1}{S_d}\ , \tag{2.80}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of a $(d-1)$-dimensional unit sphere.

2.7 Quantization

So far we have only discussed correlation functions from a path integral (or statistical mechanics) point of view. It is also convenient to consider the theory in the canonical quantization formalism. In this context, we must choose a foliation of spacetime by spatial slices and there will be a Hilbert space associated to each spatial slice. Usually one chooses spatial slices according to the symmetries of the theory such that the Hilbert spaces at different slices are the same and that we can move between slices with a unitary operator representing time evolution. Then, correlation functions can be thought of as time-ordered (with respect to our foliation) expectation values

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \langle 0|T\left\{\hat{\mathcal{O}}_1(\tau_1,\mathbf{x}_1)\dots\hat{\mathcal{O}}_n(\tau_n,\mathbf{x}_n)\right\}|0\rangle\ , \tag{2.81}$$

where $|0\rangle$ os the vacuum and $\hat{\mathcal{O}}_i(x)$ are quantum operators acting on the Hilbert space. Notice that the Hilbert space is different for different foliations. The same set of correlators can be given different quantum mechanical interpretations by choosing different foliations.

We shall only consider euclidean time evolution using a vector field associated to a symmetry. In other words, the hamiltonian $\hat{H}$ that generates ∂_τ is a constant of motion. We can then write

$$\hat{\mathcal{O}}(\tau,\mathbf{x}) = e^{\tau\hat{H}}\hat{\mathcal{O}}(0,\mathbf{x})e^{-\tau\hat{H}}\ . \tag{2.82}$$

Notice that this makes the expectation value (2.81) finite if the hamiltonian as a spectrum (with real part) bounded from below.

[9] You can try to derive this formula using the embedding space formalism of section 2.12.

For Poincaré invariant field theories, one may choose to foliate spacetime with surfaces of constant $x^1 = \tau$. Then, the action of the quantum operator $\hat{H}$ corresponds to the insertion of

$$H = -\int d\mathbf{x}\, T^{11}(\tau, \mathbf{x}) \tag{2.83}$$

in the path integral language. Notice that the Ward identitiy (2.75) = (2.77) with $\epsilon_\mu(x) = \delta^1_\mu$ implies that the insertion of H does not depend on the euclidean time τ as long as it does not cross the Euclidean time of the other operator insertions. This corresponds to the operator ordering in the quantum mechanical language. Moreover, we can use the same Ward identity to compute the commutator of $\hat{H}$ with $\hat{\mathcal{O}}$. This gives

$$\left[\hat{H}, \hat{\mathcal{O}}(\tau, \mathbf{x})\right] = \partial_\tau \hat{\mathcal{O}}(\tau, \mathbf{x}) \,, \tag{2.84}$$

in agreement with (2.82). Similarly, we can also define the momentum operator $\hat{P}^\mu$ by the insertion of

$$P^\mu = -\int d\mathbf{x}\, T^{1\mu}(0, \mathbf{x}) \,, \tag{2.85}$$

in the correlator. This definition imples

$$\left[\hat{P}^\mu, \hat{\mathcal{O}}(\tau, \mathbf{x})\right] = \partial_\mu \hat{\mathcal{O}}(\tau, \mathbf{x}) \,. \tag{2.86}$$

2.7.1 Conjugation

In quantum mechanics the hamiltonian is an hermitian operator. Therefore,

$$\hat{\mathcal{O}}(\tau, \mathbf{x})^\dagger = e^{-\tau \hat{H}} \hat{\mathcal{O}}(0, \mathbf{x})^\dagger e^{\tau \hat{H}} = \hat{\mathcal{O}}(-\tau, \mathbf{x}) \,. \tag{2.87}$$

if $\hat{\mathcal{O}}(0, \mathbf{x})$ is an hermitian operator (representing an observable). In other words, for real scalar operators conjugation acts like a reflection $\tau \to -\tau$. The same is true for operators with spin:

$$\hat{\mathcal{O}}_{\mu_1 \dots \mu_l}(\tau, \mathbf{x})^\dagger = \Theta^{\nu_1}_{\mu_1} \dots \Theta^{\nu_l}_{\mu_l} \hat{\mathcal{O}}_{\nu_1 \dots \nu_l}(-\tau, \mathbf{x}) \,, \qquad \Theta^\nu_\mu = \delta^\nu_\mu - 2\delta^1_\mu \delta^\nu_1 \,, \tag{2.88}$$

where the euclidean time τ corresponds to the index $\mu, \nu = 1$. [10]

2.7.2 Reflection positivity

One can also think of the correlator as computing an inner product in the Hilbert space. For example,

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \langle \psi_{out} | \psi_{in} \rangle \,, \tag{2.90}$$

[10] Notice that this is consistent with the particular case of operators with indices created by derivatives:

$$\partial_\tau \hat{\mathcal{O}}(\tau, \mathbf{x})^\dagger = -\partial_\tau \hat{\mathcal{O}}(-\tau, \mathbf{x}) \,, \qquad \frac{\partial}{\partial x^i} \hat{\mathcal{O}}(\tau, \mathbf{x})^\dagger = \frac{\partial}{\partial x^i} \hat{\mathcal{O}}(-\tau, \mathbf{x}) \,, \qquad i = 2, \dots, d \,. \tag{2.89}$$

where

$$|\psi_{in}\rangle = \hat{\mathcal{O}}_{k+1}(\tau_{k+1}, \mathbf{x}_{k+1}) \ldots \hat{\mathcal{O}}_n(\tau_n, \mathbf{x}_n)|0\rangle \tag{2.91}$$

$$|\psi_{out}\rangle = \hat{\mathcal{O}}_k^\dagger(\tau_k, \mathbf{x}_k) \ldots \hat{\mathcal{O}}_1^\dagger(\tau_1, \mathbf{x}_1)|0\rangle \tag{2.92}$$

assuming the operators are time-ordered $\tau_1 > \tau_2 > \cdots > \tau_n$. In a unitary theory, we have

$$\langle\psi|\psi\rangle \geq 0\,, \tag{2.93}$$

for all states $|\psi\rangle$. Choosing

$$|\psi\rangle = \hat{\mathcal{O}}_1(\tau_1, \mathbf{x}_1) \ldots \hat{\mathcal{O}}_n(\tau_n, \mathbf{x}_n)|0\rangle\,, \qquad 0 > \tau_1 > \tau_2 > \cdots > \tau_n\,, \tag{2.94}$$

and using the conjugation formula (2.87), we find

$$\langle\psi|\psi\rangle = \langle\mathcal{O}_n(-\tau_n, \mathbf{x}_n) \ldots \mathcal{O}_1(-\tau_1, \mathbf{x}_1)\mathcal{O}_1(\tau_1, \mathbf{x}_1) \ldots \mathcal{O}_n(\tau_n, \mathbf{x}_n)\rangle \geq 0\,. \tag{2.95}$$

This condition is known as reflection positivity and it follows from unitarity of the Lorentzian theory obtained by Wick rotation $\tau \to it$. The Osterwalder–Schrader theorem states that the converse is also true (with a few extra assumptions).

Exercise 2.7.1 *The two point function of a scalar primary operator can be written as an inner product*

$$\langle\mathcal{O}(-\tau_1, \boldsymbol{x}_1)\mathcal{O}(\tau_2, \boldsymbol{x}_2)\rangle = \langle\mathcal{O}(\tau_1, \boldsymbol{x}_1)|\mathcal{O}(\tau_2, \boldsymbol{x}_2)\rangle\,, \tag{2.96}$$

where

$$|\mathcal{O}(\tau, \boldsymbol{x})\rangle = \mathcal{O}(\tau, \boldsymbol{x})|0\rangle\,, \qquad \tau < 0\,. \tag{2.97}$$

In the quantization with constant $x^1 = \tau$ surfaces, it is natural to decompose the state $|\mathcal{O}(\tau, \boldsymbol{x})\rangle$ into an eigenbasis of momenta $\hat{P}^\mu$. Notice that the conjugation rule (2.88) *implies that P^1 is hermitian and $\hat{P}^j$ for $j = 2, \ldots, d$ is anti-hermitian. In fact, $\hat{P}^1 = H$ is the hamiltonian given in* (2.83) *and $\hat{P}_L^j = -i\hat{P}^j$ is the hermitian operator representing spatial momentum in the Lorentzian theory. We can then write*

$$|\mathcal{O}(\tau, \boldsymbol{x})\rangle = \sum_\alpha \int \frac{E d\boldsymbol{k}}{(2\pi)^d}\, \psi_\alpha(\tau, \boldsymbol{x}; E, \boldsymbol{k})|E, \boldsymbol{k}, \alpha\rangle\,, \tag{2.98}$$

where $|E, \boldsymbol{k}, \alpha\rangle$ is an eigenstate of momentum

$$\hat{H}|E, \boldsymbol{k}, \alpha\rangle = E|E, \boldsymbol{k}, \alpha\rangle\,, \qquad \hat{P}^j|E, \boldsymbol{k}, \alpha\rangle = ik^j|E, \boldsymbol{k}, \alpha\rangle\,, \tag{2.99}$$

and the label α distinguishes states with the same momentum eigenvalue. Use (2.86) *to show that*

$$\psi_\alpha(\tau, \boldsymbol{x}; E, \boldsymbol{k}) = e^{\tau E + i\boldsymbol{k}\cdot\boldsymbol{x}} q_\alpha(E, \boldsymbol{k})\,. \tag{2.100}$$

Using the normalization [11] $\langle E, \boldsymbol{k}, \alpha | E', \boldsymbol{k}', \alpha' \rangle = (2\pi)^d \delta_{\alpha\alpha'} \delta^{d-1}(\boldsymbol{k} - \boldsymbol{k}') \delta(E - E')$, *show that the two point function becomes*

$$\langle \mathcal{O}(-\tau_1, \boldsymbol{x}_1) \mathcal{O}(\tau_2, \boldsymbol{x}_2) \rangle = \frac{1}{(\tau^2 + \boldsymbol{x}^2)^\Delta} = \int \frac{dE d\boldsymbol{k}}{(2\pi)^d} e^{-E\tau + i\boldsymbol{k}\cdot\boldsymbol{x}} \rho(E, \boldsymbol{k}) \,, \qquad \tau > 0 \,, \tag{2.101}$$

where $\tau = -(\tau_1 + \tau_2)$, $\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ *and the spectral density is* $\rho(E, \boldsymbol{k}) = \sum_\alpha |q_\alpha(E, \boldsymbol{k})|^2$. *Show that*

$$\rho(E, \boldsymbol{k}) = \frac{2\pi^{\frac{d}{2}+1}}{\Gamma(\Delta)\Gamma\left(\Delta - \frac{d}{2} + 1\right)} \Theta\left(E - |\boldsymbol{k}|\right) \left(\frac{E^2 - \boldsymbol{k}^2}{4}\right)^{\Delta - \frac{d}{2}} . \tag{2.102}$$

Conclude that there is no particle interpretation of this spectral density for generic Δ. *Conclude also that reflection positivity (or unitarity) implies that* $\Delta \geq \frac{d}{2} - 1$. *Give a particle interpretation to the spectral density for* $\Delta = \frac{d}{2} - 1$ *and* $\Delta = d - 2$.

Local operators are divided into two types: primary and descendant. Descendant operators are operators that can be written as (linear combinations of) derivatives of other local operators. Primary operators can not be written as derivatives of other local operators. Primary operators at the origin are annihilated by the generators of special conformal transformations. Moreover, they are eigenvectors of the dilatation generator and form irreducible representations of the rotation group $SO(d)$,

$$[K_\mu, \mathcal{O}(0)] = 0 \;, \qquad [D, \mathcal{O}(0)] = \Delta\, \mathcal{O}(0) \;, \qquad [M_{\mu\nu}, \mathcal{O}_A(0)] = [M_{\mu\nu}]_A^B\, \mathcal{O}_B(0) \;.$$

Exercise 2.7.2 ***Analytic structure of the two-point function***

Correlation functions in a Euclidean QFT are always *time ordered: see eq.* (2.81). *The reason for this is that out-of-time ordered correlators do not exist in general in Euclidean signature:*

a) *indeed, apply eq.* (2.82) *to a two-point function of operators which are odered as written:*

$$\langle 0 | \, \hat{\mathcal{O}}(\tau_1, \vec{x}_1) \hat{\mathcal{O}}(\tau_2, \vec{x}_2) \, | 0 \rangle \;. \tag{2.103}$$

In general H is only bounded from below, not from above. By using this fact, show that the two-point function is infinite if the operators are anti-time ordered. Correlation functions computed from a path-integral indeed compute Euclidean time ordered correlators.

b) *Consider the two point function of a primary operator* $\mathcal{O}$ *as a function of time, at fixed distance* $\vec{x}$:

$$F(\tau) = \langle \mathcal{O}(\tau, \vec{x}) \mathcal{O}(0) \rangle = \frac{1}{(\vec{x}^2 + \tau^2)^\Delta}. \tag{2.104}$$

Define $F(\tau)$ for complex values of τ by analytic continuation, and find the singularities of $F(\tau)$ in the complex τ-plane, for generic Δ.

c) *Set*

$$\tau = it + \epsilon, \quad t, \epsilon \in \mathbb{R}. \tag{2.105}$$

Then as $\epsilon \to 0$, you get a correlation function in Lorentzian time. Compute the vacuum expectation value of $[\hat{\mathcal{O}}(it, \vec{x}), \hat{\mathcal{O}}(0)]$. *Is it compatible with causality?* Hint*: give a small real part to $\tau = it + \epsilon$, with $\epsilon \to 0^\pm$. The ordering of operators depends on the sign of ϵ.*

[11] If the normalization includes a positive factor depending on E and $\mathbf{k}$, that will not change the positivity properties of the spectral density.

2.8 Radial quantization and the state-operator map

Consider $\mathbb{R}^d$ in spherical coordinates. Writing the radial coordinate as $r = e^\tau$ we find

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2 = e^{2\tau}\left(d\tau^2 + d\Omega_{d-1}^2\right) \ . \tag{2.106}$$

Thus, the cylinder $\mathbb{R} \times S^{d-1}$ can be obtained as a Weyl transformation of euclidean space $\mathbb{R}^d$.

Exercise 2.8.1 *Compute the two-point function of a scalar primary operator on the cylinder using the Weyl transformation property (2.73).*

A local operator inserted at the origin of $\mathbb{R}^d$ prepares a state at $\tau = -\infty$ on the cylinder. On the other hand, a state on a constant time slice of the cylinder can be propagated backwards in time until it corresponds to a boundary condition on a arbitrarily small sphere around the origin of $\mathbb{R}^d$, which defines a local operator. Furthermore, time translations on the cylinder correspond to dilatations on $\mathbb{R}^d$. This teaches us that the spectrum of the dilatation generator on $\mathbb{R}^d$ is the same as the energy spectrum for the theory on $\mathbb{R} \times S^{d-1}$. [12]

2.9 Unitarity bounds

Exercise 2.9.1 *The generators of the conformal algebra can be represented as follows*

$$\begin{aligned} \hat{P}_\mu &= -i\partial_\mu \ , & \hat{L}_{\mu\nu} &= -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\ \hat{D} &= -\, x \cdot \partial \ , & \hat{K}_\mu &= i(2x_\mu \, x\cdot\partial - x^2 \partial_\mu) \end{aligned} \tag{2.107}$$

a*) In a unitary representation, there is a positive definite inner product such that*

$$\hat{D}^\dagger = \hat{D} \ , \qquad \hat{K}_\mu^\dagger = \hat{P}_\mu \ , \qquad \hat{L}_{\mu\nu}^\dagger = \hat{L}_{\mu\nu} \ . \tag{2.108}$$

Show that unitarity implies that the dimension (or eigenvalue of $\hat{D}$) of a scalar primary state $|\mathcal{O}\rangle$ can not be lower than $\frac{d-2}{2}$, and that the bound is saturated by a state created by a free massless scalar field (obeying the equation of motion $\partial^2 \mathcal{O}(x) = 0$).

b*) Show that a vector primary state $|\mathcal{O}^\alpha\rangle$ contained in a unitary representation must have dimension larger or equal to $d-1$. Show that when the bound is saturated, the state is created by a conserved current. Recall that for a spin 1 state,*

$$\hat{L}_{\mu\nu}|\mathcal{O}^\alpha\rangle = (M_{\mu\nu})^\alpha{}_\beta |\mathcal{O}^\beta\rangle \ , \qquad (M_{\mu\nu})^\alpha{}_\beta = i(\eta_{\nu\beta}\delta_\mu^\alpha - \eta_{\mu\beta}\delta_\nu^\alpha) \ . \tag{2.109}$$

Hint: Compute the norm of $P_\mu|\mathcal{O}^\mu\rangle$.

c*) Verify that the operator*

$$\hat{C} = \hat{D}^2 - \frac{1}{2}\left(\hat{K}_\mu \hat{P}^\mu + \hat{P}_\mu \hat{K}^\mu\right) + \frac{1}{2}\hat{L}_{\mu\nu}\hat{L}^{\mu\nu} \tag{2.110}$$

[12] More precisely, there can be a constant shift equal to the Casimir energy of the vacuum on S^{d-1}, which is related with the Weyl anomaly. In $d = 2$, this gives the usual energy spectrum $\left(\Delta - \frac{c}{12}\right)\frac{1}{L}$ where c is the central charge and L is the radius of S^1.

is a Casimir of the conformal algebra (i.e. it commutes with all its generators). Determine its value for a scalar and a vector primary state.

d) *Generalize questions* ***b)*** *and* ***c)*** *for symmetric traceless primary states* $|\mathcal{O}^{\alpha_1\ldots\alpha_l}\rangle$*. Recall that for spin* l *states,*

$$\hat{L}_{\mu\nu}|\mathcal{O}^{\alpha_1\ldots\alpha_l}\rangle = \sum_{i=1}^{l}(M_{\mu\nu})^{\alpha_i}_{\ \beta}|\mathcal{O}^{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+1}\ldots\alpha_l}\rangle \ . \tag{2.111}$$

You should find that the dimension of such a state in a unitary theory must be greater or equal to $d-2+l$*.*

2.10 Operator Product Expansion

The Operator Product Expansion (OPE) between two scalar primary operators takes the following form

$$\mathcal{O}_i(x)\mathcal{O}_j(0) = \sum_k C_{ijk}|x|^{\Delta_k-\Delta_i-\Delta_j}\left[\mathcal{O}_k(0) + \underbrace{\beta\, x^\mu\partial_\mu\mathcal{O}_k(0)+\ldots}_{\text{descendants}}\right] \tag{2.112}$$

where β denotes a number determined by conformal symmetry. For simplicity we show only the contribution of a scalar operator $\mathcal{O}_k$. In general, in the OPE of two scalars there are primary operators of all spins.

Exercise 2.10.1 *Compute* β *by using this OPE inside a three-point function.*

The OPE has a finite radius of convergence inside correlation functions. This follows from the state operator map with an appropriate choice of origin for radial quantization.

OPE for tensor operators... leading OPE can always be completed into conformal invariant 3pt-function (very useful for counting OPE coefficients) ... connection with embedding formalism

2.11 Conformal Bootstrap

Using the OPE successively one can reduce any $n-$point function to a sum of one-point functions, which all vanish except for the identity operator. Thus, knowing the operator content of the theory, *i.e.* the scaling dimensions Δ and $SO(d)$ irreps $\mathcal{R}$ of all primary operators, and the OPE coefficients C_{ijk},[13] one can determine all correlation functions of local operators. This set of data is called CFT data because it essentially defines the theory. [14] The CFT data is not arbitrary, it must satisfy several constraints:

[13]For primary operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ transforming in non-trivial irreps of $SO(d)$ there are several OPE coefficients C_{123}. The number of OPE coefficients C_{123} is given by the number of symmetric traceless tensor representations that appear in the tensor product of the 3 irreps of $SO(d)$ associated to $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

[14]However, there are observables besides the vacuum correlation functions of local operators. It is also interesting to study non-local operators (line operators, surface operators, boundary conditions, etc) and correlation functions in spaces with non-trivial topology (for example, correlators at finite temperature).

- **OPE associativity** - Different ways of using the OPE to compute a correlation function must give the same result. This leads to the conformal bootstrap equations described below.
- **Existence of stress-energy tensor** - The stress-energy tensor $T_{\mu\nu}$ is a conserved primary operator (with $\Delta = d$) whose correlation functions obey the conformal Ward identities.
- **Unitarity** - In the Euclidean context this corresponds to reflection positivity and it implies lower bounds on the scaling dimensions. It also implies that one can choose a basis of real operators where all OPE coefficients are real. In the context of statistical physics, there are interesting non-unitary CFTs.

It is sufficient to impose OPE associativity for all four-point functions of the theory. For a four-point function of scalar operators, the bootstrap equation reads

$$\sum_k C_{12k}C_{k34}G^{(12)(34)}_{\Delta_k,l_k}(x_1,\dots,x_4) = \sum_q C_{13q}C_{q24}G^{(13)(24)}_{\Delta_q,l_q}(x_1,\dots,x_4)\,,$$

where $G_{\Delta,l}$ are conformal blocks, which encode the contribution from a primary operator of dimension Δ and spin l and all its descendants.

2.12 Embedding Space Formalism

The conformal group $SO(d+1,1)$ acts naturally on the space of light rays through the origin of $\mathbb{R}^{d+1,1}$,

$$-\left(P^0\right)^2 + \left(P^1\right)^2 + \cdots + \left(P^{d+1}\right)^2 = 0\ . \tag{2.113}$$

A section of this light-cone is a $d-$dimensional manifold where the CFT lives. For example, it is easy to see that the Poincaré section $P^0 + P^{d+1} = 1$ is just $\mathbb{R}^d$. To see this parametrize this section using

$$P^0(x) = \frac{1+x^2}{2}\ , \qquad P^\mu(x) = x^\mu\ , \qquad P^{d+1}(x) = \frac{1-x^2}{2}\ , \tag{2.114}$$

with $\mu = 1,\dots,d$ and $x^\mu \in \mathbb{R}^d$ and compute the induced metric. In fact, any conformally flat manifold can be obtained as a section of the light-cone in the embedding space $\mathbb{R}^{d+1,1}$. Using the parametrization $P^A = \Omega(x)P^A(x)$ with $x^\mu \in \mathbb{R}^d$, one can easily show that the induced metric is simply given by $ds^2 = \Omega^2(x)\delta_{\mu\nu}dx^\mu dx^\nu$. With this is mind, it is natural to extend a primary operator from the physical section to the full light-cone with the following homogeneity property

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta}\mathcal{O}(P)\ , \qquad \lambda \in \mathbb{R}\ . \tag{2.115}$$

This implements the Weyl transformation property (2.73). One can then compute correlation functions directly in the embedding space, where the constraints of conformal symmetry are just homogeneity and $SO(d+1,1)$ Lorentz invariance. Physical correlators are simply obtained by restricting to the section of the light-cone associated with the physical space of interest. This idea goes back to Dirac [20] and has been further develop by many authors [21, 22, 23, 24, 25, 26, 27].

Exercise 2.12.1 *Rederive the form of two and three point functions of scalar primary operators in* $\mathbb{R}^d$ *using the embedding space formalism.*

Vector primary operators can also be extended to the embedding space. In this case, we impose

$$P^A \mathcal{O}_A(P) = 0 , \qquad \mathcal{O}_A(\lambda P) = \lambda^{-\Delta} \mathcal{O}_A(P) , \qquad \lambda \in \mathbb{R} , \tag{2.116}$$

and the physical operator is obtained by projecting the indices to the section,

$$\mathcal{O}_\mu(x) = \frac{\partial P^A}{\partial x^\mu} \mathcal{O}_A(P) \bigg|_{P^A = P^A(x)} . \tag{2.117}$$

Notice that this implies a redundancy: $\mathcal{O}_A(P) \to \mathcal{O}_A(P) + P_A \Lambda(P)$ gives rise to the same physical operator $\mathcal{O}(x)$, for any scalar function $\Lambda(P)$ such that $\Lambda(\lambda P) = \lambda^{-\Delta-1}\Lambda(P)$. This redundancy together with the constraint $P^A \mathcal{O}_A(P) = 0$ remove 2 degrees of freedom of the $(d+2)$-dimensional vector $\mathcal{O}_A$.

Exercise 2.12.2 *Show that the two-point function of vector primary operators is given by*

$$\left\langle \mathcal{O}^A(P_1) \mathcal{O}^B(P_2) \right\rangle = const \frac{\eta^{AB} \left(P_1 \cdot P_2\right) - P_2^A P_1^B}{\left(-2P_1 \cdot P_2\right)^{\Delta+1}} , \tag{2.118}$$

up to redundant terms.

Exercise 2.12.3 *Consider the parametrization* $P^A = \left(P^0, P^\mu, P^{d+1}\right) = (\cosh\tau, \Omega^\mu, -\sinh\tau)$ *of the global section* $\left(P^0\right)^2 - \left(P^{d+1}\right)^2 = 1$*, where* Ω^μ *(*$\mu = 1, \ldots, d$*) parametrizes a unit* $(d-1)-$*dimensional sphere,* $\Omega \cdot \Omega = 1$*. Show that this section has the geometry of a cylinder exactly like the one used for the state-operator map.*

Conformal correlation functions extended to the light-cone of $\mathbb{R}^{1,d+1}$ are annihilated by the generators of $SO(1, d+1)$

$$\sum_{i=1}^{n} J_{AB}^{(i)} \left\langle \mathcal{O}_1(P_1) \ldots \mathcal{O}_n(P_n) \right\rangle = 0 , \tag{2.119}$$

where $J_{AB}^{(i)}$ is the generator

$$J_{AB} = -i \left(P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A} \right) , \tag{2.120}$$

acting on the point P_i. For a given choice of light cone section, some generators will preserve the section and some will not. The first are Killing vectors (isometry generators) and the second are conformal Killing vectors. The commutation relations give the usual Lorentz algebra

$$[J_{AB}, J_{CD}] = i \left(\eta_{AC} J_{BD} + \eta_{BD} J_{AC} - \eta_{BC} J_{AD} - \eta_{AD} J_{BC} \right) . \tag{2.121}$$

Exercise 2.12.4 *Check that the conformal algebra (2.28) follows from (2.121) and*

$$\begin{aligned} D &= -i\, J_{0,d+1} \ , & P_\mu &= J_{\mu 0} - J_{\mu,d+1} \ , \\ M_{\mu\nu} &= J_{\mu\nu} \ , & K_\mu &= J_{\mu 0} + J_{\mu,d+1} . \end{aligned} \tag{2.122}$$

Exercise 2.12.5 *Show that equation (2.119) for $J_{AB} = J_{0,d+1}$ implies time translation invariance on the cylinder*

$$\sum_{i=1}^{n} \frac{\partial}{\partial \tau_i} \langle \mathcal{O}_1(\tau_1, \Omega_1) \dots \mathcal{O}_n(\tau_n, \Omega_n) \rangle = 0 \ , \tag{2.123}$$

and dilatation invariance on $\mathbb{R}^d$

$$\sum_{i=1}^{n} \left(\Delta_i + x_i^\mu \frac{\partial}{\partial x_i^\mu} \right) \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = 0 \ . \tag{2.124}$$

In this case, you will need to use the differential form of the homogeneity property $P^A \frac{\partial}{\partial P^A} \mathcal{O}_i(P) = -\Delta_i \mathcal{O}_i(P)$. It is instructive to do this exercise for the other generators as well.

2.13 Conformal anomalies

In these lectures, we are exploring consequences of exactly realized conformal symmetry in flat space. We also pointed out that Weyl invariant theories are automatically conformal invariant when restricted to flat space. In this section, we discuss the sources of breaking of conformal symmetry, with special attention to the so called *conformal anomalies.* You already know from chapter 1 that conformal symmetry is broken when a relevant coupling is turned on in the action. Correlation functions cease to be conformal invariant and their dependence on the distance is governed by the Callan-Symanzik equations. On the other hand, we know that if a stress-tensor with vanishing trace can be defined,[15] conformal invariance is guaranteed. We conclude that the trace of the stress-tensor is a legitimate non vanishing scalar operator along the renormalization group flow. We can be more specific. Recall eq. (1.73), which we rewrite here while coupling the theory to a background metric:

$$Z[g_{\mu\nu}] = \int [D\phi]\, e^{-S[\phi, g_{\mu\nu}]} \ , \quad S[\phi, g_{\mu\nu}] = S^*[\phi, g_{\mu\nu}] + \sum_i g_i(\mu) \int d^d x \sqrt{g} \mathcal{O}_i(x) \ , \tag{2.125}$$

$$g_i(\mu) = g_i^0 \mu^{d-\Delta_i} \ . \tag{2.126}$$

Here $S^*[\phi, g_{\mu\nu}]$ is assumed to be Weyl invariant. Weyl invariance of the fixed point action means that there is a change of variables in the path integral $\phi \to \phi_\Omega$ such that $S^*[\phi_\Omega, \Omega^2(x) g_{\mu\nu}] = S[\phi, g_{\mu\nu}]$. We also take $\mathcal{O}_i$ to be scalar primary operators at the

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15

fixed point,[16] *i.e.* $\mathcal{O}_i \to \Omega^{-\Delta}\mathcal{O}_i$ when $\phi \to \phi_\Omega$. Finally, we also assume for now that the path-integral measure is invariant under Weyl transformations:[17] $[D\phi_\Omega] = [D\phi]$ – but more on this later! We can now study the variation of the free energy under an infinitesimal Weyl transformation with $\Omega \simeq 1+\sigma$. Let us denote the functional variation as

$$\delta_\sigma \equiv \int d^d x\, \sigma(x)\, \frac{\delta}{\delta\sigma(x)}\bigg|_{\sigma=0} \; . \tag{2.127}$$

Then

$$\begin{aligned}
&\delta_\sigma \log Z[(1+2\sigma)g_{\mu\nu}] \\
&\quad = \frac{1}{Z}\,\delta_\sigma \int [D\phi_{\Omega(\sigma)}]\exp\left\{-S[\phi, g_{\mu\nu}] - \sum_i g_i(\mu)\int d^d y\sqrt{g(y)}\,\sigma(y)(d-\Delta_i)\,\mathcal{O}_i(y)\right\} \\
&\qquad = \frac{1}{Z}\int [D\phi] e^{-S[\phi,g_{\mu\nu}]}\left\{-\sum_i g_i(\mu)(d-\Delta_i)\int d^d y\sqrt{g(y)}\sigma(y)\,\mathcal{O}_i(y)\right\} \; .
\end{aligned} \tag{2.128}$$

On the other hand,

$$\delta_\sigma \log Z[(1+2\sigma)g_{\mu\nu}] = \int d^d x\sqrt{g(x)}\sigma(x) T^\mu_\mu(x) \; . \tag{2.129}$$

Using eq (2.126), we finally obtain

$$\langle T^\mu_\mu(x)\ldots\rangle = -\sum_i \beta_i(g_j)\,\langle \mathcal{O}_i(x)\ldots\rangle \; , \qquad \beta_i(g_j) = \frac{d\, g_i}{d\log\mu} \; , \tag{2.130}$$

In the flat space limit, one can write $T^\mu_\mu = -\partial_\mu j^\mu_D$, where $j_{D\,\mu}$ is the dilatation current. Therefore, eq. (2.130) relates the breaking of scale invariance to the beta functions of the theory. It is valid as an operator equation: we added dots as place-holder of other local operators *inserted at separated points*. This follows from repeating the same chain of equalities with other background currents turned on in the path-integral.

If the divergence of a current is a local operator, like in eq. (2.130), the symmetry is said to be explicitly broken. The symmetry is simply not there. There are two more subtle ways in which the symmetry might be broken. One possibility is that while the current is conserved, the vacuum is not invariant under the symmetry. This phenomenon is called spontaneous symmetry breaking and has enormous importance, but you will see it up close in your QFT courses.

The third way to break a symmetry is again realized when $\partial_\mu j^\mu \neq 0$, but this time on the right hand side there is a function of the background fields rather than a local operator: this is called anomalous breaking. In our case, again the background fields

[16] This is not a restriction: we chose a basis of *scaling variables*, which at this point in the course we can call eigenstates of the dilatation operator. They cannot be descendants, because they would turn into total derivatives in the action.

[17] Notice that the fixed point action could be adsorbed into the path integral measure, so the invariance of the former is not an additional assumption.

contribute to the trace of the stress-tensor, and this is the contribution that we are going to focus on. Let us first summarize:[18]

$$T^\mu_\mu(x) = \begin{cases} -\beta^i(g)\,\mathcal{O}_i(x)\ , & \text{Explicit breaking,} \\ 0, \quad \text{but } x^\mu T_{\mu\nu}\,|0\rangle \neq 0\ , & \text{Spontaneous breaking,} \\ \mathcal{A}(x)\ , & \text{Anomalous breaking,} \end{cases} \tag{2.131}$$

where $\mathcal{A}(x)$ is called the *conformal (or Weyl) anomaly*, and is a function of the metric and its derivatives, which are the only background fields that we are going to consider. One way to understand the conformal anomalies is to note that the assumption made above that the path-integral measure is Weyl invariant was not justified: the anomaly appears when this assumption is wrong. Here we will take a more abstract approach.

Before inquiring whether there are theories for which $\mathcal{A}(x) \neq 0$, let us list some general properties that the anomaly has to satisfy. $T^\mu_\mu(x)$ is a local scalar (*i.e.* diffeomorphism invariant) operator of dimension d. The anomaly is then constrained by the same rules:

1. the anomaly is local,

2. it is a scalar under diffeomorphisms,

3. it has dimension d.

Rule 1 means that $\mathcal{A}(x)$ is a function of the metric and of finitely many derivatives of the metric evaluated at the same point x where T^μ_μ is evaluated.[19] Rule 2 implies that the anomaly is a function of the covariant derivative ∇_μ and of the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, with all indices contracted. Finally, we should only consider functions that have a well defined Taylor expansion when the metric is nearly flat, *i.e.* they can be expanded in powers of $\delta g_{\mu\nu} \equiv g_{\mu\nu} - \delta_{\mu\nu}$. This excludes things like R^α where α is not a positive integer.[20] A first consequence of these rules is that $\mathcal{A}(x)$ is a polynomial built out of contractions of the building blocks $(\nabla_\mu, R^\mu{}_{\nu\rho\sigma})$. Therefore, it vanishes in flat space.

An important additional rule follows from the Abelian nature of the Weyl group. Two consecutive Weyl transformations must commute, which leads to the

4. Wess-Zumino consistency condition:

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] \log Z = \int d^dx \sqrt{g(x)}\,(\sigma_2(x)\delta_{\sigma_1}\mathcal{A}(x) - \sigma_1(x)\delta_{\sigma_2}\mathcal{A}(x)) = 0\ . \tag{2.132}$$

These four rules allow to classify the possible anomaly polynomials for each spacetime dimension. First of all, since the Riemann tensor has dimension 2, and it always takes an even number of derivatives to build a scalar, there are no invariants in odd spacetime

[18] In fact, terminology varies: sometimes it is called anomalous the breaking of a symmetry of the action by quantum corrections. This would include cases in which the β functions are zero at tree level but not at one- (or higher-) loops.

[19] Knowledge of derivatives of the metric up to arbitrarily high order is equivalent to knowledge of the value of the metric at some point other than x.

[20]

dimension: *there is no Weyl anomaly in odd dimensions.*[21] Let us write the most general anomaly polynomial in dimensions 2 and 4:[22]

$$d = 2 \qquad \mathcal{A} = \frac{c}{24\pi} R \ , \tag{2.133}$$

$$d = 4 \qquad \mathcal{A} = \frac{a}{64\pi^2} E_4 + \frac{c}{64\pi^2} W_{\lambda\mu\nu\rho} W^{\lambda\mu\nu\rho} + e_1 R^2 + e_2 \Box R \ . \tag{2.134}$$

In eqs. (2.133) and (2.134), the normalizations are chosen to match the literature. For reasons that we are not going to touch in these notes, the coefficient c in the two-dimensional anomaly (2.133) is called the *central charge* of the CFT. In eq. (2.134), we defined the *Euler density*

$$E_4 = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2 \ , \tag{2.135}$$

while the square of the Weyl tensor reads

$$W_{\lambda\mu\nu\rho} W^{\lambda\mu\nu\rho} = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 2 R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2 \ . \tag{2.136}$$

It is an exercise to check the following:

Exercise 2.13.1 *In $d = 4$, R^2 does not obey the Wess-Zumino consistency condition* i.e. *rule 4 above. Therefore*

$$e_1 = 0 \ . \tag{2.137}$$

The fate of e_2 is similar, for a different reason. Suppose that a local function of the metric – let's call it $\mathcal{B}$ – is added to the action:

$$S[\phi, g] \to S[\phi, g] + \int d^d x \, \sqrt{g(x)} \mathcal{B}(x) \ . \tag{2.138}$$

The only effect of the addition on the correlators at separated points is a shift in the definition of the stress-tensor. The shift is proportional to the c-number $\delta\mathcal{B}/\delta g^{\mu\nu}$, which only gives a disconnected contribution at separated points. The distinction between the theory before and after adding $\mathcal{B}$ is therefore purely conventional, and we consider two theories equivalent if they differ only by counterterms which are local functions of the background fields.

Exercise 2.13.2 *Prove that*

$$\delta_\sigma \int d^4 x \, \sqrt{g} R^2 = -12 \int d^4 x \, \sqrt{g} \, \sigma \Box R \ . \tag{2.139}$$

We say that the anomaly proportional to e_2 in eq. (2.134) is *(cohomologically) trivial*, and we can set

$$e_2 = 0 \ , \tag{2.140}$$

[21]

[22] We actually restrict ourselves to parity invariant anomalies. One more term, the signature invariant, is allowed in $d = 4$ if we give up parity.

by adding a counterterm $\mathcal{B} \propto R^2$ to the action.

The existence of trivial anomalies raises the following question: why are the surviving terms in eqs. (2.133) and (2.134) non trivial? After all, they also are c-numbers, and the trace of the stress-tensor does not contribute to connected correlators at separated points. The answer is that this disconnected effect is *not* the only one: the coefficients a and c appear in connected correlation functions as well. We shall now see this in detail in $d = 2$, which will also allow us to prove that the anomaly in eq. (2.133) is always present if the theory is unitary.

2.13.1 The story in $d = 2$

In two dimesnions, the metric is determined by a single scalar function up to diffeomorphisms. In particular, one can choose the *conformal gauge*:[23]

$$g_{\mu\nu}(x) = e^{2\sigma(x)}\delta_{\mu\nu} \ . \tag{2.141}$$

Since the Weyl anomaly controls the dependence of the path-integral on $\sigma(x)$, it completely determines the dependence of the theory on the metric. In order to compute this explicitly, let us take a derivative wrt σ, but notice that here σ is not infinitesimal:

$$\frac{\delta}{\delta\sigma(x)} \log Z\left[e^{2\sigma}\delta_{\mu\nu}\right] = \frac{c}{24\pi}\sqrt{g}R \ . \tag{2.142}$$

Here we used eq. (2.133), and $g_{\mu\nu}$ on the right hand side is still (2.141). In conformal gauge,

$$R = -2e^{-2\sigma(x)}\delta^{\mu\nu}\partial_\mu\partial_\nu\sigma(x) \ . \tag{2.143}$$

Eqs. (2.142) and (2.143) together give a differential equation which is easy to integrate:

$$Z\left[e^{2\sigma}\delta_{\mu\nu}\right] = Z[\delta_{\mu\nu}] \exp\left\{-\frac{c}{24\pi}\int d^2x\, \sigma(x)\delta^{\mu\nu}\partial_\mu\partial_\nu\sigma(x)\right\} \ . \tag{2.144}$$

In order to recast this equation in a covariant form, we rewrite eq. (2.143) as

$$R = -2\Box\sigma \ , \tag{2.145}$$

and use a Green function to invert for σ:

$$\sigma(x) = \frac{1}{2}\int d^2x'\,\sqrt{g(x')}G(x,x')R(x') \ , \quad \Box G(x,x') = -\frac{1}{\sqrt{g}}\delta^2(x-x') \ . \tag{2.146}$$

Putting all together, we find the *Polyakov effective action*:

$$\begin{aligned} &Z[g_{\mu\nu}] = Z[\delta_{\mu\nu}]e^{S_P} \ , \\ &\qquad S_P = \frac{c}{96\pi}\int d^2x\sqrt{g(x)}\int d^2x'\,\sqrt{g(x')}R(x)G(x,x')R(x') \ . \end{aligned} \tag{2.147}$$

[23] One can show that it is possible to bring every two-dimensional metric in this form by a change of coordinates, see *e.g.* Polchinski.

Sometimes the Polyakov action is written $S_P = -c/96\pi \int d^2x\sqrt{g}R\frac{1}{\Box}R$. Notice that the Polyakov action is non local, which means that the anomaly (2.133) is non-trivial. As we explore in the next exercise, eq. (2.147) implies that all n-point functions of the stress-tensor are completely fixed in terms of the anomaly coefficient c.

Exercise 2.13.3 ***Correlation functions of the stress-tensor from the Polyakov effective action***

Use the Polyakov action to compute the two-point function of the stress-tensor in flat space. Hints: *It is convenient to use complex coordinates:*

$$z = x + iy, \quad \bar{z} = x - iy. \tag{2.148}$$

Verify that the only non zero component of the metric in flat space is $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. *In complex coordinates therefore* $T_{z\bar{z}}$ *is proportional to the trace. So you only need to compute correlators of* T_{zz} *and* $T_{\bar{z}\bar{z}}$. *To get the Green function, use the following formula (can you justify it?):*

$$\partial_z \frac{1}{\bar{z}} = \pi\delta(x)\delta(y). \tag{2.149}$$

Here is one of the components of the two-point function you want to get:

$$\langle T_{zz}(z)T_{zz}(z')\rangle = \frac{c/(8\pi^2)}{(z-z')^4}. \tag{2.150}$$

(In much of the 2d CFT literature, the stress tensor is renormalized: $T(z) = -2\pi T_{zz}(z)$. *Also notice that* $c = 1$ *for a free scalar in 2 dimensions).*

Eq. (2.150) has an immediate consequence: reflection positivity implies that $c \geq 0$, and if $c = 0$ the stress-tensor itself vanishes as an operator. Therefore, *every unitary local conformal field theory has a Weyl anomaly.*

A second important consequence of eq. (2.147) is that the stress-tensor does not transform homogeneously under Weyl transformations. At the same time, the stress-tensor does transform as a primary under globally defined conformal transformations in flat space. Let us see how this happens.

Exercise 2.13.4 ***The Polyakov effective action and the Schwartzian derivative***

a*. Compute the one-point function of the stress-tensor in a non trivial background metric* $g_{\mu\nu}$ *from the Polyakov effective action. Without loss of generality, you can choose the conformal gauge* $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$, *and consider only the variation that gives you the* T_{zz} *component. The result is*

$$\langle T_{zz}\rangle = \frac{c}{12\pi}\left(\partial_z^2\sigma - (\partial_z\sigma)^2\right) \tag{2.151}$$

b*. Verify that in 2 dimensions, the conformal Killing vectors are generated by all holomorphic changes of coordinates:* $w = f(z)$, $\bar{w} = \bar{f}(\bar{z})$. *Under a conformal transformation, the stress-tensor changes like a primary, up to a inhomogeneous piece due to the anomaly:*

$$T'_{ww}(w) = \left(\frac{dz}{dw}\right)^2\left(T_{zz} + \frac{c}{24\pi}\{w, z\}\right). \tag{2.152}$$

$\{w, z\}$ *is called the Schwartzian derivative and reads*

$$\{w, z\} = \frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2 . \tag{2.153}$$

Derive eq. (2.153) *using the result of point* ***a****.* Hint *Choose a Weyl transformation that compensates for the diffeomorphism* $w = f(z)$, $\bar{w} = \bar{f}(\bar{z})$.

c*. Not all of the holomorphic maps are globally invertible: if we ask for invertibility, we go back to the usual conformal group. Show that the conformal transformations can be cast in the form:*

$$f(z) = \frac{az+b}{cz+d}, \quad \text{with } ad - bc = 1. \tag{2.154}$$

This realizes the isomorphism $SL(2,\mathbb{C}) \simeq SO(3,1)$*. Finally, show that the Schwartzian derivative vanishes for these globally invertible conformal transformations.*

The last computation in this section is an easy and interesting consequence of eqs. (2.152) and (2.153). Consider a cylinder parametrized by an infinite coordinate σ_1 and a periodic one $\sigma_2 \in [0, \beta)$, and define the complex coordinate $w = \sigma_1 + \mathrm{i}\sigma_2$. Then the holomorphic map

$$z = e^{2\pi w/\beta} \tag{2.155}$$

maps the cylinder into the complex z-plane minus the origin and the point at infinity. On the plane, $\langle T_{zz} \rangle = 0$, and so the one-point function of the stress-tensor on the cylinder is determined by the Schwartzian derivative (2.153):

$$\langle T_{ww} \rangle = \langle T_{\bar{w}\bar{w}} \rangle = \frac{c\,\pi}{12\beta^2} , \qquad \text{on the cylinder.} \tag{2.156}$$

Going back to the coordinates σ_1 and σ_2, we get

$$\langle T^{11} \rangle = -\langle T^{22} \rangle = \frac{c\,\pi}{6\beta^2} . \tag{2.157}$$

There are two ways of interpreting this result. First, take σ_1 to be the time coordinates, and do the Wick rotation accordingly. Then,

$$-T^{11} = T^{tt} = \mathcal{E} = -\frac{c\,\pi}{6\beta^2} , \qquad \text{Casimir energy density on the circle.} \tag{2.158}$$

The Casimir energy density $\mathcal{E}$ of a $2d$ CFT on a circle of length β is negative and proportional to c. On the other hand, if we choose σ_2 as time, we get a CFT at finite temperature $T = 1/\beta$. Eq. (2.157) then becomes the equation of state of the CFT, which thus turns out to be also determined by the anomaly:

$$-T^{22} = T^{tt} = \mathcal{E} = \frac{c\,\pi T^2}{6} \qquad \text{Equation of state.} \tag{2.159}$$

2.14 Large N Factorization

Consider a $U(N)$ gauge theory with fields valued in the adjoint representation. Schematically, we can write the action as

$$S = \frac{N}{\lambda}\int dx \; \mathrm{Tr}\left[(D\Phi)^2 + c_3\Phi^3 + c_4\Phi^4 + \dots\right] \tag{2.160}$$

where we introduced the 't Hooft coupling $\lambda = g_{YM}^2 N$ and c_i are other coupling constants independent of N. Following 't Hooft [28], we consider the limit of large N with λ kept fixed. The propagator of an adjoint field obeys

$$\left\langle \Phi^i_j \Phi^k_l \right\rangle \propto \frac{\lambda}{N}\delta^i_l \delta^k_j \tag{2.161}$$

where we used the fact that the adjoint representation can be represented as the direct product of the fundamental and the anti-fundamental representation. This suggests that one can represent a propagator by a double line, where each line denotes the flow of a fundamental index. Start by considering the vacuum diagrams in this language. A diagram with V vertices, E propagators (or edges) and F lines (or faces) scales as

$$\left(\frac{N}{\lambda}\right)^V \left(\frac{\lambda}{N}\right)^E N^F = \left(\frac{N}{\lambda}\right)^\chi \lambda^F \,, \tag{2.162}$$

where $\chi = V + F - E = 2 - 2g$ is the minimal Euler character of the two dimensional surface where the double line diagram can be embedded and g is the number of handles of this surface. Therefore, the large N limit is dominated by diagrams that can be drawn on a sphere ($g = 0$). These diagrams are called planar diagrams. For a given topology, there is an infinite number of diagrams that contribute with increasing powers of the coupling λ, corresponding to tesselating the surface with more and more faces. Figure 2.1 shows two examples of vacuum diagrams in the double line notation. This topological expansion has the structure of string perturbation theory with λ/N playing the role of the string coupling. As we shall see this is precisely realized in maximally supersymmetric Yang-Mills theory (SYM).

Figure 2.1 Vacuum diagrams in the double line notation. Interaction vertices are marked with a small blue dot. The left diagram is planar while the diagram on the right has the topology of a torus (genus 1 surface).

Let us now consider single-trace local operators of the form $\mathcal{O} = c_J \mathrm{Tr}\left(\Phi^J\right)$, where c_J is a normalization constant independent of N. Adapting the argument above, it is easy to conclude that the connected correlators are given by a large N expansion of the form

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_c = \sum_{g=0}^{\infty} N^{2-n-2g} f_g(\lambda) \,, \tag{2.163}$$

which is dominated by the planars diagrams ($g = 0$). Moreover, we see that the planar two-point function is independent of N while connected higher point functions are suppressed by powers of N. This is large N factorization. In particular it implies that the two-point function of a multi-trace operator $\tilde{\mathcal{O}}(x) =: \mathcal{O}_1(x) \dots \mathcal{O}_k(x) :$ is dominated by the product of the two-point functions of its single-trace constituents

$$\left\langle \tilde{\mathcal{O}}(x)\tilde{\mathcal{O}}(y) \right\rangle \approx \prod_i \langle \mathcal{O}_i(x)\mathcal{O}_i(y) \rangle = \frac{1}{(x-y)^{2\sum_i \Delta_i}} \,, \tag{2.164}$$

where we assumed that the single-trace operators were scalar conformal primaries properly normalized. We conclude that the scaling dimension of the multi-trace operator $\tilde{\mathcal{O}}$ is given by $\sum_i \Delta_i + O(1/N^2)$. In other words, the space of local operators in a large N CFT has the structure of a Fock space with single-trace operators playing the role of single particle states of a weakly coupled theory. This is the form of large N factorization relevant for AdS/CFT. However, notice that conformal invariance was not important for the argument. It is well known that large N factorization also occurs in confining gauge theories. Physically, it means that colour singlets (like glueballs or mesons) interact weakly in large N gauge theories (see [29] for a clear summary).

The stress tensor has a natural normalization that follows from the action, $T_{\mu\nu} \sim \frac{N}{\lambda}\mathrm{Tr}\left(\partial_\mu \Phi \partial_\nu \Phi\right)$. This leads to the large N scaling

$$\langle T_{\mu_1\nu_1}(x_1) \dots T_{\mu_n\nu_n}(x_n) \rangle_c \sim N^2 \,, \tag{2.165}$$

which will be important below. This normalization of $T_{\mu\nu}$ is also fixed by the Ward identities.

2.15 Problems

Exercise 2.15.1 ***Conformal transformations and airplane wings***

Conformal transformations are useful in many physics problems. Consider the case of irrotational flow (no vorticity) of an incompressible fluid.

a) *Show that we can write the velocity field* $\boldsymbol{v} = \nabla\phi$ *where the velocity potential* ϕ *obeys the Laplace equation* $\nabla^2\phi = 0$ *with Newman boundary conditions* $\boldsymbol{n} \cdot \nabla\phi = 0$ *at the surface of any obstacle.*

b) *Consider the case where the flow is two dimensional (more precisely, the flow is translational invariant in the third direction). In this case, show that the velocity potential can be written as*

$$\phi(x,y) = Re\, \Phi(x+iy) \,, \tag{2.166}$$

where $\Phi(z)$ is holomorphic for $z \in \Sigma$ the domain of the flow. Moreover, show that we can choose $Im\, \Phi(z) = 0$ for $z \in \partial\Sigma$. [24]

c) *The pressure p can be obtained from the Navier-Stokes equation*

$$\rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p\,, \tag{2.167}$$

where ρ is the fluid density. Show that for an incompressible and irrotational flow this leads to Bernoulli's equation

$$p + \frac{1}{2}\rho \boldsymbol{v}^2 = const\,. \tag{2.168}$$

Combine this with the previous result to show that the total force on an obstacle with boundary $\partial\Sigma$ can be written as

$$F \equiv F_x + iF_y = -\frac{i}{2}\rho \oint_{\partial\Sigma} dz |\Phi'(z)|^2\,. \tag{2.169}$$

d) *Consider the case of a flow past a cylinder of radius R. Show that*

$$\Phi_{cyl}(z) = u\left(z + \frac{R^2}{z}\right)\,, \tag{2.170}$$

where $u\boldsymbol{e}_x$ is the fluid velocity infinitely far away from the cylinder. What is the force exerted on the cylinder.

e) *Consider an holomorphic map $z \to w = f(z)$ that maps the region $\Sigma = \{z \in \mathbb{C} : |z| > R\}$ to the a region $\tilde{\Sigma}$ corresponding to the flow region outside an obstacle with shape given by the curve $f(Re^{i\theta})$ with $\theta \in [0, 2\pi]$. Show that the velocity potential for this new obstacle is given by*

$$\Phi(w) = \Phi_{cyl}(f^{-1}(w))\,. \tag{2.171}$$

Use this idea to compute the flow past an airplane wing described by the Zhukovsky map

$$w = \zeta + \frac{1}{\zeta}\,, \qquad \zeta = z + \frac{i-1}{2}\left(\sqrt{2R^2 - 1} - 1\right)\,. \tag{2.172}$$

Plot the shape of the wing for $R = 1.05$ and determine the force exerted by the fluid on the wing.

f) Challenge: *What is the optimal shape Σ of the cross section of an airplane wing so that it maximizes the lifting force for a fixed length of the boundary $\partial\Sigma$.*

Exercise 2.15.2 ***Scale + Unitarity ⇒ Conformal***

Show that a scale invariant, unitary two-dimensional field theory is conformal invariant.

The original derivation is due to Polchinski, following the work of Zamolodchikov. The more recent paper Arxiv:0910.1087 gives a very clear review of the argument in section 2.1.

[24] If the boundary $\partial\Sigma$ has several disconnected components then $\mathrm{I}m\,\Phi$ will be constant on each component but not necessarily zero.

Exercise 2.15.3 ***Operator Product Expansion - scalar case***

The general form of the OPE of two scalar operators is

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+l}} \left[F^{(12k)}_{a_1\dots a_l}(x,\partial_y)\, \mathcal{O}_k^{a_1\dots a_l}(y) \right]_{y=0} \tag{2.173}$$

where the sum runs over all primary operators $\mathcal{O}_k$ with spin l and dimension Δ.

***a**. Show that scale invariance implies that*

$$F^{(12k)}_{a_1\dots a_l}\left(\lambda x, \lambda^{-1}\partial_y\right) = \lambda^l F^{(12k)}_{a_1\dots a_l}(x,\partial_y) \tag{2.174}$$

***b**. Compute the three-point function of scalar primary operators,*

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_3(w)\rangle = \frac{C_{123}}{|x|^{\Delta_1+\Delta_2-\Delta_3}|w|^{\Delta_3+\Delta_2-\Delta_1}|x-w|^{\Delta_1+\Delta_3-\Delta_2}}\ , \tag{2.175}$$

using the OPE above, and derive

$$\left[F^{(123)}(x,\partial_y)\left(1+\frac{y^2-2y\cdot w}{w^2}\right)^{-\Delta_3}\right]_{y=0} = \left(1+\frac{x^2-2x\cdot w}{w^2}\right)^{\frac{\Delta_2-\Delta_1-\Delta_3}{2}}\ . \tag{2.176}$$

***c**.* Write a* `Mathematica` *program that uses the last equation to compute the coefficients $a_{n,m}$ for $n+2m\le 10$ in the derivative expansion*

$$F^{(123)}(x,\partial_y) = \sum_{n,m=0}^{\infty} a_{n,m}(x\cdot\partial_y)^n(x^2\partial_y^2)^m \tag{2.177}$$

Suggestion: choose $w^2=1$ in equation (2.176).

***d**.* Make a table of your results and try to guess an analytic formula for $a_{n,m}$. The function*

$$\texttt{Pochhammer[t,k]} = (t)_k = \frac{\Gamma(t+k)}{\Gamma(t)} = t(t+1)\dots(t+k-1) \tag{2.178}$$

will be very useful.

Exercise 2.15.4 ***Operator Product Expansion - vector case***

***a**.* In order to study the OPE terms that involve operators with non-zero spin it is convenient to introduce a polarization vector ϵ_a. The idea is that we can encode a symmetric traceless tensor in a harmonic polynomial. If we define*

$$\mathcal{O}(x,\epsilon) = \epsilon^{a_1}\dots\epsilon^{a_l}\mathcal{O}_{a_1\dots a_l}(x) \tag{2.179}$$

we can recover the tensor from the polynomial using

$$\mathcal{O}_{a_1\dots a_l}(x) = \frac{1}{l!(h-1)_l} D_{a_1}\dots D_{a_l}\mathcal{O}(x,\epsilon) \tag{2.180}$$

where $2h$ *is the dimension of (Euclidean) spacetime and*

$$D_a = \left(h - 1 + \epsilon \cdot \frac{\partial}{\partial \epsilon}\right) \frac{\partial}{\partial \epsilon^a} - \frac{1}{2}\epsilon_a \frac{\partial^2}{\partial \epsilon \cdot \partial \epsilon} \ . \tag{2.181}$$

Show (possibly using `Mathematica`*) that*

$$[D_a, D_b] = 0 \ , \qquad D^2 \propto \epsilon^2 \ , \qquad D_a \, \epsilon^2 = \epsilon^2 \left(D_a + 2\frac{\partial}{\partial \epsilon^a}\right) \ . \tag{2.182}$$

These properties guarantee that the tensor (2.180) is symmetric and traceless and that we can set $\epsilon^2 = 0$ *in* $\mathcal{O}(x, \epsilon)$ *(because* D_a *is an interior operator to this constraint).*

Check that, for unit vectors x *and* y*, we have*

$$(x \cdot D)^l (\epsilon \cdot y)^l = 2^{-l} (l!)^2 C_l^{h-1}(x \cdot y) \tag{2.183}$$

where $C_l^{h-1}(t) =$ `GegenbauerC[l,h-1,t]` *is the Gegenbauer polynomial.*

b*. In this formalism, the OPE can be written as*

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_k \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+l}} \left[F^{(12k)}\left(x, \partial_y, D\right) \mathcal{O}_k(y, \epsilon)\right]_{y=0} \tag{2.184}$$

where

$$F^{(12k)}\left(\lambda x, \lambda^{-1}\partial_y, \alpha D\right) = (\alpha)^l F^{(12k)}\left(x, \partial_y, D\right) \ . \tag{2.185}$$

Compute the three-point function of two scalar primary operators with a spin l *operator,*

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\mathcal{O}_k(w, \epsilon')\rangle = C_{12k} \frac{\left(\epsilon' \cdot w \, (x - w)^2 - \epsilon' \cdot (w - x) \, w^2\right)^l}{|x|^{\Delta_1+\Delta_2-\Delta+l} |w|^{\Delta+\Delta_2-\Delta_1+l} |x - w|^{\Delta_1+\Delta-\Delta_2+l}} \ , \tag{2.186}$$

using the OPE (2.184) and the two-point function

$$\langle \mathcal{O}_k(y, \epsilon)\mathcal{O}_k(w, \epsilon')\rangle = \frac{\left(\epsilon \cdot \epsilon' \, (y - w)^2 - 2\epsilon \cdot (y - w) \, \epsilon' \cdot (y - w)\right)^l}{(y - w)^{2(\Delta+l)}} \tag{2.187}$$

and derive

$$\left[F^{(12k)}\left(x, \partial_y, D\right) \frac{\left(\epsilon \cdot \epsilon' - 2\frac{\epsilon\cdot(y-w) \, \epsilon'\cdot(y-w)}{1-2y\cdot w+y^2}\right)^l}{(1 - 2y \cdot w + y^2)^{\Delta}}\right]_{y=0} = \frac{\left(\epsilon' \cdot x + \epsilon' \cdot w \, (x^2 - 2w \cdot x)\right)^l}{(1 - 2x \cdot w + x^2)^{\frac{\Delta_1+\Delta-\Delta_2+l}{2}}} \tag{2.188}$$

where we have chosen $w^2 = 1$.

c*.** Write a Mathematica program that uses the last equation to compute the coefficients* $a_{n,m}$ *and* $b_{n,m}$ *for* $n + 2m \leq 4$ *in the derivative expansion of the spin 1 case,*

$$F^{(12k)}\left(x, \partial_y, D\right) = \sum_{n,m=0}^{\infty} \left[a_{n,m} \, x \cdot D + b_{n,m} \, x^2 \, \partial_y \cdot D\right] (x \cdot \partial_y)^n (x^2 \partial_y^2)^m \ . \tag{2.189}$$

d. *You can also study the case of general spin using the expansion*

$$F^{(12k)}(x,\partial_y,D)=\sum_{n,m=0}^{\infty}\sum_{q=0}^{l}a_{n,m,q}\,(x\cdot D)^{l-q}(x^2\,\partial_y\cdot D)^q(x\cdot\partial_y)^n(x^2\partial_y^2)^m\ . \tag{2.190}$$

Show that the leading term in the OPE gives

$$a_{0,0,0}=\frac{1}{l!(h-1)_l}\ . \tag{2.191}$$

or equivalently

$$\mathcal{O}_1(x)\mathcal{O}_2(0)=\sum_k\frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta+l}}\left[x_{a_1}\dots x_{a_l}\mathcal{O}_k^{a_1\dots a_l}(0)+\dots\right] \tag{2.192}$$

Exercise 2.15.5 ***Conformal Blocks from OPE***

The four-point function of scalar primary operators can be expanded using the OPE (2.173). This leads to the conformal block decomposition

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle=\sum_k C_{12k}C_{k34}\,G^{(12)(34)}_{\Delta_k,l_k}(x_1,\dots,x_4) \tag{2.193}$$

where

$$G^{(12)(34)}_{\Delta,l}(x_1,\dots,x_4)=\frac{F^{(12k)}(x_{12},\partial_{x_2},D)\,\langle\mathcal{O}_k(x_2,\epsilon)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle}{|x_{12}|^{\Delta_1+\Delta_2-\Delta+l}C_{k34}} \tag{2.194}$$

$$=\frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}}\left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_{12}}\left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_{34}}g_{\Delta,l}(u,v) \tag{2.195}$$

Here, $\Delta_{ij}=\Delta_i-\Delta_j$ *and* u,v *are conformal invariant cross ratios*

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\ ,\qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}\ . \tag{2.196}$$

a. Use the expansion (2.177) of the scalar OPE with*

$$a_{n,m}=\frac{(-1)^m\left(\frac{\Delta_k+\Delta_{12}}{2}\right)_m\left(\frac{\Delta_k-\Delta_{12}}{2}\right)_{m+n}}{4^m m!n!\left(\Delta_k+1-\frac{d}{2}\right)_m(\Delta_k)_{2m+n}}\ , \tag{2.197}$$

to compute the first terms of the double series expansion of the scalar conformal block

$$g_{\Delta,0}(u,v)=u^{\frac{\Delta}{2}}\sum_{p,q=0}^{\infty}b_{p,q}u^p(1-v)^q\ . \tag{2.198}$$

Suggestion: choose $x_4\to\infty$ *and* $x_{13}^2=1$ *to show that*

$$g_{\Delta,0}(x_{12}^2,1-2x_{12}\cdot x_{13}+x_{12}^2)=|x_{12}|^{\Delta}F^{(12k)}(x_{12},\partial_{x_2})\,|x_{23}|^{-\Delta-\Delta_{34}} \tag{2.199}$$

and

$$\sum_{p,q=0}^{\infty} b_{p,q} x^{2p} (2x \cdot w - x^2)^q = \left[F^{(12k)}\left(x, \partial_y\right) |y|^{-\Delta-\Delta_{34}} \right]_{y=w-x} \tag{2.200}$$

where we have written $x_{12} = x$ *and* $x_{13} = w$. *Then, expand at small* x *to determine the coefficients* $b_{p,q}$ *for* $q + 2p \leq 6$. *Can you guess the general formula?*

b*. In the non-zero spin case, choose* $x_4 \to \infty$ *and* $x_{13}^2 = 1$ *to show that*

$$g_{\Delta,l}(x_{12}^2, 1 - 2x_{12} \cdot x_{13} + x_{12}^2) = |x_{12}|^{\Delta-l} F^{(12k)}\left(x_{12}, \partial_{x_2}, D\right) \frac{(\epsilon \cdot x_{23})^l}{x_{23}^{\Delta+\Delta_{34}+l}} \tag{2.201}$$

c*. It is convenient to parametrize the cross ratios by*

$$u = z\bar{z} \ , \qquad v = (1-z)(1-\bar{z}) \ , \tag{2.202}$$

where z *and* $\bar{z}$ *are independent variables. Show that for the choice* $x_4 \to \infty$ *and* $x_{13}^2 = 1$ *in Euclidean space, we have* $z = |z|e^{i\theta}$ *and* $\bar{z} = |z|e^{-i\theta}$ *with* $|z|^2 = x_{12}^2$ *and* θ *the angle between the vectors* x_{12} *and* x_{13}.

d*. Use the leading order term in the OPE*

$$F^{(12k)}\left(x, \partial_y, D\right) = \frac{1}{l!(h-1)_l}(x \cdot D)^l + \dots \tag{2.203}$$

to derive the small $|z|$ *behaviour of the conformal block*

$$g_{\Delta,l} \approx \frac{|x_{12}|^{\Delta-l}}{l!(h-1)_l}(x_{12} \cdot D)^l (\epsilon \cdot x_{13})^l = \frac{l!}{2^l (h-1)_l} |z|^{\Delta} C_l^{h-1}(\cos\theta) \tag{2.204}$$

where $C_l^{h-1}(\cos\theta)$ *is the Gegenbauer polynomial. Notice that this limit is particularly simple in two and four dimensions*

$$g_{\Delta,l} \approx \frac{1}{2^l}|z|^{\Delta} \frac{e^{il\theta} + e^{-il\theta}}{1 + \delta_{l,0}} \ , \qquad d = 2 \ , \tag{2.205}$$

$$g_{\Delta,l} \approx \frac{1}{2^l}|z|^{\Delta} \frac{e^{i(l+1)\theta} - e^{-i(l+1)\theta}}{e^{i\theta} - e^{-i\theta}} \ , \qquad d = 4 \ . \tag{2.206}$$

Note that the result in $d = 2$ *is defined as the limit* $d \to 2$ *of the expression in general dimension.*

Exercise 2.15.6 ***Conformal Blocks from Casimir differential equation***

In the embedding formalism, each primary operator is promoted to an homogeneous field on the future light-cone of the origin of $\mathbb{M}^{d+2}$,

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta} \mathcal{O}(P) \ , \qquad P^2 = 0, \quad \lambda > 0 \ . \tag{2.207}$$

In this formalism, conformal transformations are just $SO(d+1,1)$ *Lorentz transformations of Minkowski space* $\mathbb{M}^{d+2}$. *The conformal block decomposition can then be written as*

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\mathcal{O}_3(P_3)\mathcal{O}_4(P_4)\rangle = \sum_k C_{12k} C_{k34}\, G^{(12)(34)}_{\Delta_k, l_k}(P_1, \dots, P_4) \tag{2.208}$$

where

$$G^{(12)(34)}_{\Delta,l}(P_1,\dots,P_4)=\frac{1}{P_{12}^{(\Delta_1+\Delta_2)/2}P_{34}^{(\Delta_3+\Delta_4)/2}}\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}}\left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_{34}}{2}}g_{\Delta,l}(u,v)\ , \tag{2.209}$$

$P_{ij}=-2P_i\cdot P_j$ *and* u,v *are conformal invariant cross ratios*

$$u=\frac{P_{12}P_{34}}{P_{13}P_{24}}\ ,\qquad v=\frac{P_{14}P_{23}}{P_{13}P_{24}}\ . \tag{2.210}$$

The conformal blocks are eigenfunctions of the conformal Casimir,

$$\frac{1}{2}(J_{1,AB}+J_{2,AB})(J_1^{AB}+J_2^{AB})G^{(12)(34)}_{\Delta,l}(P_1,\dots,P_4)=\mathcal{C}_{\Delta,l}\,G^{(12)(34)}_{\Delta,l}(P_1,\dots,P_4)\ , \tag{2.211}$$

with eigenvalue $\mathcal{C}_{\Delta,l}=\Delta(\Delta-d)+l(l+d-2)$*, where*

$$J_{AB}=i\left(P_A\frac{\partial}{\partial P^B}-P_B\frac{\partial}{\partial P^A}\right) \tag{2.212}$$

are the Lorentz generators in $\mathbb{M}^{d+2}$ *with indices* $A,B=0,1,\dots,d+1$.

a. Show (using* `Mathematica`*) that (2.211) together with (2.209) is equivalent to*

$$\mathcal{D}\,g_{\Delta,l}(u,v)=\frac{1}{2}\mathcal{C}_{\Delta,l}\,g_{\Delta,l}(u,v) \tag{2.213}$$

where

$$\mathcal{D}=(1-u-v)\frac{\partial}{\partial v}\left(v\frac{\partial}{\partial v}+a+b\right)+u\frac{\partial}{\partial u}\left(2u\frac{\partial}{\partial u}-d\right) \tag{2.214}$$

$$-(1+u-v)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+a\right)\left(u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+b\right) \tag{2.215}$$

and $a=(\Delta_2-\Delta_1)/2$ *and* $b=(\Delta_3-\Delta_4)/2$.

***b**. Transform to the coordinates* z *and* $\bar z$ *defined in (2.202) and obtain*

$$\mathcal{D}=\mathcal{D}_z+\mathcal{D}_{\bar z}+(d-2)\frac{z\bar z}{z-\bar z}\left((1-z)\frac{\partial}{\partial z}-(1-\bar z)\frac{\partial}{\partial \bar z}\right) \tag{2.216}$$

with

$$\mathcal{D}_z=z^2(1-z)\frac{\partial^2}{\partial z^2}-(a+b+1)z^2\frac{\partial}{\partial z}-abz\ . \tag{2.217}$$

***c**. In two dimensions, the partial differential equation separates in two ordinary differential equations. Show that*

$$g_{\Delta,l}=\frac{k_{\Delta+l}(z)k_{\Delta-l}(\bar z)+k_{\Delta+l}(\bar z)k_{\Delta-l}(z)}{2^l(1+\delta_{l,0})} \tag{2.218}$$

satisfies the boundary condition (2.205) if $k_\beta(z) \approx z^{\beta/2}$ *for small* z*, and the Casimir differential equation if*

$$\mathcal{D}_z k_\beta(z) = \frac{\beta}{2}\left(\frac{\beta}{2}+1\right) k_\beta(z) \ . \tag{2.219}$$

Conclude that

$$k_\beta(z) = z^{\beta/2} \ {}_2F_1\left(\frac{\beta}{2}+a, \frac{\beta}{2}+b, \beta, z\right) \ . \tag{2.220}$$

d. *Check that*

$$g_{\Delta,l} = \frac{z\bar{z}}{2^l(z-\bar{z})}\big(k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta+l}(\bar{z})k_{\Delta-l-2}(z)\big) \tag{2.221}$$

satisfies both the differential equation and the boundary condition in $d = 4$*.*

Exercise 2.15.7 ***Stress tensor three-point function***

The goal of this exercise is to determine how many independent tensor structures are available for the three point function of the stress-energy tensor in a conformal field theory. You should use the embedding formalism to encode the operator $T^{ab}(x)$ *in a field* $T(P,Z)$ *obeying* $T(\lambda P, \alpha Z + \beta P) = \lambda^d \alpha^2 T(P,Z)$*. Then, the general solution compatible with conformal symmetry (and permutation symmetry and parity invariance) is*

$$\langle T(P_1,Z_1)T(P_2,Z_2)T(P_3,Z_3)\rangle = \frac{a_1 G_{000} + a_2 G_{100} + a_3 G_{110} + a_4 G_{200} + a_5 G_{111}}{(P_{12}P_{13}P_{23})^{\frac{d+2}{2}}} \tag{2.222}$$

where $P_{ij} = -2P_i \cdot P_j$ *and*

$$G_{000} = V_1^2 V_2^2 V_3^2 \tag{2.223}$$
$$G_{100} = H_{12}V_1V_2V_3^2 + \textit{permutations} \tag{2.224}$$
$$G_{110} = H_{12}H_{13}V_2V_3 + \textit{permutations} \tag{2.225}$$
$$G_{200} = H_{12}^2 V_3^2 + \textit{permutations} \tag{2.226}$$
$$G_{111} = H_{12}H_{13}H_{23} \tag{2.227}$$

with $V_1 = V_{1,23}$*,* $V_2 = V_{2,31}$ *and* $V_3 = V_{3,12}$*. The coefficients* a_k *are further constrained by requiring conservation of the stress-energy tensor. This corresponds to the condition*

$$\left[\left(h - 1 + Z\cdot\frac{\partial}{\partial Z}\right)\frac{\partial}{\partial Z}\cdot\frac{\partial}{\partial P} - \frac{1}{2}Z\cdot\frac{\partial}{\partial P}\frac{\partial^2}{\partial Z\cdot\partial Z}\right]\langle T(P,Z)T(P_2,Z_2)T(P_3,Z_3)\rangle = 0 \tag{2.228}$$

a. *Implement this condition in Mathematica and show that it is equivalent to*

$$0 = a_1 - h(h+3)a_3 + 2h(h+5)a_4 - 4(h^2-1)a_5 \tag{2.229}$$
$$0 = a_2 - (h+1)a_3 + 4ha_4 - 2(h^2-1)a_5 \tag{2.230}$$

Tips

1. *Define a scalar product to represent* $Z_i \cdot Z_j$*,* $Z_i \cdot P_j$ *and* $P_i \cdot P_j$*. You can use the function* `CenterDot` *and give it some useful properties like* $Z_i \cdot Z_i = 0$*.*

2. *Define a derivative operator* $\frac{\partial}{\partial M^A}$ *with respect to a vector with an open index* (M^A *could be* Z_i^A *or* P_i^A). *The basic rules that you need to give are* $\frac{\partial}{\partial M^A} M_B = \eta_{AB}$ *and* $\frac{\partial}{\partial M^A} M \cdot Q = Q_A$.

3. *Implement rules for index contraction.*

4. *After acting on the ansatz (2.222) with the differential operator that takes the divergence, as in (2.228), you will need to identify the independent building blocks in the result. produced by the action of the operator in (2.228). After performing all index contractions, you should be able to rewrite all* $Z_i \cdot Z_j$ *in terms of* H_{ij} *and all* $Z_i \cdot P_j$ *in terms of* V_i.

b. *In 3 dimensions not all building blocks are independent. This follows from the fact that the 6 vectors* Z_i *and* P_i *can not be linearly independent in* $3+2=5$ *dimensions. Show that this reduces the number of independent tensor structures of the 3pt function of the stress-energy tensor from 3 to 2.* Hint*: show that the determinant of the* 6×6 *matrix of dot products* $Z_i \cdot Z_j$, *with* $Z_i \to P_{i-3}$ *for* $i = 4, 5, 6$, *is proportional to the numerator of (2.222) with* $(a_1, a_2, a_3, a_4, a_5) = (4, 4, 2, 1, 2)$.

Exercise 2.15.8 ***Conformal Ward Identity in 2D***

Consider the Ward identity

$$\delta_\epsilon \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle = \frac{1}{2\pi i} \oint \epsilon(z) \langle T(z) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle \tag{2.231}$$

where δ_ϵ *denotes the variation under a conformal transformation parametrized by* $\epsilon(z)$, $T(z)$ *is the holomorphic part of the stress-energy tensor and the integration contour encircles the insertion points* z_i *of the local operators* $\mathcal{O}_i$.

a) *Derive the variation* $\delta_\epsilon \mathcal{O}$ *of a Virasoro primary field under a conformal transformation, from the Ward identity above and the OPE*

$$T(z)\mathcal{O}(w) \sim \frac{h}{(z-w)^2}\mathcal{O}(w) + \frac{1}{z-w}\partial\mathcal{O}(w) \tag{2.232}$$

b) *Derive the OPE of the stress tensor* $T(z)$ *with itself from*

$$\delta_\epsilon T(z) = \epsilon(z)\partial T(z) + 2\epsilon'(z)T(z) + \frac{c}{12}\,\epsilon'''(z) \tag{2.233}$$

c) *Use this OPE to derive the following recursion formula for n-point functions of* $T(z)$,

$$\begin{aligned}\langle T(z)T(z_1)...T(z_n)\rangle =& \sum_{i=1}^{n} \frac{c}{2(z-z_i)^4} \langle T(z_1)...T(z_{i-1})T(z_{i+1})\ldots T(z_n)\rangle \\ &+ \sum_{i=1}^{n} \left(\frac{2}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i} \right) \langle T(z_1)...T(z_n)\rangle\ .\end{aligned} \tag{2.234}$$

Notice that this shows that all n-point functions of the stress tensor $T(z)$ are entirely fixed given the central charge c. Do you think this simplicity extends to higher dimensional CFTs?

***d**) Use the recursion formula (2.234) to determine the 3-point function*

$$\langle T(z_1)T(z_2)T(z_3)\rangle \ . \tag{2.235}$$

Exercise 2.15.9 ***Compactified Free Boson***

Consider the Euclidean action for a real scalar field φ on a two dimensional torus

$$S[\varphi] = \frac{1}{2}\int_0^\beta d\tau \int_0^L d\sigma \left[(\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2\right] \ . \tag{2.236}$$

The partition function is given by the functional integral

$$Z = \int [d\varphi] e^{-S[\varphi]} \ , \tag{2.237}$$

where we sum over all possible field configurations on the torus. The field φ takes values on a circle of radius R. In other words, we identify φ with $\varphi + 2\pi R$ and, in the path integral, we sum over all field configurations that obey the periodicity conditions

$$\varphi(\tau,\sigma) = \varphi(\tau,\sigma+L) + 2\pi R\, w_1 = \varphi(\tau+\beta,\sigma) + 2\pi R\, w_2 \ , \qquad w_1, w_2 \in \mathbb{Z} \ . \tag{2.238}$$

In order to evaluate the path integral it is convenient to expand the field in Fourier modes

$$\varphi(\tau,\sigma) = 2\pi R\, w\frac{\sigma}{L} + \sum_{n=-\infty}^{\infty} a_n(\tau) e^{2\pi i n\frac{\sigma}{L}} \tag{2.239}$$

where $a_{-n}(\tau) = a_n^*(\tau)$ and $w \in \mathbb{Z}$ is often called the winding number.

***a**) Show that the partition function factorizes into a quantum mechanical path integral for each mode,*

$$Z = \sum_{w\in\mathbb{Z}} e^{-\beta\frac{2\pi^2R^2w^2}{L}} \int [da_0] e^{-\frac{L}{2}\int_0^\beta d\tau\, (\partial_\tau a_0)^2} \prod_{n=1}^{\infty} Z_n \ , \tag{2.240}$$

where

$$Z_n = \int [da_n da_n^*] e^{-L\int_0^\beta d\tau\, \left[|\partial_\tau a_n|^2 + \left(\frac{2\pi n}{L}\right)^2 |a_n|^2\right]} \ . \tag{2.241}$$

***b**) Show that*

$$\frac{Z_n}{Z_m} = \frac{\sinh^2\frac{\pi\beta m}{L}}{\sinh^2\frac{\pi\beta n}{L}} = \frac{\sum_{N_n=0}^{\infty}\sum_{\bar{N}_n=0}^{\infty} e^{-\beta\frac{2\pi n}{L}(N_n+\bar{N}_n+1)}}{\sum_{N_m=0}^{\infty}\sum_{\bar{N}_m=0}^{\infty} e^{-\beta\frac{2\pi m}{L}(N_m+\bar{N}_m+1)}} \ . \tag{2.242}$$

Suggestion: Expand $a_n(\tau)$ and $a_m(\tau)$ in Fourier modes. The following identity might be useful

$$\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right) = \frac{\sinh(\pi x)}{\pi x} \ . \tag{2.243}$$

c) *Notice that the zero mode $a_0(\tau)$ behaves like a free particle in a periodic box of size $2\pi R$. More precisely, argue that*

$$Z_0 = \int [da_0] e^{-\frac{L}{2}\int_0^\beta d\tau\,(\partial_\tau a_0)^2} = Tr\, e^{-\beta\hat{H}} \qquad \hat{H} = \frac{1}{2L}\hat{p}^2 \ , \tag{2.244}$$

with $\hat{p}$ the momentum conjugate to a_0. Use this canonical language to show that

$$Z_0 = \sum_{k\in\mathbb{Z}} e^{-\beta\frac{k^2}{2LR^2}} \ . \tag{2.245}$$

d) *From (2.242) and the expected partition function for an harmonic oscillator, we conclude that*

$$Z_n = \sum_{N_n=0}^{\infty}\sum_{\bar{N}_n=0}^{\infty} e^{-\beta\frac{2\pi n}{L}(N_n+\bar{N}_n+1)} \ . \tag{2.246}$$

The product over all non-zero modes then gives

$$\prod_{n=1}^{\infty} Z_n = e^{-\beta E_0}\prod_{n=1}^{\infty}\sum_{N_n=0}^{\infty}\sum_{\bar{N}_n=0}^{\infty} e^{-\beta\frac{2\pi n}{L}(N_n+\bar{N}_n)} \ , \tag{2.247}$$

where $E_0 = \frac{2\pi}{L}\sum_{n=1}^{\infty} n$ is the sum over all zero-point energies of the harmonic oscillators. Renormalize this vacuum energy by regulating the sum $\sum_{n=1}^{\infty} n \to \sum_{n=1}^{\infty} n e^{-\epsilon n}$, with $\epsilon > 0$, and keeping only the constant term in the series expansion around $\epsilon = 0$.

e) *By putting everything together, determine the spectrum of the original theory (2.236) and show that it is invariant under $R \to 1/(2\pi R)$ (this is called T-duality in String Theory).*

Chapter 3

Anti-de Sitter Spacetime

Euclidean AdS spacetime is the hyperboloid

$$-\left(X^0\right)^2+\left(X^1\right)^2+\cdots+\left(X^{d+1}\right)^2=-R^2\;,\qquad X^0>0\;,\tag{3.1}$$

embedded in $\mathbb{R}^{d+1,1}$. For large values of X^0 this hyperboloid approaches the light-cone of the embedding space that we discussed in section 2.12. It is clear from the definition that Euclidean AdS is invariant under $SO(d+1,1)$. The generators are given by

$$J_{AB}=-i\left(X_A\frac{\partial}{\partial X^B}-X_B\frac{\partial}{\partial X^A}\right)\;.\tag{3.2}$$

Poincaré coordinates are defined by

$$\begin{aligned}X^0&=R\frac{1+x^2+z^2}{2z}\\X^\mu&=R\frac{x^\mu}{z}\\X^{d+1}&=R\frac{1-x^2-z^2}{2z}\end{aligned}\tag{3.3}$$

where $x^\mu\in\mathbb{R}^d$ and $z>0$. In these coordinates, the metric reads

$$ds^2=R^2\frac{dz^2+\delta_{\mu\nu}dx^\mu dx^\nu}{z^2}\;.\tag{3.4}$$

This shows that AdS is conformal to $\mathbb{R}^+\times\mathbb{R}^d$ whose boundary at $z=0$ is just $\mathbb{R}^d$. These coordinates make explicit the subgroup $SO(1,1)\times ISO(d)$ of the full isometry group of AdS. These correspond to dilatation and Poincaré symmetries inside the $d-$dimensional conformal group. In particular, the dilatation generator is

$$D=-i\,J_{0,d+1}=-X_0\frac{\partial}{\partial X^{d+1}}+X_{d+1}\frac{\partial}{\partial X^0}=-z\frac{\partial}{\partial z}-x^\mu\frac{\partial}{\partial x^\mu}\;.\tag{3.5}$$

Another useful coordinate system is

$$\begin{aligned}X^0&=R\cosh\tau\cosh\rho\\X^\mu&=R\,\Omega^\mu\sinh\rho\\X^{d+1}&=-R\sinh\tau\cosh\rho\end{aligned}\tag{3.6}$$

where Ω^μ $(\mu = 1, \dots, d)$ parametrizes a unit $(d-1)-$dimensional sphere, $\Omega \cdot \Omega = 1$. The metric is given by

$$ds^2 = R^2 \left[\cosh^2 \rho \, d\tau^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_{d-1}^2\right] \ . \tag{3.7}$$

To understand the global structure of this spacetime it is convenient to change the radial coordinate via $\tanh \rho = \sin r$ so that $r \in [0, \frac{\pi}{2}[$. Then, the metric becomes

$$ds^2 = \frac{R^2}{\cos^2 r} \left[d\tau^2 + dr^2 + \sin^2 r \, d\Omega_{d-1}^2\right] \ , \tag{3.8}$$

which is conformal to a solid cylinder whose boundary at $r = \frac{\pi}{2}$ is $\mathbb{R} \times S^{d-1}$. In these coordinates, the dilatation generator $D = -i\, J_{0,d+1} = -\frac{\partial}{\partial \tau}$ is the hamiltonian conjugate to global time.

Exercise 3.0.1 ***Anderson localization and diffusion on the Poincaré disk***

Consider a conducting wire. We model the small defects in the wire by a random potential $V(x)$. We shall consider that the defects are rare and spaced by the typical length L. Away from the defects, we assume the potential vanishes and therefore the electron wavefunction is given by

$$a\, e^{ikx} + b\, e^{-ikx} \, . \tag{3.9}$$

Show that the effect of the n-th defect can be described by the matching conditions

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = U_n \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} \qquad\qquad U_n = \begin{bmatrix} 1/t_n & r_n^*/t_n^* \\ r_n/t_n & 1/t_n^* \end{bmatrix} \tag{3.10}$$

where t_n and r_n are the transmission and reflection coefficients of the n-th defect. Recall that conservation of probability implies that $|t_n|^2 + |r_n|^2 = 1$. Show that the effective reflection coefficient R_n of the first n defects obeys the following recursion relation

$$R_{n+1} = \frac{R_n + w_n}{1 + R_n^* \, w_n} \, , \qquad\qquad w_n = \frac{T_n}{T_n^*} r_{n+1} \, . \tag{3.11}$$

We model each defect by a random (complex) reflection coefficient with average $\langle r \rangle = 0$ and variance $\langle r r^* \rangle = \epsilon^2 \ll 1$. Notice that w_n is a random variable with the same statistical properties as r_{n+1}. For $\epsilon \ll 1$, we can approximate

$$R_{n+1} = R_n + w_n \left(1 - |R_n|^2\right) + O(w_n^2) \, . \tag{3.12}$$

Show that this random process leads to the following diffusion equation

$$\frac{1}{D} \frac{\partial P}{\partial z} = \frac{1}{4} \frac{\partial^2}{\partial R \partial R^*} \left(1 - |R|^2\right)^2 P \, , \qquad\qquad D = \frac{4\epsilon^2}{L} \, , \tag{3.13}$$

where $P(R, z)$ is the probability that the reflection coefficient for a length z of the wire takes the value R. Notice that this can be written more geometrically has a diffusion equation on the Poincare disk (or Euclidean AdS_2)

$$\frac{1}{D} \frac{\partial F}{\partial z} = \nabla^2 F \, , \qquad\qquad P = \sqrt{g} F \, , \tag{3.14}$$

where the metric is given by

$$ds^2 = \frac{4dRdR^*}{(1-|R|^2)^2} = d\rho^2 + \sinh^2\rho\, d\phi^2 = \frac{du^2}{u^2-1} + (u^2-1)d\phi^2\,, \tag{3.15}$$

with the parametrizations $R = e^{i\phi}\tanh\frac{\rho}{2}$ *and* $u = \cosh\rho$.

Show that

$$\frac{1}{D}\frac{d}{dz}\langle u\rangle = 2\langle u\rangle\,, \qquad \langle u\rangle = \int_1^\infty du \int_0^{2\pi} d\phi\, F(z,u,\phi)\, u\,, \tag{3.16}$$

Using the initial condition $\langle u\rangle = 1$ *for* $z=0$ *(zero reflection coefficient for a short wire), this gives*

$$\langle u\rangle = e^{2zD}\,, \tag{3.17}$$

which implies that the average reflection coefficient tends to 1 exponentially fast with the length of the wire. Therefore, any small disorder generates localized states in 1D. This is called Anderson localization and the length scale $1/(2D)$ *is called the localization length.*

Challenge: Find (an integral representation of) the solution of the diffusion equation with initial condition

$$F(z=0,u,\phi) = \delta(u-1)\,. \tag{3.18}$$

3.1 Particle dynamics in AdS

For most purposes it is more convenient to work in Euclidean signature and analytically continue to Lorentzian signature only at the end of the calculation. However, it is important to discuss the Lorentzian signature to gain some intuition about real time dynamics. In this case, AdS is defined by the universal cover of the manifold

$$-\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + \left(X^d\right)^2 - \left(X^{d+1}\right)^2 = -R^2\,, \tag{3.19}$$

embedded in $\mathbb{R}^{d,2}$. The universal cover means that we should unroll the non-contractible (timelike) cycle. To see this explicitly it is convenient to introduce global coordinates[1]

$$\begin{aligned} X^0 &= R\cos t\cosh\rho \\ X^\mu &= R\,\Omega^\mu \sinh\rho \\ X^{d+1} &= -R\sin t\cosh\rho \end{aligned} \tag{3.20}$$

where Ω^μ ($\mu = 1,\dots,d$) parametrizes a unit $(d-1)-$dimensional sphere. The original hyperboloid is covered with $t \in [0,2\pi[$ but we consider $t\in\mathbb{R}$. The metric is given by

$$ds^2 = R^2\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2\right]\,. \tag{3.21}$$

[1]Notice that this is just the analytic continuation $\tau \to i\,t$ and $X^{d+1} \to i\,X^{d+1}$ of the Euclidean global coordinates (3.6).

To understand the global structure of this spacetime it is convenient to change the radial coordinate via $\tanh\rho = \sin r$ so that $r \in [0, \frac{\pi}{2}[$. Then, the metric becomes

$$ds^2 = \frac{R^2}{\cos^2 r}\left[-dt^2 + dr^2 + \sin^2 r\, d\Omega_{d-1}^2\right] \,, \tag{3.22}$$

which is conformal to a solid cylinder whose boundary at $r = \frac{\pi}{2}$ is $\mathbb{R} \times S^{d-1}$.

Geodesics are given by the intersection of AdS with 2-planes through the origin of the embedding space. In global coordinates, the simplest timelike geodesic describes a particle sitting at $\rho = 0$. This corresponds to (the universal cover of) the intersection of $X^\mu = 0$ for $\mu = 1, \ldots, d$ with the hyperboloid (3.19). Performing a boost in the X^1, X^{d+1} plane we can obtain an equivalent timelike geodesic $X^1 \cosh\beta = X^{d+1}\sinh\beta$ and $X^\mu = 0$ for $\mu = 2, \ldots, d$. In global coordinates, this gives an oscillating trajectory

$$\tanh\rho = \tanh\beta\, \sin t \,, \tag{3.23}$$

with period 2π. In fact, all timelike geodesics oscillate with period 2π in global time. One can say AdS acts like a box that confines massive particles. However, it is a very symmetric box that does not have a center because all points are equivalent.

Null geodesics in AdS are also null geodesics in the embedding space. For example, the null ray $X^{d+1} - X^1 = X^0 - R = X^\mu = 0$ for $\mu = 2, \ldots, d$ is a null ray in AdS which in global coordinates is given by $\cosh\rho = \frac{1}{\cos t}$. This describes a light ray that passes through the origin at $t = 0$ and reaches the conformal boundary $\rho = \infty$ at $t = \pm\frac{\pi}{2}$. All light rays in AdS start and end at the conformal boundary traveling for a global time interval equal to π.

One can also introduce Poincaré coordinates

$$\begin{aligned} X^\mu &= R\frac{x^\mu}{z} \\ X^d &= \frac{R}{2}\frac{1 - x^2 - z^2}{z} \\ X^{d+1} &= \frac{R}{2}\frac{1 + x^2 + z^2}{z} \end{aligned} \tag{3.24}$$

where now $\mu = 0, 1, \ldots, d-1$ and $x^2 = \eta_{\mu\nu}x^\mu x^\nu$. However, in Lorentzian signature, Poincaré coordinates do not cover the entire spacetime. Surfaces of constant z approach the light-like surface $X^d + X^{d+1} = 0$ when $z \to \infty$. This null surface is often called the Poincaré horizon.

We have seen that AdS acts like a box for classical massive particles. Quantum mechanically, this confining potential gives rise to a discrete energy spectrum. Consider the Klein-Gordon equation

$$\nabla^2\phi = m^2\phi \,, \tag{3.25}$$

in global coordinates (3.21). In order to emphasize the correspondence with CFT we will solve this problem using an indirect route. Firstly, consider the action of the quadratic Casimir of the AdS isometry group on a scalar field

$$\frac{1}{2}J_{AB}J^{BA}\phi = \left[-X^2\partial_X^2 + X\cdot\partial_X\left(d + X\cdot\partial_X\right)\right]\phi \,. \tag{3.26}$$

Formally, we are extending the function ϕ from AdS, defined by the hypersurface $X^2 = -R^2$, to the embedding space. However, the action of the quadratic Casimir is independent of this extension because the generators J_{AB} are interior to AdS, i.e. $\left[J_{AB}, X^2+R^2\right]=0$. If we foliate the embedding space $\mathbb{R}^{d,2}$ with AdS surfaces of different radii R, we obtain that the laplacian in the embedding space can be written as

$$\partial_X^2 = -\frac{1}{R^{d+1}}\frac{\partial}{\partial R}R^{d+1}\frac{\partial}{\partial R} + \nabla^2_{AdS} \ . \tag{3.27}$$

Substituting this in (3.26) and noticing that $X\cdot\partial_X = R\partial_R$ we conclude that

$$\frac{1}{2}J_{AB}J^{BA}\phi = R^2\nabla^2_{AdS}\phi \ . \tag{3.28}$$

Therefore, we should identify m^2R^2 with the quadratic Casimir of the conformal group.

The Lorentzian version of the conformal generators (2.122) is

$$D = -J_{0,d+1} \ , \qquad\qquad P_\mu = J_{\mu 0} + i\, J_{\mu,d+1} \ , \tag{3.29}$$

$$M_{\mu\nu} = J_{\mu\nu} \ , \qquad\qquad K_\mu = J_{\mu 0} - i\, J_{\mu,d+1} \ . \tag{3.30}$$

Exercise 3.1.1 *Show that, in global coordinates, the conformal generators take the form*

$$D = i\frac{\partial}{\partial t} \ , \qquad M_{\mu\nu} = -i\left(\Omega_\mu\frac{\partial}{\partial\Omega^\nu} - \Omega_\nu\frac{\partial}{\partial\Omega^\mu}\right) \ ,$$

$$P_\mu = -ie^{-it}\left[\Omega_\mu\left(\partial_\rho - i\tanh\rho\,\partial_t\right) + \frac{1}{\tanh\rho}\nabla_\mu\right] \ ,$$

$$K_\mu = ie^{it}\left[\Omega_\mu\left(-\partial_\rho - i\tanh\rho\,\partial_t\right) - \frac{1}{\tanh\rho}\nabla_\mu\right] \ ,$$

where $\nabla_\mu = \frac{\partial}{\partial\Omega^\mu} - \Omega_\mu\Omega^\nu\frac{\partial}{\partial\Omega^\nu}$ *is the covariant derivative on the unit sphere* S^{d-1}*.*

In analogy with the CFT construction we can look for primary states, which are annihilated by K_μ and are eigenstates of the hamiltonian, $D\phi = \Delta\phi$. The condition $K_\mu\phi = 0$ splits in one term proportional to Ω_μ and one term orthogonal to Ω_μ. The second term implies that ϕ is independent of the angular variables Ω^μ. The first term gives $\left(\partial_\rho + \Delta\tanh\rho\right)\phi = 0$, which implies that

$$\phi \propto \left(\frac{e^{-it}}{\cosh\rho}\right)^\Delta = \left(\frac{R}{X^0 - iX^{d+1}}\right)^\Delta \ . \tag{3.31}$$

This is the lowest energy state. One can get excited states acting with P_μ. Notice that all this states will have the same value of the quadratic Casimir

$$\frac{1}{2}J_{AB}J^{BA}\phi = \Delta(\Delta-d)\phi \ . \tag{3.32}$$

This way one can generate all normalizable solutions of $\nabla^2\phi = m^2\phi$ with $m^2R^2 = \Delta(\Delta-d)$. This shows that the one-particle energy spectrum is given by $\omega = \Delta + l + 2n$ where $l = 0,1,2,\dots$ is the spin, generated by acting with $P_{\mu_1}\dots P_{\mu_l} - traces$, and $n = 0,1,2,\dots$ is generated by acting with $\left(P^2\right)^n$.

Exercise 3.1.2 *Given the symmetry of the metric (3.22) we can look for solutions of the form*

$$\phi = e^{i\omega t} Y_l(\Omega) F(r) \ , \tag{3.33}$$

where $Y_l(\Omega)$ is a spherical harmonic with eigenvalue $-l(l+d-2)$ of the laplacian on the unit sphere S^{d-1}. Derive a differential equation for $F(r)$ and show that it is solved by

$$F(r) = (\cos r)^{\Delta} (\sin r)^{l} \ {}_2F_1\left(\frac{l+\Delta-\omega}{2}, \frac{l+\Delta+\omega}{2}, l+\frac{d}{2}, \sin^2 r\right) \ , \tag{3.34}$$

with $2\Delta = d + \sqrt{d^2 + 4(mR)^2}$. We chose this solution because it is smooth at $r=0$. We also need to impose another boundary condition at the boundary of AdS $r = \frac{\pi}{2}$. Imposing that there is no energy flux through the boundary leads to the quantization of the energies $|\omega| = \Delta + l + 2n$ with $n = 0, 1, 2, \dots$ (see reference [9]).

If there are no interactions between the particles in AdS, then the Hilbert space is a Fock space and the energy of a multi-particle state is just the sum of the energies of each particle. Turning on small interactions leads to small energy shifts of the multi-particle states. This structure is very similar to the space of local operators in large N CFTs if we identify single-particle states with single-trace operators.

3.2 Quantum Field Theory in AdS

Let us now return to Euclidean signature and consider QFT on the AdS background. For simplicity, consider a free scalar field with action

$$S = \int_{AdS} dX \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2\right] \ . \tag{3.35}$$

The two-point function $\langle \phi(X)\phi(Y)\rangle$ is given by the propagator $\Pi(X,Y)$, which obeys

$$\left[\nabla_X^2 - m^2\right] \Pi(X,Y) = -\delta(X,Y) \ . \tag{3.36}$$

From the symmetry of the problem it is clear that the propagator can only depend on the invariant $X \cdot Y$ or equivalently on the chordal distance $\zeta = (X-Y)^2/R^2$. From now on we will set $R=1$ and all lengths will be expressed in units of the AdS radius.

Exercise 3.2.1 *Use (3.26) and (3.28) to show that*

$$\Pi(X,Y) = \frac{\mathcal{C}_\Delta}{\zeta^\Delta} \ {}_2F_1\left(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}, 2\Delta - d + 1, -\frac{4}{\zeta}\right) \ , \tag{3.37}$$

where $2\Delta = d + \sqrt{d^2 + (2m)^2}$ and

$$\mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}} \Gamma\left(\Delta - \frac{d}{2} + 1\right)} \ . \tag{3.38}$$

For a free field, higher point functions are simply given by Wick contractions. For example,

$$\begin{aligned}\langle\phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4)\rangle &= \Pi(X_1,X_2)\Pi(X_3,X_4)+\Pi(X_1,X_3)\Pi(X_2,X_4)\\ &\quad+\Pi(X_1,X_4)\Pi(X_2,X_3)\ . \end{aligned} \tag{3.39}$$

Weak interactions of ϕ can be treated perturbatively. Suppose the action includes a cubic term,

$$S=\int_{AdS} dX\left[\frac{1}{2}\left(\nabla\phi\right)^2+\frac{1}{2}m^2\phi^2+\frac{1}{3!}g\phi^3\right]\ . \tag{3.40}$$

Then, there is a non-vanishing three-point function

$$\langle\phi(X_1)\phi(X_2)\phi(X_3)\rangle=-g\int_{AdS} dY\,\Pi(X_1,Y)\Pi(X_2,Y)\Pi(X_3,Y)+O(g^3)\ ,$$

and a connected part of the four-point function of order g^2. This is very similar to perturbative QFT in flat space.

Given a correlation function in AdS we can consider the limit where we send all points to infinity. More precisely, we introduce

$$\mathcal{O}(P)=\frac{1}{\sqrt{C_\Delta}}\lim_{\lambda\to\infty}\lambda^\Delta\,\phi\left(X=\lambda P+\dots\right), \tag{3.41}$$

where P is a future directed null vector in $\mathbb{R}^{d+1,1}$ and the $\dots$ denote terms that do not grow with λ whose only purpose is to enforce the AdS condition $X^2=-1$. In other words, the operator $\mathcal{O}(P)$ is the limit of the field $\phi(X)$ when X approaches the boundary point P of AdS. Notice that, by definition, the operator $\mathcal{O}(P)$ obeys the homogeneity condition (2.115). Correlation functions of $\mathcal{O}$ are naturally defined by the limit of correlation functions of ϕ in AdS. For example, the two-point function is given by

$$\langle\mathcal{O}(P_1)\mathcal{O}(P_2)\rangle=\frac{1}{\left(-2P_1\cdot P_2\right)^\Delta}+O(g^2)\ , \tag{3.42}$$

which is exactly the CFT two-point function of a primary operator of dimension Δ. The three-point function $\langle\mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\rangle$ is given by

$$-\,g\,C_\Delta^{-\frac{3}{2}}\int_{AdS} dX\,\Pi(X,P_1)\Pi(X,P_2)\Pi(X,P_3)+O(g^3)\ , \tag{3.43}$$

where

$$\Pi(X,P)=\lim_{\lambda\to\infty}\lambda^\Delta\,\Pi\left(X,Y=\lambda P+\dots\right)=\frac{C_\Delta}{\left(-2P\cdot X\right)^\Delta} \tag{3.44}$$

is the bulk to boundary propagator.

Exercise 3.2.2 *Write the bulk to boundary propagator in Poincaré coordinates.*

Exercise 3.2.3 *Compute the following generalization of the integral in (3.43),*

$$\int_{AdS} dX \prod_{i=1}^{3} \frac{1}{(-2P_i \cdot X)^{\Delta_i}} \ , \tag{3.45}$$

and show that it reproduces the expected formula for the CFT three-point function $\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\mathcal{O}_3(P_3)\rangle$. *It is helpful to use the integral representation*

$$\frac{1}{(-2P \cdot X)^{\Delta}} = \frac{1}{\Gamma(\Delta)} \int_0^\infty \frac{ds}{s} s^{\Delta} e^{2sP\cdot X} \tag{3.46}$$

to bring the AdS integral to the form

$$\int_{AdS} dX e^{2X\cdot Q} \tag{3.47}$$

with Q a future directed timelike vector. Choosing the X^0 direction along Q and using the Poincaré coordinates (3.3) it is easy to show that

$$\int_{AdS} dX e^{2X\cdot Q} = \pi^{\frac{d}{2}} \int_0^\infty \frac{dz}{z} z^{-\frac{d}{2}} e^{-z+Q^2/z} \ . \tag{3.48}$$

To factorize the remaining integrals over s_1, s_2, s_3 and z it is helpful to change to the variables t_1, t_2, t_3 and z using

$$s_i = \frac{\sqrt{z\, t_1 t_2 t_3}}{t_i} \ . \tag{3.49}$$

3.2.1 State-Operator Map

We have seen that the correlation functions of the boundary operator (3.41) have the correct homogeneity property and $SO(d+1,1)$ invariance expected of CFT correlators of a primary scalar operator with scaling dimension Δ. We will now argue that they also obey an associative OPE. The argument is very similar to the one used in CFT. We think of the correlation functions as vacuum expectation values of time ordered products

$$\langle \phi(X_1)\phi(X_2)\phi(X_3)\dots\rangle = \langle 0|\dots\hat{\phi}(\tau_3,\rho_3,\Omega_3)\hat{\phi}(\tau_2,\rho_2,\Omega_2)\hat{\phi}(\tau_1,\rho_1,\Omega_1)|0\rangle \ ,$$

where we assumed $\tau_1 < \tau_2 < 0 < \tau_3 < \dots$. We then insert a complete basis of states at $\tau = 0$,

$$\begin{aligned} &\langle \phi(X_1)\phi(X_2)\phi(X_3)\dots\rangle \\ =&\sum_\psi \langle 0|\dots\hat{\phi}(\tau_3,\rho_3,\Omega_3)|\psi\rangle\langle\psi|\hat{\phi}(\tau_2,\rho_2,\Omega_2)\hat{\phi}(\tau_1,\rho_1,\Omega_1)|0\rangle \ . \end{aligned} \tag{3.50}$$

Using $\hat{\phi}(\tau,\rho,\Omega) = e^{\tau D}\hat{\phi}(0,\rho,\Omega)e^{-\tau D}$ and choosing an eigenbasis of the Hamiltonian $D = -\frac{\partial}{\partial\tau}$ it is clear that the sum converges for the assumed time ordering. The next step, is to establish a one-to-one map between the states $|\psi\rangle$ and boundary operators. It is clear that every boundary operator (3.41) defines a state. Inserting the boundary operator at $P^A = \left(P^0, P^\mu, P^{d+1}\right) = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$, which is the boundary point defined by $\tau \to -\infty$ in global coordinates, we can write

$$\langle \dots \phi(X_3)\mathcal{O}(P)\rangle = \langle 0|\dots \hat{\phi}(\tau_3, \rho_3, \Omega_3)\,|\mathcal{O}\rangle \ , \tag{3.51}$$

where

$$\begin{aligned} |\mathcal{O}\rangle &= \lim_{\tau\to-\infty} \left(e^{-\tau}\cosh\rho\right)^\Delta \hat{\phi}(\tau,\rho,\Omega)\,|0\rangle \\ &= \sum_\psi |\psi\rangle\,(\cosh\rho)^\Delta \lim_{\tau\to-\infty} \langle\psi|\, e^{\tau(D-\Delta)}\hat{\phi}(0,\rho,\Omega)\,|0\rangle \ . \end{aligned} \tag{3.52}$$

The limit $\tau \to -\infty$ projects onto the primary state with wave function (3.31).

The map from states to boundary operators can be established using global time translation invariance,

$$\begin{aligned} &\langle 0|\dots \hat{\phi}(\tau_3,\rho_3,\Omega_3)\,|\psi(0)\rangle \\ = &\lim_{\tau\to-\infty} \langle 0|\dots \hat{\phi}(\tau_3,\rho_3,\Omega_3)e^{\tau D}\,|\psi(\tau)\rangle \equiv \langle \dots \phi(X_3)\mathcal{O}_\psi(P)\rangle \end{aligned} \tag{3.53}$$

where $|\psi(\tau)\rangle = e^{-\tau D}|\psi\rangle$ and $P^A = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ is again the boundary point defined by $\tau \to -\infty$ in global coordinates. The idea is that $|\psi(\tau)\rangle$ prepares a boundary condition for the path integral on a surface of constant τ and this surface converges to a small cap around the boundary point $P^A = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$ when $\tau \to -\infty$. This is depicted in figure 3.1.

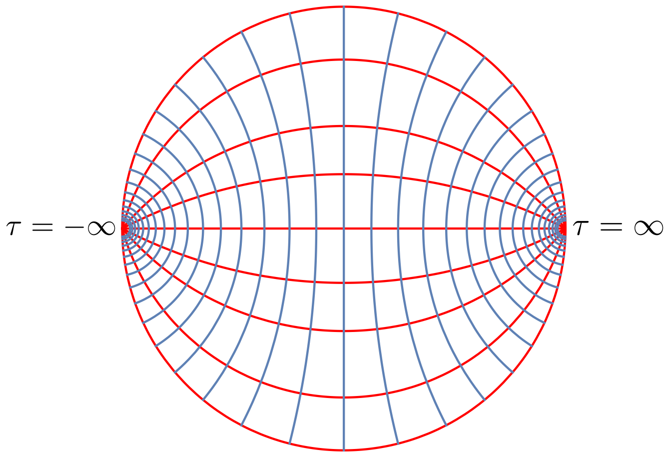


Figure 3.1 Curves of constant τ (in blue) and constant ρ (in red) for AdS$_2$ stereographically projected to the unit disk (Poincaré disk). This shows how surfaces of constant τ converge to a boundary bound when $\tau \to -\infty$. The cartesian coordinates in the plane of the figure are given by $\vec{w} = \frac{(\cosh\rho\sinh\tau, \sinh\rho)}{1+\cosh\rho\cosh\tau}$ which puts the AdS$_2$ metric in the form $ds^2 = \frac{4d\vec{w}^2}{1-\vec{w}^2}$.

The Hilbert space of the bulk theory can be decomposed in irreducible representations of the isometry group $SO(d+1,1)$. These are the highest weight representations of the conformal group, labelled by the scaling dimension and $SO(d)$ irrep of the the primary state. Therefore, the CFT conformal block decomposition of correlators follows from the partial wave decomposition in AdS, i.e. the decomposition in intermediate eigenstates of the Hamiltonian organized in irreps of the isometry group $SO(d+1,1)$. For example, the conformal block decomposition of the disconnected part of the four-point function,

$$\langle \mathcal{O}(P_1)\dots\mathcal{O}(P_4)\rangle = \frac{1}{(P_{12}P_{34})^\Delta} + \frac{1}{(P_{13}P_{24})^\Delta} + \frac{1}{(P_{14}P_{23})^\Delta} \ , \tag{3.54}$$

where $P_{ij} = -2P_i \cdot P_j$, is given by a sum of conformal blocks associated with the vacuum and two-particle intermediate states

$$\langle \mathcal{O}(P_1)\dots\mathcal{O}(P_4)\rangle = G_{0,0}(P_1,\dots,P_4) + \sum_{\substack{l=0\\ even}}^{\infty}\sum_{n=0}^{\infty} c_{n,l}\, G_{2\Delta+2n+l,l}(P_1,\dots,P_4) \ .$$

Exercise 3.2.4 *Check this statement in $d=2$ using the formula [30]*

$$G_{E,l}(P_1,P_2,P_3,P_4) = \frac{k(E+l,z)k(E-l,\bar z) + k(E-l,z)k(E+l,\bar z)}{(-2P_1\cdot P_2)^\Delta\,(-2P_3\cdot P_4)^\Delta\,(1+\delta_{l,0})} \tag{3.55}$$

where

$$k(2\beta,z) = (-z)^\beta\ {}_2F_1(\beta,\beta,2\beta,z) \ . \tag{3.56}$$

Determine the coefficients $c_{n,l}$ for $n \le 1$ by matching the Taylor series expansion around $z=\bar z=0$. ***Extra:*** *using a computer you can compute many coefficients and guess the general formula.*

3.2.2 Generating function

There is an equivalent way of defining CFT correlation functions from QFT in AdS. We introduce the generating function

$$W[\phi_b] = \left\langle e^{\int_{\partial AdS} dP \phi_b(P)\mathcal{O}(P)} \right\rangle \ , \tag{3.57}$$

where the integral over ∂AdS denotes an integral over a chosen section of the null cone in $\mathbb{R}^{d+1,1}$ with its induced measure. We impose that the source obeys $\phi_b(\lambda P) = \lambda^{\Delta-d}\phi_b(P)$ so that the integral is invariant under a change of section, i.e. conformal invariant. For example, in the Poincaré section the integral reduces to $\int d^d x \phi_b(x)\mathcal{O}(x)$. Correlation functions are easily obtained with functional derivatives

$$\langle \mathcal{O}(P_1)\dots\mathcal{O}(P_n)\rangle = \frac{\delta}{\delta\phi_b(P_1)}\cdots\frac{\delta}{\delta\phi_b(P_n)} W[\phi_b]\bigg|_{\phi_b=0} \ . \tag{3.58}$$

If we set the generating function to be equal to the path integral over the field ϕ in AdS

$$W[\phi_b] = \frac{\int_{\phi\to\phi_b}[d\phi]\,e^{-S[\phi]}}{\int_{\phi\to 0}[d\phi]\,e^{-S[\phi]}} \ , \tag{3.59}$$

with the boundary condition that it approaches the source ϕ_b at the boundary,

$$\lim_{\lambda\to\infty} \lambda^{d-\Delta}\phi(X=\lambda P+\dots) = \frac{1}{2\Delta-d}\frac{1}{\sqrt{C_\Delta}}\phi_b(P) \ , \tag{3.60}$$

then we recover the correlation functions of $\mathcal{O}$ defined above as limits of the correlation functions of ϕ.

For a quadratic bulk action, tha ratio of path intagrals in (3.59) is given e^{-S} computed on the classical solution obeying the required boundary conditions. A natural guess for this solution is

$$\phi(X) = \sqrt{C_\Delta}\int_{\partial AdS} dP \frac{\phi_b(P)}{(-2P\cdot X)^\Delta} \ . \tag{3.61}$$

This automatically solves the AdS equation of motion $\nabla^2\phi = m^2\phi$, because it is an homogeneous function of weight $-\Delta$ and it obeys $\partial_A\partial^A\phi = 0$ in the embedding space (see equations (3.26) and (3.28)). To see that it also obeys the boundary condition (3.60) it is convenient to use the Poincaré section.

Exercise 3.2.5 *In the Poincaré section (2.114) and using Poincaré coordinates (3.3), formula (3.61) reads*

$$\phi\,(z,x) = \sqrt{C_\Delta}\int d^dy \frac{z^\Delta\phi_b(y)}{(z^2+(x-y)^2)^\Delta} \tag{3.62}$$

and (3.60) reads

$$\lim_{z\to 0} z^{\Delta-d}\phi(z,x) = \frac{1}{2\Delta-d}\frac{1}{\sqrt{C_\Delta}}\phi_b(x) \ . \tag{3.63}$$

Show that (3.63) follows from (3.62). You can assume $2\Delta > d$.

The cubic term $\frac{1}{3!}g\phi^3$ in the action will lead to (calculable) corrections of order g in the classical solution (3.61). To determine the generating function $W[\phi_b]$ in the classical limit we just have to compute the value of the bulk action (3.40) on the classical solution. However, before doing that, we have to address a small subtlety. We need to add a boundary term to the action (3.40) in order to have a well posed variational problem.

Exercise 3.2.6 *The coefficient β should be chosen such that the quadratic action* [2]

$$S_2 = \int_{AdS} dw\sqrt{G}\left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] + \beta\int_{AdS} dw\sqrt{G}\,\nabla_\alpha\left(\phi\nabla^\alpha\phi\right) \tag{3.64}$$

is stationary around a classical solution obeying (3.63) for any variation $\delta\phi$ that preserves the boundary condition, i.e.

$$\delta\phi(z,x) = z^\Delta\left[f(x)+O(z)\right] \ . \tag{3.65}$$

[2] Here w stands for a generic coordinate in AdS and the index α runs over the $d+1$ dimensions of AdS.

Show that $\beta = \frac{\Delta - d}{d}$ *and that the on-shell action is given by a boundary term*

$$S_2 = \frac{2\Delta - d}{2d} \int_{AdS} dw \sqrt{g}\, \nabla_\alpha \left(\phi \nabla^\alpha \phi \right) \ . \tag{3.66}$$

Finally, show that for the classical solution (3.62) this action is given by [3]

$$S_2 = -\frac{1}{2} \int d^d y_1 d^d y_2 \phi_b(y_1) \phi_b(y_2) K(y_1, y_2) \ , \tag{3.67}$$

where

$$\begin{aligned} K(y_1, y_2) &= C_\Delta \frac{2\Delta - d}{d} \lim_{z \to 0} \int \frac{d^d x}{z^{d-1}} \frac{z^\Delta}{(z^2 + (x - y_1)^2)^\Delta} \partial_z \frac{z^\Delta}{(z^2 + (x - y_2)^2)^\Delta} \\ &= \frac{1}{(y_1 - y_2)^{2\Delta}} \end{aligned} \tag{3.68}$$

is the CFT two point function (3.42).

Exercise 3.2.7 *Using* $\phi = \phi_0 + O(g)$ *with* ϕ_0 *given by (3.61), show that the complete on-shell action is given by*

$$S = -\frac{1}{2} \int d^d y_1 d^d y_2 \phi_b(y_1) \phi_b(y_2) K(y_1, y_2) + \frac{1}{3!} g \int_{AdS} dX \left[\phi_0(X) \right]^3 + O(g^2) \ ,$$

and that this leads to the three-point function (3.43). ***E****xtra: Compute the terms of* $O(g^2)$ *in the on-shell action.*

We have seen that QFT on an AdS background naturally defines conformal correlation functions living on the boundary of AdS. Moreover, we saw that a weakly coupled theory in AdS gives rise to factorization of CFT correlators like in a large N expansion. However, there is one missing ingredient to obtain a full-fledged CFT: a stress-energy tensor. In the next section, we will see that this requires dynamical gravity in AdS. The next exercise also shows that a free QFT in AdS$_{d+1}$ can not be dual to a local CFT$_d$.

Exercise 3.2.8 *Compute the free-energy of a gas of free scalar particles in AdS. Since particles are free and bosonic one can create multi-particle states by populating each single particle state an arbitrary number of times. That means that the total partition function is a product over all single particle states and it is entirely determined by the single particle partition function. More precisely, show that*

$$F = -T \log Z = -T \log \prod_{\psi_{sp}} \left(\sum_{k=0}^{\infty} q^{k E_{\psi_{sp}}} \right) = -T \sum_{n=1}^{\infty} \frac{1}{n} Z_1 \left(q^n \right) \ , \tag{3.69}$$

$$Z_1(q) = \sum_{\psi_{sp}} q^{E_{\psi_{sp}}} = \frac{q^\Delta}{(1 - q)^d} \ , \tag{3.70}$$

[3] This integral is divergent if the source ϕ_b is a smooth function and $\Delta > \frac{d}{2}$. The divergence comes from the short distance limit $y_1 \to y_2$ and does not affect the value of correlation functions at separate points. Notice that a small value of $z > 0$ provides a UV regulator.

where $q = e^{-\frac{1}{RT}}$ *and we have used the single-particle spectrum of the hamiltonian* $D = -\frac{\partial}{\partial \tau}$ *of AdS in global coordinates. Show that*

$$F \approx -\zeta(d+1) R^d T^{d+1} \tag{3.71}$$

in the high temperature regime and compute the entropy using the thermodynamic relation $S = -\frac{\partial F}{\partial T}$. *Compare this result with the expectation*

$$S \sim (RT)^{d-1} \, , \tag{3.72}$$

for the high temperature behaviour of the entropy of a CFT on a sphere S^{d-1} *of radius* R. *See section 4.3 of reference [31] for more details.*

3.3 Gravity with AdS boundary conditions

Consider general relativity in the presence of a negative cosmological constant

$$I[G] = \frac{1}{\ell_P^{d-1}} \int d^{d+1}w \sqrt{G}\, [\mathcal{R} - 2\Lambda] \ . \tag{3.73}$$

The AdS geometry

$$ds^2 = R^2 \frac{dz^2 + dx_\mu dx^\mu}{z^2} \ , \tag{3.74}$$

is a maximally symmetric classical solution with $\Lambda = -\frac{d(d-1)}{2R^2}$. When the AdS radius R is much larger than the Planck length ℓ_P the metric fluctuations are weakly coupled and form an approximate Fock space of graviton states. One can compute the single graviton states and verify that they are in one-to-one correspondence with the CFT stress-tensor operator and its descendants (with AdS energies matching scaling dimensions). One can also obtain CFT correlation functions of the stress-energy tensor using Witten diagrams in AdS. The new ingredients are the bulk to boundary and bulk to bulk graviton propagators [32, 33, 34, 35, 36].

In the gravitational context, it is nicer to use the partition function formulation

$$Z[g_{\mu\nu}, \phi_b] = \int_{\substack{G \to g \\ \phi \to \phi_b}} [dG]\, [d\phi]\, e^{-I[G,\phi]} \tag{3.75}$$

where

$$I[G,\phi] = \frac{1}{\ell_P^{d-1}} \int d^{d+1}w \sqrt{G} \left[\mathcal{R} - 2\Lambda + \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right] \tag{3.76}$$

and the boundary condition are

$$\begin{aligned} ds^2 &= G_{\alpha\beta} dw^\alpha dw^\beta = R^2 \frac{dz^2 + dx^\mu dx^\nu \left[g_{\mu\nu}(x) + O(z) \right]}{z^2} \ , \\ \phi &= \frac{z^{d-\Delta}}{2\Delta - d} \left[\phi_b(x) + O(z) \right] \ . \end{aligned} \tag{3.77}$$

By construction the partition function is invariant under diffeomorphisms of the boundary metric $g_{\mu\nu}$. Therefore, this definition implies the Ward identity (2.72). The generating function is also invariant under Weyl transformations

$$Z\left[\Omega^2 g_{\mu\nu}, \Omega^{\Delta-d}\phi_b\right] = Z\left[g_{\mu\nu}, \phi_b\right] \qquad (naive) \tag{3.78}$$

This follows from the fact that the boundary condition

$$\begin{aligned} ds^2 &= R^2 \frac{dz^2 + dx^\mu dx^\nu \left[\Omega^2(x) g_{\mu\nu}(x) + O(z)\right]}{z^2} \\ \phi &= \frac{z^{d-\Delta}}{2\Delta - d}\left[\Omega^{\Delta-d}(x)\phi_b(x) + O(z)\right] \end{aligned} \tag{3.79}$$

can be mapped to (3.77) by the following coordinate transformation

$$\begin{aligned} z &\to z\,\Omega - \frac{1}{4} z^3 \Omega \left(\partial_\mu \log \Omega\right)^2 + O(z^5) \\ x^\mu &\to x^\mu - \frac{1}{2} z^2 \partial^\mu \log \Omega + O(z^4) \end{aligned} \tag{3.80}$$

where indices are raised and contracted using the metric $g_{\mu\nu}$ and its inverse. In other words, a bulk geometry that satisfies (3.77) also satisfies (3.79) with an appropriate choice of coordinates. If the partition function (3.75) was a finite quantity this would be the end of the story. However, even in the classical limit, where $Z \approx e^{-I}$, the partition function needs to be regulated. The divergences originate from the $z \to 0$ region and can be regulated by cutting off the bulk integrals at $z = \epsilon$ (as it happened for the scalar case discussed above). Since the coordinate transformation (3.80) does not preserve the cutoff, the regulated partition function is not obviously Weyl invariant. This has been studied in great detail in the context of holographic renormalization [37, 38]. In particular, it leads to the Weyl anomaly $g^{\mu\nu}T_{\mu\nu} \neq 0$ when d is even. The crucial point is that this is a UV effect that does not affect the connected correlation functions of operators at separate points. In particular, the integrated form (2.75)=(2.77) of the conformal Ward identity is valid.

We do not now how to define the quantum gravity path integral in (3.75). The best we can do is a semiclassical expansion when $\ell_P \ll R$. This semiclassical expansion gives rise to connected correlators of the stress tensor $T_{\mu\nu}$ that scale as

$$\langle T_{\mu_1\nu_1}(x_1) \dots T_{\mu_n\nu_n}(x_n)\rangle_c \sim \left(\frac{R}{\ell_P}\right)^{d-1} . \tag{3.81}$$

This is exactly the scaling (2.165) we found from large N factorization if we identify $N^2 \sim \left(\frac{R}{\ell_P}\right)^{d-1}$. This suggests that CFTs related to semiclassical Einstein gravity in AdS, should have a large number of local degrees of freedom. This can be made more precise. The two-point function of the stress tensor in a CFT is given by

$$\langle T_{\mu\nu}(x) T_{\sigma\rho}(0)\rangle = \frac{C_T}{S_d^2}\frac{1}{x^{2d}}\left[\frac{1}{2} I_{\mu\sigma} I_{\nu\rho} + \frac{1}{2} I_{\mu\rho} I_{\nu\sigma} - \frac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho}\right] , \tag{3.82}$$

where $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of a $(d-1)$-dimensional unit sphere and

$$I_{\mu\nu} = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \ . \tag{3.83}$$

The constant C_T provides an (approximate) measure of the number of degrees of freedom.[4] For instance, for n_φ free scalar fields and n_ψ free Dirac fields we find [39]

$$C_T = n_\varphi \frac{d}{d-1} + n_\psi 2^{\left[\frac{d}{2}\right]-1} d \ , \tag{3.84}$$

where $[x]$ is the integer part of x. If the CFT is described by Einstein gravity in AdS, we find [32]

$$C_T = 8\frac{d+1}{d-1}\frac{\pi^{\frac{d}{2}}\Gamma(d+1)}{\Gamma^3\left(\frac{d}{2}\right)}\frac{R^{d-1}}{\ell_P^{d-1}} \ , \tag{3.85}$$

which shows that the CFT dual of a semiclassical gravitational theory with $R \gg \ell_P$, must have a very large number of degrees of freedom.

In summary, semiclassical gravity with AdS boundary conditions gives rise to a set of correlation functions that have all the properties (conformal invariance, Ward identities, large N factorization) expected for the correlation functions of the stress tensor of a large N CFT. Therefore, it is natural to ask if a CFT with finite N is a quantum theory of gravity.

[4] However, for $d > 2$, C_T is not a c-function that always decreases under Renormalization Group flow.

Chapter 4

The AdS/CFT Correspondence

4.1 Quantum Gravity as CFT

What is quantum gravity? The most conservative answer is a standard quantum mechanical theory whose low energy dynamics around a weakly curved background is well described by general relativity (or some other theory with a dynamical metric). This viewpoint is particularly useful with asymptotically AdS boundary conditions. In this case, we can view the AdS geometry with a radius much larger than the Planck length as a background for excitations (gravitons) that are weakly coupled at low energies. Therefore, we just need to find a quantum system that reproduces the dynamics of low energy gravitons in a large AdS. In fact, we should be more precise about the word "reproduces". We should define observables in quantum gravity that our quantum system must reproduce. This is not so easy because the spacetime geometry is dynamical and we can not define local operators. In fact, the only well defined observables are defined at the (conformal) boundary like the partition function (3.75) and the associated correlation functions obtained by taking functional derivatives. But in the previous section we saw that these observables have all the properties expected for the correlation functions of a large N CFT. Thus, quantum gravity with AdS boundary conditions is equivalent to a CFT.

There are many CFTs and not all of them are useful theories of quantum gravity. Firstly, it is convenient to consider a family of CFTs labeled by N, so that we can match the bulk semiclassical expansion using $N^2 \sim \left(\frac{R}{\ell_P}\right)^{d-1}$. In the large N limit, every CFT single-trace primary operator of scaling dimension Δ gives rise to a weakly coupled field in AdS with mass $m \sim \Delta/R$. Therefore, if are looking for a UV completion of pure gravity in AdS without any other low energy fields, then we need to find a CFT where all single-trace operators have parametrically large dimension, except the stress tensor. This requires strong coupling and seems rather hard to achieve. Notice that a weakly coupled CFT with gauge group $SU(N)$ and fields in the adjoint representation has an infinite number of primary single-trace operators with order 1 scaling dimension. It is natural to conjecture that large N factorization and correct spectrum of single-trace operators are sufficient conditions for a CFT to provide a UV completion of General Relativity (GR) [40]. However, this is not obvious because we still have to check if the CFT correlation functions of $T_{\mu\nu}$ match the prediction from GR in AdS. For example, the stress tensor three-point function is fixed by conformal symmetry to be a linear combination of 3 independent conformal invariant structures. [1] On the other hand, the action (3.73) predicts a specific linear combination. It is not obvious that all large N CFTs with the correct spectrum will automatically give rise to the same three-point function. There has been some recent progress in this respect. The authors of [41] used causality to show that this is the case. Unfortunately, their argument uses the bulk theory and can not be formulated entirely in CFT language. In any case, this is just the three-point fuction and GR predicts the leading large N behaviour of all n-point functions. It is an important open problem to prove the following conjecture:

Any large N CFT where all single-trace operators, except the stress tensor, have parametrically large scaling dimensions, has the stress tensor correlation functions predicted by General Relativity in AdS.

Perhaps the most pressing question is if such CFTs exist at all. At the moment, we do not know the answer to this question but in the next section we will discuss closely related examples that are realized in the context of string theory.

If some CFTs are theories of quantum gravity, it is natural to ask if there are other CFT observables with a nice gravitational interpretation. One interesting example that will be extensively discussed in this school is the entanglement entropy of a subsystem. In section 4.3, we will discuss how CFT thermodynamics compares with black hole thermodynamics in AdS. In addition, in section 4.4 we will give several examples of QFT phenomena that have beautiful geometric meaning in the holographic dual.

4.2 String Theory

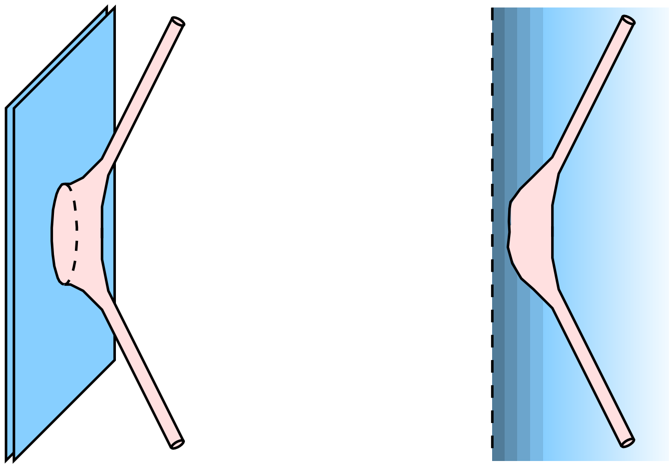
The logical flow presented above is very different from the historical route that led to the AdS/CFT correspondence. Moreover, from what we said so far AdS/CFT looks like a very abstract construction without any concrete examples of CFTs that have simple gravitational duals. If this was the full story probably I would not be writing these lecture notes. The problem is that we have stated properties that we want for our CFTs but we have said nothing about how to construct these CFTs besides the fact that they should be strongly coupled and obey large N factorization. Remarkably, string theory provides a "method" to find explicit examples of CFTs and their dual gravitational theories.

The basic idea is to consider the low energy description of D-brane systems from the perspective of open and closed strings. Let us illustrate the argument by quickly summarizing the prototypical example of AdS/CFT [5]. Consider N coincident D3-branes of type IIB string theory in 10 dimensional Minkowski spacetime. Closed strings propagating in 10 dimensions can interact with the D3-branes. This interaction can be described in two equivalent ways:

(a) D3-branes can be defined as a submanifold where open strings can end. This means that a closed string interacts with the D3-branes by breaking the string loop into an open string with endpoints attached to the D3-branes.

[1] Here we are assuming $d \geq 4$. For $d = 3$ there are only 2 independent structures.

(b) D3-branes can be defined as solitons of closed string theory. In other words, they create a non-trivial curved background where closed strings propagate.



(a) (b)

Figure 4.1 (a) Closed string scattering off branes in flat space. (b) Closed string propagating in a curved background.

These two alternatives are depicted in figure 4.1. Their equivalence is called open/closed duality. The AdS/CFT correspondence follows from the low-energy limit of open/closed duality. We implement this low-energy limit by taking the string length $\ell_s \to 0$, keeping the string coupling g_s, the number of branes N and the energy fixed. In description (a), the low energy excitations of the system form two decoupled sectors: massless closed strings propagating in 10 dimensional Minkoski spacetime and massless open strings attached to the D3-branes, which at low energies are well described by $\mathcal{N}=4$ Supersymmetric Yang-Mills (SYM) with gauge group $SU(N)$. In description (b), the massless closed strings propagate in the following geometry

$$ds^2 = \frac{1}{\sqrt{H(r)}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{H(r)}\left[dr^2 + r^2 d\Omega_5^2\right] \ , \tag{4.1}$$

where $\eta_{\mu\nu}$ is the metric of the 4 dimensional Minkowski space along the branes and

$$H(r) = 1 + \frac{R^4}{r^4} \ , \qquad R^4 = 4\pi g_s N \ell_s^4 \ . \tag{4.2}$$

Naively, the limit $\ell_s \to 0$ just produces 10 dimensional Minkowski spacetime. However, one has to be careful with the region close to the branes at $r=0$. A local high energy excitation in this region will be very redshifted from the point of view of the observer at infinity. In order to determine the correct low-energy limit in the region around $r=0$ we introduce a new coordinate $z = R^2/r$, which we keep fixed as $\ell_s \to 0$. This leads to

$$ds^2 = R^2\frac{dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu}{z^2} + R^2 d\Omega_5^2 \ , \tag{4.3}$$

which is the metric of $\mathrm{AdS}_5 \times S^5$ both with radius R. Therefore, description (b) also leads to 2 decoupled sectors of low energy excitations: massless closed strings in 10D and full type IIB string theory on $\mathrm{AdS}_5 \times S^5$. This led Maldacena to conjecture that

$$\begin{array}{ccc} SU(N)\ \mathrm{SYM} & \Leftrightarrow & \text{IIB strings on } \mathrm{AdS}_5 \times S^5 \\ g_{YM}^2 = 4\pi g_s & & \frac{R^4}{\ell_s^4} = g_{YM}^2 N \equiv \lambda \end{array}$$

SYM is conformal for any value of N and the coupling constant g_{YM}^2. The lagrangian of the theory involves the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right] \ , \tag{4.4}$$

6 scalars fields Φ^m and 4 Weyl fermions Ψ^a, which are all valued in the adjoint representation of $SU(N)$. The lagrangian is given by

$$\begin{aligned} \frac{1}{g_{YM}^2}\mathrm{Tr}\Bigg[&\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(D^\mu\Phi^m\right)^2 + \bar{\Psi}^a\sigma^\mu D_\mu\Psi_a \\ &-\frac{1}{4}\left[\Phi^m,\Phi^n\right]^2 - C_m^{ab}\Psi_a\left[\Phi^m,\Psi_b\right] - \bar{C}_{mab}\bar{\Psi}^a\left[\Phi^m,\bar{\Psi}^b\right]\Bigg] \ , \end{aligned} \tag{4.5}$$

where D_μ is the gauge covariant derivative and C_m^{ab} and $\bar{C}_{mab}$ are constants fixed by the $SO(6) = SU(4)$ global symmetry of the theory. Notice that the isometry group of $\mathrm{AdS}_5 \times S^5$ is $SO(5,1) \times SO(6)$, which matches precisely the bosonic symmetries of SYM: conformal group $\times$ global $SO(6)$. There are many interesting things to say about SYM. In some sense, SYM is the simplest interacting QFT in 4 dimensions [42]. However, this is not the focus of these lectures and we refer the reader to the numerous existing reviews about SYM [11, 43].

The remarkable conjecture of Maldacena has been extensively tested since it was first proposed in 1997 [5]. To test this conjecture one has to be able to compute the same observable on both sides of the duality. This is usually a very difficult task. On the SYM side, the regime accessible to perturbation theory is $g_{YM}^2 N \ll 1$. This implies $g_s \ll 1$, which on the string theory side suppresses string loops. However, it also implies that the AdS radius of curvature R is much smaller than the string length ℓ_s. This means that the string worldsheet theory is very strongly coupled. In fact, the easy regime on the string theory side is $g_s \ll 1$ and $R \gg \ell_s$, so that (locally) strings propagate in an approximately flat space. Thus, directly testing the conjecture is a formidable task. There are three situations where a direct check can be made analitycally.

The first situation arises when some observable is independent of the coupling constant. In this case, one can compute it at weak coupling $\lambda \ll 1$ using the field theory description and at strong coupling $\lambda \gg 1$ using the string theory description. Usually this involves completely different techniques but in the end the results agree. Due to the large supersymmetry of SYM there are many observables that do not depend on the coupling constant. Notable examples include the scaling dimensions of half BPS single-trace operators and their three-point functions [44].

The second situation involves observables that depend on the coupling constant λ but preserve enough supersymmetry that can be computed at any value of λ using a technique called localization. Important examples of this type are the sphere partition function and the expectation value of circular Wilson loops [**?**, 45].

Finally, the third situation follows from the conjectured integrability of SYM in the planar limit. Assuming integrability one can compute the scaling dimension of non-protected single-trace operators at any value of λ and match this result with SYM perturbative calculations for $\lambda \ll 1$ and with weakly coupled string theory for $\lambda \gg 1$ (see figure 1 from [46]). Planar scattering amplitudes an three-point functions of single-trace operators can also be computed using integrability [47, 48].

There are also numerical tests of the gauge/gravity duality. The most impressive study in this context, was the Monte-Carlo simulation of the BFSS matrix model [49] at finite temperature that reproduced the predictions from its dual black hole geometry [50, 51, 52, 53, 54, 55, 56].

How does the Maldacena conjecture fit into the general discussion of the previous sections? One important novelty is the presence of a large internal sphere on the gravitational side. We can perform a Kaluza-Klein reduction on S^5 and obtain an effective action for AdS_5

$$\frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{g_{10}} e^{-2\Phi} \left[\mathcal{R}_{10} + \dots\right] \to \frac{R^5}{8(2\pi)^4 g_s^2 \ell_s^8} \int d^5x \sqrt{g_5} \left[\mathcal{R}_5 + \dots\right] \,.$$

This allows us to identify the 5 dimensional Planck length

$$\ell_P^3 = \frac{8(2\pi)^4 g_s^2 \ell_s^8}{R^5} \tag{4.6}$$

and verify the general prediction $N^2 \sim R^3/\ell_P^3$. Remarkably, at strong coupling $\lambda \gg 1$ all single-trace non-protected operators of SYM have parametrically large scaling dimensions. This is simple to understand from the string point of view. Massive string states have masses $m \sim 1/\ell_s$. But we saw in the previous sections that the dual operator to an AdS field of mass m has a scaling dimension $\Delta \sim mR \sim R/\ell_s \sim \lambda^{\frac{1}{4}}$. The only CFT operators that have small scaling dimension for $\lambda \gg 1$ are dual to massless string states that constitute the fields of type IIB supergravity (SUGRA). Therefore, one can say that SYM (with $N \gg \lambda \gg 1$) provides a UV completion of IIB SUGRA with $\mathrm{AdS}_5 \times S^5$ boundary conditions.

String theory provides more concrete examples of AdS/CFT dual pairs. These examples usually involve SCFTs (or closely related non-supersymmetry theories). This is surprising because SUSY played no role in our general discussion. At the moment, it is not known if SUSY is an essential ingredient of AdS/CFT or if it is only a useful property that simplifies the calculations. The latter seems more likely but notice that SUSY might be essential to stabilize very strong coupling and allow the phenomena of large scaling dimensions for almost all single-trace operators. Another observation is that it turns out to be very difficult to construct AdS duals with small internal spaces (for SYM we got a 5-sphere with the same radius of AdS_5). It is an open problem to find CFTs with gravity duals in less than 10 dimensions (see [57, 58] for attempts in this direction).

Another interesting class of examples are the dualities between vector models and Higher Spin Theories (HST) [59, 60]. Consider for simplicity the free $O(n)$ model in 3 dimensions

$$S = \int d^3x \sum_{i=1}^{n} \frac{1}{2} \partial_\mu \varphi^i \partial^\mu \varphi^i \ . \tag{4.7}$$

In this case, the analogue of single-trace operators are the $O(n)$ singlets $\mathcal{O}_l = \sum_i \varphi^i \partial_{\mu_1} \dots \partial_{\mu_l} \varphi^i$ with even spin l and dimension $\Delta = 1 + l$. At large n, the correlation functions of these operators factorize with n playing the role of N^2 in a $SU(N)$ gauge theory with adjoint fields. The AdS dual of this CFT is a theory with one massless field for each even spin. These theories are rather non-local and they can not be defined in flat spacetime. Even if we introduce the relevant interaction $\left(\varphi^i \varphi^i\right)^2$ and flow to the IR fixed point (Wilson-Fisher fixed point), the operators $\mathcal{O}_l$ with $l > 2$ get anomalous dimensions of order $\frac{1}{n}$ and therefore the classical AdS theory still contains the same number of massless higher spin fields. This duality has been extended to theories with fermions and to theories where the global $O(n)$ symmetry is gauged using Chern-Simons gauge fields. It is remarkable that HST in AdS seems to have the correct structure to reproduce the CFT observables that have been computed so far. Notice that in these examples of AdS/CFT supersymmetry plays no role. However, it is unclear if the AdS description is really useful in this case.[2] In practice, the large n limit of these vector models is solvable and the dual HST in AdS is rather complicated to work with even at the classical level. There are also analogous models in $\text{AdS}_3/\text{CFT}_2$ duality [63].

4.3 Finite Temperature

In section 3.3, we argued that holographic CFTs must have a large number of local degrees of freedom, using the two-point function of the stress tensor. Another way of counting degrees of freedom is to look at the entropy density when the system is put at finite temperature. For a CFT in flat space and infinite volume, the temperature dependence of the entropy density is fixed by dimensional analysis because there is no other scale available,

$$s = c_s T^{d-1} \ . \tag{4.8}$$

The constant c_s is a physical measure of the number of degrees of freedom.

The gravitational dual of the system at finite temperature is a black brane in asymptotically AdS space. The Euclidean metric is given by

$$ds^2 = \frac{R^2}{z^2} \left[\frac{dz^2}{1 - (z/z_H)^d} + \left(1 - \frac{z^d}{z_H^d}\right) d\tau^2 + \delta_{ij} dx^i dx^j \right] \ . \tag{4.9}$$

Exercise 4.3.1 *Show that in order to avoid a conical defect at the horizon $z = z_H$, we need to identify Euclidean time τ with period $\frac{4\pi z_H}{d}$. This fixes the Hawking temperature $T = \frac{d}{4\pi z_H}$.*

[2] In practice it was very useful because it led to an intensive study of Chern-Simons matter theories, which gave rise to the remarkable conjecture of fermion/boson duality in 3 dimensions [61, 62].

The formula $T = \frac{d}{4\pi z_H}$ illustrates a general phenomena in holography: high energy corresponds to the region close to the boundary and low energy corresponds to the deep interior of the dual geometry.

The entropy of the system is given by the Bekenstein-Hawking formula

$$S = \frac{A_H}{4G_N} = \frac{4\pi}{\ell_P^{d-1}} \frac{R^{d-1}}{z_H^{d-1}} \int d^{d-1}x \qquad \Rightarrow \qquad c_s = \frac{(4\pi)^d}{d^{d-1}} \frac{R^{d-1}}{\ell_P^{d-1}} \,. \tag{4.10}$$

As expected c_s is very large in the bulk classical limit $R \gg \ell_P$. Interestingly, the ratio

$$\frac{c_s}{C_T} = \frac{\pi^{\frac{d}{2}}}{8} \left(\frac{4}{d}\right)^d \frac{d-1}{d+1} \frac{\Gamma^3\left(\frac{d}{2}\right)}{\Gamma(d)} \tag{4.11}$$

only depends on the spacetime dimension d if the CFT has a classical bulk dual [64]. It would be very nice to prove that all large N CFTs where all single-trace operators, except the stress tensor, have parametrically large scaling dimensions, satisfy (4.11). Notice that (4.11) is automatic in $d = 2$ because $C_T = 2c$ and $c_s = \frac{\pi}{3}c$ are uniquely fixed in terms of the central charge c. In planar SYM, $C_T = 40N^2$ is independent of the 't Hooft coupling but c_s varies with λ (although not much, $c_s(\lambda = \infty) = \frac{3}{4}c_s(\lambda = 0)$). In this case, (4.11) is only satisfied at strong coupling, when all primary operators with spin greater than 2 have parametrically large scaling dimensions.

Exercise 4.3.2 *Consider a CFT on a sphere of radius L and at temperature T. In this case, the entropy is a non-trivial function of the dimensioless combination LT. Let us compute this function assuming the CFT is well described by Einstein gravity with asymptotically AdS boundary conditions. There are two possible bulk geometries that asymptote to the Euclidean boundary $S^1 \times S^{d-1}$. The first is pure AdS*

$$ds^2 = R^2 \left[\frac{dr^2}{1+r^2} + \left(1+r^2\right) d\tau^2 + r^2 d\Omega_{d-1}^2\right] \tag{4.12}$$

with Euclidean time periodically identified and the second is Schwarzschild-AdS

$$ds^2 = R^2 \left[\frac{dr^2}{f(r)} + f(r) d\tilde{\tau}^2 + r^2 d\Omega_{d-1}^2\right] \,, \tag{4.13}$$

where $f(r) = 1 + r^2 - \frac{m}{r^{d-2}}$. At the boundary $r = r_{max} \gg 1$, both solutions should be conformal to $S^1 \times S^{d-1}$ with the correct radii. Show that this fixes the periodicities

$$\Delta\tau = \frac{1}{TL} \frac{r_{max}}{\sqrt{1+r_{max}^2}} \,, \qquad \Delta\tilde{\tau} = \frac{1}{TL} \frac{r_{max}}{\sqrt{f(r_{max})}} \,. \tag{4.14}$$

Show also that regularity of the metric (4.13) implies the periodicity

$$\Delta\tilde{\tau} = \frac{4\pi}{f'(r_H)} = \frac{4\pi}{r_H d + \frac{d-2}{r_H}} \,, \tag{4.15}$$

where $r = r_H$ is the largest zero of $f(r)$. Notice that this implies a minimal temperature for Schwarzschild black holes in AdS, $T > \frac{\sqrt{d(d-2)}}{2\pi L}$.

Both (4.12) and (4.13) are stationary points of the Euclidean action (3.73). Therefore, we must compute the value of the on-shell action in order to decide which one dominates the path integral. Show that the difference of the on-shell actions is given by

$$I_{BH} - I_{AdS} = -2S_d \frac{R^{d-1}}{\ell_P^{d-1}} \left[r_{max}^d \Delta\tau - \left(r_{max}^d - r_H^d \right) \Delta\tilde{\tau} \right] \tag{4.16}$$

$$\longrightarrow S_d \frac{R^{d-1}}{\ell_P^{d-1}} \frac{1}{TL} r_H^{d-2}(1 - r_H^2) \tag{4.17}$$

where S_d is the area of a unit $(d-1)$-dimensional sphere and in the last step we took the limit $r_{max} \to \infty$. Conclude that the black hole only dominates the bulk path integral when $r_H > 1$, which corresponds to $T > \frac{d-1}{2\pi L}$. This is the Hawking-Page phase transition [65]. It is natural to set the free-energy of the AdS phase to zero because this phase corresponds to a gas of gravitons around the AdS background whose free energy does not scale with the large parameter $(R/\ell_P)^{d-1}$. Therefore, the free energy of the black hole phase is given by

$$F_{BH} = \frac{1}{L} S_d \frac{R^{d-1}}{\ell_P^{d-1}} r_H^{d-2}(1 - r_H^2) \ . \tag{4.18}$$

Verify that the thermodynamic relation $\frac{\partial F}{\partial T} = -S$ agrees with the Bekenstein-Hawking formula for the black hole entropy. Since this a first order phase transition you can also compute its latent heat.

In the last exercise, we saw that for a holographic CFT on a sphere of radius L, the entropy is a discontinuous function of the temperature. In fact, we found that for sufficiently high temperatures $T > \frac{d-1}{2\pi L}$, the entropy was very large $S \sim C_T$, while for lower temperatures the entropy was small because it did not scale with C_T. This can be interpreted as deconfinement of the numerous degrees of freedom measured by $C_T \gg 1$ which do not contribute to the entropy below the transition temperature $T_c = \frac{d-1}{2\pi L}$. How can this bevavior be understood from the point of view of a large N gauge CFT?

4.4 Applications

The AdS/CFT correspondence (or the gauge/gravity duality more generally) is a useful framework for thinking about strong coupling phenomena in QFT. Besides the specific examples of strongly coupled CFTs that can be studied in great detail using the gravitational dual description, AdS/CFT provides a geometric reformulation of many effects in QFT. Usually, we do not know the precise gravitational dual of a given QFT of interest (like QCD) but it is still very useful to study gravitational toy models that preserve the main features we are interested in. These models enlarge our intuition because they are very different from QFT models based on weakly interacting quasi-particles. There are many examples of QFT observables that have a nice geometric interpretation in the dual gravitational description. Perhaps the most striking one is the computation of entanglement entropy as the area of a minimal surface in the dual geometry [66]. Let us illustrate this approach in the context of confinig gauge theories like pure Yang-Mills theory.

Confinement means that the quark anti-quark potential between static quarks grows linearly with the distance L at large distances

$$V(L) \approx \sigma L \ , \qquad L \to \infty \ , \tag{4.19}$$

where σ is the tension of the flux tube or effective string. This potential can be defined through the expectation value of a Wilson loop (in the fundamental representation)

$$W[C] = \mathrm{T}r \ P \exp \oint_C A_\mu dx^\mu \ , \tag{4.20}$$

for a rectangular contour C with sides $T \times L$,

$$\langle W[C] \rangle \sim e^{-TV(L)} \ , \qquad T \to \infty \ . \tag{4.21}$$

This is equivalent to the area law $\langle W[C] \rangle \sim e^{-\sigma Area[C]}$ for large contours. In the gauge/string duality there is a simple geometric rule to compute expectation values of Wilson loops [67]. One should evaluate the path integral

$$\langle W[C] \rangle = \int_{\partial\Sigma = C} [d\Sigma] e^{-S_s[\Sigma]} \tag{4.22}$$

summing over all surfaces Σ in the dual geometry that end at the contour C at the boundary. The path integral is weighted using the dual string world-sheet action. At large N, we expect that the dominant contribution comes from surfaces Σ with disk topology. In specific examples, like SYM, this can be made very precise. For example, at large 't Hooft coupling the world-sheet action reduces to [3]

$$S_s[\Sigma] = \frac{1}{4\pi \ell_s^2} Area[\Sigma] \ . \tag{4.23}$$

In this case, since the theory is conformal, there is no confinement and the quark anti-quark potential is Coulomb like,

$$V(L) = \frac{a(N, \lambda)}{L} \ . \tag{4.24}$$

For most confining gauge theories (e.g. pure Yang-Mills theory) we do not know neither the dual geometry nor the dual string world-sheet action. However, we can get a nice qualitative picture if we assume (4.23) and only change the background geometry. The most general $(d+1)$-dimensional geometry that preserves d-dimensional Poincaré invariance can be written as

$$ds^2 = R^2 \left[\frac{dz^2}{z^2} + A^2(z) dx^\mu dx_\mu \right] \ . \tag{4.25}$$

The profile of the function $A^2(z)$ encodes many properties of the dual QFT. For a CFT, scale invariance fixes $A(z) \propto z^{-1}$. For asymptotically free gauge theories, we still expect

[3] In fact, the total area of Σ is infinite but the divergence comes from the region close to the boundary of AdS. This can be regulated by cutting of AdS at $z = \epsilon$, and renormalized by subtracting a divergent piece proportional to the length of the contour C.

that $A(z)$ diverges for $z \to 0$ however the function should be very different for larger values of z. In particular, it should have a minimum for some value $z = z_\star > 0$. Let us see what this implies for the expectation value of a large Wilson loop. The string path integral (4.22) will be dominated by the surface Σ with minimal area. For large contours C, this surface will sink inside AdS until the value $z = z_\star$ that minimizes $A^2(z)$ and the worldsheet area will be given by

$$R^2 A^2(z_\star) Area[C] + O(Length[C]) \; . \tag{4.26}$$

Therefore, we find a confining potential with flux tube tension

$$\sigma = \frac{A^2(z_\star)}{4\pi} \frac{R^2}{\ell_s^2} \; . \tag{4.27}$$

What happens if we put the QFT at finite temperature? In this case, we can probe confinement by computing

$$\left\langle W(C_x) \bar{W}(C_{x+L}) \right\rangle_\beta = e^{-\beta F_{q\bar{q}}(\beta,L)} \tag{4.28}$$

where C_x is the contour around the Euclidean time circle at the spatial position x (Polyakov loop). $F_{q\bar{q}}(\beta, L)$ denotes the free energy of a static quark anti-quark pair at distance L and temperature $1/\beta$. If $F_{q\bar{q}}(\beta, L) \to \infty$ as we separate the pair, then we are in the confined phase. On the other hand, if $F_{q\bar{q}}(\beta, L)$ remains finite when $L \to \infty$, we are in the deconfined phase. Let us see how this works in the holographic dual. For low temperatures, the dual geometry is simply given by (4.25) with Euclidean time identified with period β. Therefore, the bulk minimal surface that ends on C_x and C_{x+L} will have a cylindrical topology and its area will scale linearly with L at large L. In fact, we find $F_{q\bar{q}}(\beta, L) \approx \sigma L$ like in the vacuum. On the other hand, for high enough temperature we expect the bulk path integral to be dominated by a black hole geometry (see exercise 4.3.2 about Hawking-Page phase transition). The metric can then be written as

$$ds^2 = R^2 \left[\frac{dz^2}{z^2 f(z)} + f(z) d\tau^2 + g(z) dx^i dx_i \right] \; , \tag{4.29}$$

where $f(z)$ vanishes for some value $z = z_H$. This means that the Euclidean time circle is contractible in the bulk. Therefore, for large L, the minimal surface has two disconnected pieces with disk topology ending on C_x and C_{x+L} whose area remains finite when $L \to \infty$. This means deconfinement

$$\lim_{L\to\infty} \left\langle W(C_x) \bar{W}(C_{x+L}) \right\rangle_\beta = \langle W(C_x) \rangle_\beta^2 = e^{-2\beta F_q(\beta)} > 0 \; . \tag{4.30}$$

Another feature of a confining gauge theory is a mass gap and a discrete spectrum of mesons and glueballs. To compute this spectrum using the bulk dual one should study fluctuations around the vacuum geometry (4.25). Consider for simplicity, a scalar field obeying $\nabla^2 \phi = m^2 \phi$. Since we are interested in finding the spectrum of the operator $P_\mu P^\mu$ we look for solutions of the form $\phi = e^{ik\cdot x} \psi(z)$, which leads to

$$\frac{z}{A^d(z)} \partial_z \left(z A^d(z) \partial_z \psi \right) - \frac{k^2}{A^2(z)} \psi = m^2 R^2 \psi \; . \tag{4.31}$$

The main idea is that this equation will only have solutions that obey the boundary conditions $\psi(0) = \psi(\infty) = 0$ for special discrete values of k^2. In other words, we obtain a discrete mass spectrum as expected for a confining gauge theory.

Exercise 4.4.1 *Consider the simplest holographic model of a confining gauge theory: the hard wall model. This is just a slice of AdS, i.e. we take $A(z) = 1/z$ and cutoff space at $z = z_\star$. Show that (4.31) reduces to the Bessel equation*

$$\left[z^2\partial_z^2 + z\partial_z - \alpha^2 - k^2 z^2\right] h(z) = 0 \ , \tag{4.32}$$

where $\alpha^2 = m^2R^2 + d^2/4$ and $h(z) = z^{-\frac{d}{2}}\psi(z)$. Finally, show that the boundary conditions $h(0) = h(z_\star) = 0$, lead to the quantization

$$h_n(z) = J_\alpha\left(\frac{z}{z_\star}u_{\alpha,n}\right) \ , \qquad m_n^2 = -k^2 = \frac{u_{\alpha,n}^2}{z_\star^2} \ , \qquad n = 1, 2, \dots \tag{4.33}$$

where $u_{\alpha,n}$ is the nth zero of the Bessel function J_α.

It is instructive to compare the lightest glueball mass m_1 with the flux tube tension $\sigma = \frac{1}{4\pi z_\star^2}\frac{R^2}{\ell_s^2}$ in the hard wall model. We find that $\frac{\sigma}{m_1^2} \sim \frac{R^2}{\ell_s^2}$. The fact that this ratio is of order 1 in pure Yang-Mills theory is another indication that its holographic dual must be very stringy (curvature radius of the same order of the string length).

Above the deconfinement temperature, the system is described by a plasma of deconfined partons (quarks and gluons in QCD). The gauge/gravity duality is also very useful to describe this strongly coupled plasma. The idea is that the hydrodynamic behavior of the plasma is dual to the long wavelength fluctuations of the black hole horizon. This map can be made very precise and has led to significant developments in the theory of relativistic hydrodynamics. One important feature of the gravitational description is that dissipation is built in because black hole horizons naturally relax to equilibrium. A famous result from this line of work was the discovery of a universal ratio of shear viscosity η to entropy density s. Any CFT dual to Einstein gravity in AdS has $\frac{\eta}{s} = \frac{1}{4\pi}$. This is a rather small number (water at room temperature has $\frac{\eta}{s} \sim 30$) but remarkably it is of the same order of magnitude of that observed in the quark-gluon plasma produced in heavy ion collisions [68].

There are also many interesting applications of the gauge/gravity duality to Condensed Matter physics [15, 10]. There are many materials that are not well described by weakly coupled quasi-particles. In this case, it is useful to have alternative models based on gravitational theories in AdS that share the same qualitative features. This can give geometric intuition about the system in question.

The study of holographic models is also very useful for the discovery of general properties of CFT (and QFT more generally). If one observes that a given property holds both in weakly coupled and in holographic CFTs, it is natural to conjecture that such property holds in all CFTs. This reasoning has led to the discovery (and sometimes proof) of several important facts about CFTs, like the generalization of Zamolodchikov's c-theorem to $d > 2$ (known as F-theorem in $d = 3$ and a-theorem in $d = 4$) [69, 70, **?**, 71] or the existence of universal bounds on the three-point function of the stress tensor and its relation to the idea of energy correlators [72, 73, 74].

4.5 Universal long range forces

Another example along this line is the existence of "double-trace" operators with large spin in any CFT. The precise statement is that in the OPE of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ there is an infinite number of operators $\mathcal{O}_{n,l}$ of spin $l \gg 1$ and scaling dimension

$$\Delta_{n,l} \approx \Delta_1 + \Delta_2 + 2n + l + \frac{\gamma_n}{l^{\tau_{min}}} \tag{4.34}$$

where τ_{min} is the minimal twist (dimension minus spin) of all the operators that appear in both OPEs $\mathcal{O}_1 \times \mathcal{O}_1$ and $\mathcal{O}_2 \times \mathcal{O}_2$. In a generic CFT, this will be the stress tensor with $\tau_{min} = d - 2$ and one can derive explicit formulas for γ_n [75, 76, 77, 78]. This statement has been proven using the conformal bootstrap equations but its physical meaning is more intuitive in the dual AdS language. Consider two particle primary states in AdS. Without interactions the energy of such states is given by $\Delta_1 + \Delta_2 + 2n + l$ where $n = 0, 1, 2, \dots$ is a radial quantum number and l is the spin. Turning on interactions will change the energies of these two-particle states. However, the states with large spin and fixed n correspond to two particles orbitating each other at large distances and therefore they will suffer a small energy shift due to the gravitational long range force. At large spin, all other interactions (corresponding to operators with higher twist) give subdominant contributions to this energy shift. In other words, the general result (4.34) is the CFT reflection of the simple fact that interactions decay with distance in the dual AdS picture.

Chapter 5

Mellin amplitudes

Correlation functions of local operators in CFT are rather complicated functions of the cross-ratios. Since these are crucial observables in AdS/CFT it is useful to find simpler representations. This is the motivation to study Mellin amplitudes. They were introduced by G. Mack in 2009 [79, 80] following earlier work [81, 82]. Mellin amplitudes share many of the properties of scattering amplitudes of dual resonance models. In particular, they are crossing symmetric and have a simple analytic structure (related to the OPE). As we shall see, in the case of holographic CFTs, we can take this analogy further and obtain bulk flat space scattering amplitudes as a limit of the dual CFT Mellin amplitudes. Independently of AdS/CFT applications, Mellin amplitudes can be useful to describe CFTs in general.

5.1 Definition

Consider the n-point function of scalar primary operators [1]

$$\langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle = \int [d\gamma] M(\gamma_{ij}) \prod_{1 \le i < j \le n} \frac{\Gamma(\gamma_{ij})}{(-2P_i \cdot P_j)^{\gamma_{ij}}} \tag{5.1}$$

Conformal invariance requires weight $-\Delta_i$ in each P_i. This leads to constraints in the Mellin variables which can be conveniently written as

$$\sum_{j=1}^{n} \gamma_{ij} = 0 \ , \qquad \gamma_{ij} = \gamma_{ji} \ , \qquad \gamma_{ii} = -\Delta_i \ . \tag{5.2}$$

Notice that for $n = 2$ and $n = 3$ the Mellin variables are entirely fixed by these constraints. In these cases, there is no integral to do and the Mellin representation just gives the known form of the conformal two and three point function. The integration measure $[d\gamma]$ is over the $n(n-3)/2$ independent Mellin variables (including a factor of $\frac{1}{2\pi i}$ for each variable) and the integration contours run parallel to the imaginary axis. The precise contour in the complex plane is dictated by the requirement that it should pass to the right/left of the semi-infinite sequences of poles of the integrand that run to the left/right. This will become clear in the following example.

[1] We shall use the notation $M(\gamma_{ij})$ to denote a function $M(\gamma_{12}, \gamma_{13}, \dots)$ of all Mellin variables.

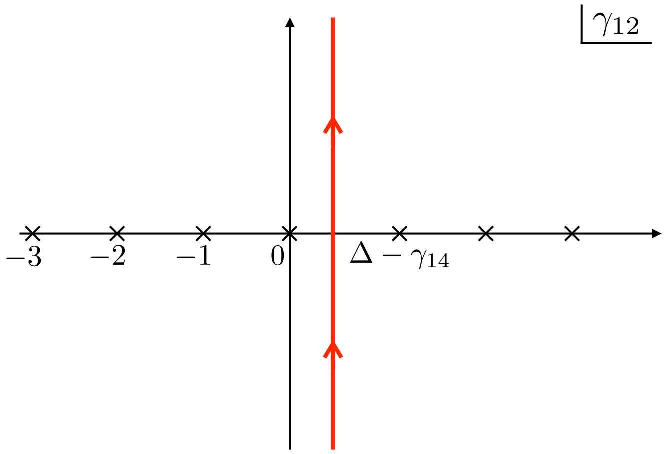


Figure 5.1 Integration contour for the Mellin variable γ_{12}. The crosses represent (double) poles of the Γ-functions given by (5.5) and (5.6). In general, the Mellin amplitude has several semi-infinite sequence of poles. Each sequence should stay entirely on one side of the contour.

Consider the case of a four-point function of a scalar operator of dimension Δ. In this case, there are two independent Mellin variables which we can choose to be γ_{12} and γ_{14}. This leads to

$$\langle \mathcal{O}(P_1)\ldots\mathcal{O}(P_4)\rangle = \frac{1}{(P_{13}P_{24})^{\Delta}}\int_{-i\infty}^{i\infty}\frac{d\gamma_{12}\gamma_{14}}{(2\pi i)^2}\hat{M}(\gamma_{12},\gamma_{14})u^{-\gamma_{12}}v^{-\gamma_{14}}\,, \tag{5.3}$$

where u and v are the cross ratios (2.39) and

$$\hat{M}(\gamma_{12},\gamma_{14}) = M(\gamma_{12},\gamma_{14})\Gamma^2(\gamma_{12})\Gamma^2(\gamma_{14})\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})\,. \tag{5.4}$$

Consider the first the complex plane of γ_{12} depicted in figure 5.1. The Γ-functions give rise to semi-infinite sequences of (double) poles at

$$\gamma_{12} = 0, -1, -2, \ldots \tag{5.5}$$
$$\gamma_{12} = \Delta-\gamma_{14}, \Delta-\gamma_{14}+1, \Delta-\gamma_{14}+2, \ldots \tag{5.6}$$

As we shall see in the next section, the Mellin amplitude $M(\gamma_{ij})$ also has the same type of semi-infinite sequences of poles. The integration contour should pass in the middle of these sequences of poles as shown in figure 5.1. Invariance of the four-point function under permutation of the insertion points P_i, leads to crossing symmetry of the Mellin amplitude

$$M(\gamma_{12},\gamma_{13},\gamma_{14}) = M(\gamma_{13},\gamma_{12},\gamma_{14}) = M(\gamma_{14},\gamma_{13},\gamma_{12})\ , \tag{5.7}$$

where we used 3 variables obeying a single constraint $\gamma_{12}+\gamma_{13}+\gamma_{14} = \Delta$. This is reminiscent of crossing symmetry of scattering amplitudes written in terms of Mandelstam invariants.

It is convenient to introduce fictitious momenta p_i such that $\gamma_{ij} = p_i \cdot p_j$. Imposing momentum conservation $\sum_{i=1}^{n} p_i = 0$ and the on-shell condition $p_i^2 = -\Delta_i$ automatically leads to the constraints (5.2). These fictitious momenta are a convenient trick but we do not know how to define them directly. In all formulas, we will only use their inner products $\gamma_{ij} = p_i \cdot p_j$. In particular, it is not clear in what vector space do the momenta p_i live. [2]

Let us be more precise about the number of independent cross ratios. The correct formula is

$$\frac{n(n-3)}{2}\,, \qquad n \le d+2 \tag{5.8}$$

$$nd - \frac{(d+1)(d+2)}{2}\,, \qquad n \ge d+2 \tag{5.9}$$

In fact, for $n > d+2$ one can write identities like

$$\det_{i,j} P_i \cdot P_j = 0 \tag{5.10}$$

using $d+3$ embedding space vectors. Notice that this makes the Mellin representation non-unique. We can shift the Mellin amplitude by the Mellin transform of

$$F(P_1, \dots, P_n) \det_{i,j} P_i \cdot P_j = 0 \tag{5.11}$$

where F is any scalar function with the appropriate homogeneity properties. This non-uniqueness of the Mellin amplitude is analogous to the non-uniqueness of the n-particle scattering amplitudes (as functions of the invariants $k_i \cdot k_j$) in $(d+1)$-dimensional spacetime if $n > d+2$.

5.2 OPE ⇒ Factorization

Consider the OPE

$$\mathcal{O}_1(x_1)\mathcal{O}_1(x_2) = \sum_k C_{12k} \left(x_{12}^2\right)^{\frac{\Delta_k - \Delta_1 - \Delta_2}{2}} \left[\mathcal{O}_k(x_2) + c\, x_{12}^2 \partial^2 \mathcal{O}_k(x_2) + \dots\right] \tag{5.12}$$

where the sum is over primary operators $\mathcal{O}_k$ and, for simplicity, we wrote the contribution of a scalar operator. The term proportional to the constant c is a descendant and is fixed by conformal symmetry like all the other terms represented by $\dots$. Let us compare this with the Mellin representation. When $x_{12}^2 \to 0$, it is convenient to integrate over γ_{12} closing the contour to the left in the γ_{12}-complex plane. This gives

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\dots\rangle = \sum_{\bar{\gamma}_{12}} \left(x_{12}^2\right)^{-\bar{\gamma}_{12}} \int [d\gamma]' \mathrm{R}es_{\bar{\gamma}_{12}} \hat{M}(\gamma_{ij}) \prod{}' \left(x_{ij}^2\right)^{-\gamma_{ij}} \tag{5.13}$$

where $[d\gamma]'$ and $\prod'$ stand for the integration measure and product excluding $ij = 12$. Comparing the two expressions we conclude that $\hat{M}$ must have poles at

$$\gamma_{12} = \frac{\Delta_1 + \Delta_2 - \Delta_k - 2m}{2}\,, \qquad m = 0, 1, 2, \dots \tag{5.14}$$

[2] The flat space limit of AdS discussed in section 5.3.2, suggests a $d+1$ dimensional space but this is unclear before the limit.

where the poles with $m > 0$ correspond to descendant contributions. If the CFT has a discrete spectrum of scaling dimensions then its Mellin amplitudes are analytic functions with single poles as its only singularities (meromorphic functions). It is also clear that the residues of these poles will be proportional to the product of the OPE coefficient C_{12k} and the Mellin amplitude of the lower point correlator $\langle \mathcal{O}_k \dots \rangle$. The precise formulas are derived in [79, 83]. Here we shall just list the main results without derivation.

5.2.1 Four-point function

In the case of the four-point function it is convenient to write the Mellin amplitude in terms of 'Mandelstam invariants'

$$s = -(p_1 + p_2)^2 = \Delta_1 + \Delta_2 - 2\gamma_{12} \tag{5.15}$$

$$t = -(p_1 + p_3)^2 = \Delta_1 + \Delta_3 - 2\gamma_{13} \tag{5.16}$$

Then, the poles and residues of the Mellin amplitude take the following form [79]

$$M(s,t) \approx C_{12k} C_{34k} \frac{Q_{l_k,m}(t)}{s - \Delta_k + l_k - 2m}\,, \qquad m = 0, 1, 2, \dots \tag{5.17}$$

where $Q_{l,m}(t)$ is a kinematical polynomial of degree l in the variable t.

This strengthens the analogy with scattering amplitudes. Each operator of spin l in the OPE $\mathcal{O}_1 \times \mathcal{O}_2$ gives rise to poles in the Mellin amplitude very similar to the poles in the scattering amplitude associated to the exchange of a particle of the same spin.

5.2.2 Planar correlators

Notice that the polynomial behaviour of the residues requires the inclusion of the Γ-functions in the definition (5.1) of Mellin amplitudes. On the other hand, the Γ-functions themselves have poles at fixed positions. For example, $\Gamma(\gamma_{12})$ gives rise to poles at $s = \Delta_1 + \Delta_2 + 2m$ with $m = 0, 1, 2, \dots$. In a generic CFT, there are no operators with these scaling dimensions and therefore the Mellin amplitude must have zeros at these values to cancel these unwanted OPE contributions. However, in correlation functions of single-trace operators in large N CFTs we expect precisely this type of contributions. At the planar level, the Γ-functions account for all multi-trace OPE contributions and the Mellin amplitude only has poles associated to single-trace operators.

5.2.3 n-point function

Considering the OPE of k scalar operators, one can derive more general factorization formulas [83]. For example, for each primary operator $\mathcal{O}$ of dimension Δ and spin l that appears in the OPEs $\mathcal{O}_1 \times \cdots \times \mathcal{O}_k$ and $\mathcal{O}_{k+1} \times \cdots \times \mathcal{O}_n$, we obtain the following sequence of poles in the n-point Mellin amplitude,

$$M_n \approx \frac{Q_m}{\gamma_{LR} - \Delta + l - 2m}\,, \qquad m = 0, 1, 2, \dots \tag{5.18}$$

where

$$\gamma_{LR} = -\left(\sum_{i=1}^{k} p_i\right)^2 = \sum_{i=1}^{k}\sum_{j>k}^{n} \gamma_{ij}\,. \tag{5.19}$$

In general, the residue can be written in terms of lower point Mellin amplitudes. For example, if $l = 0$ the residue factorizes

$$Q_0 = -2\Gamma(\Delta) M^L_{k+1}\, M^R_{n-k+1}\,, \tag{5.20}$$

with M^L_{k+1} the Mellin amplitude of $\langle \mathcal{O}_1 \dots \mathcal{O}_k \mathcal{O}\rangle$ and M^R_{n-k+1} the Mellin amplitude of $\langle \mathcal{O}\mathcal{O}_{k+1} \dots \mathcal{O}_n\rangle$. The satellite poles also factorize but give rise to more complicated formulae

$$Q_m = \frac{-2\Gamma(\Delta) m!}{\left(\Delta - \frac{d}{2} + 1\right)_m} L_m\, R_m\,, \tag{5.21}$$

with

$$L_m = \sum_{\substack{n_{ab}\geq 0 \\ \sum n_{ab}=m}} M^L(\gamma_{ab} + n_{ab}) \prod_{1\leq a<b\leq k} \frac{(\gamma_{ab})_{n_{ab}}}{n_{ab}!} \tag{5.22}$$

and similarly for R_m.

There also factorization formulas for the residues associated with operators with non-zero spin [83]. However, the general case including external operators with spin has not been worked out.

5.3 Holographic CFTs

As discussed in section 4.1, holographic CFTs have two special properties: large N factorization and a small number of low dimension single-trace operators. Therefore, one should expect that the corresponding Mellin amplitudes are particularly simple, at least at the planar level. We shall now confirm this expectation with a few simple examples.

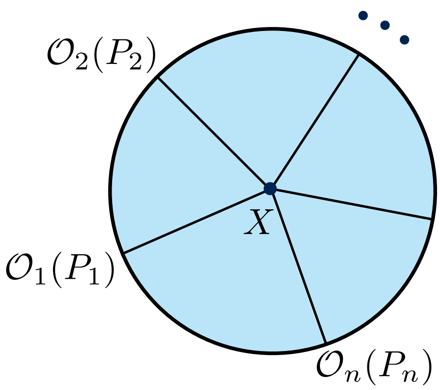


Figure 5.2 Witten diagram for a n-point contact interaction in AdS. The interior of the disk represents the bulk of AdS and the circumference represents its conformal boundary. The lines connecting the boundary points P_i to the bulk point X represent bulk to boundary propagators.

5.3.1 Witten diagrams

Consider the contact Witten diagram of figure 5.2. It corresponds to an interaction vertex $\lambda\phi_1 \dots \phi_n$ in the bulk lagrangian and it contributes [3]

$$\langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n)\rangle = \lambda \int_{AdS} dX \prod_{i=1}^{n} \frac{\sqrt{\mathcal{C}_{\Delta_i}}}{(-2P_i \cdot X)^{\Delta_i}} \tag{5.23}$$

to the dual CFT correlation function. One can show that this corresponds to a constant Mellin amplitude,

$$M = \lambda \frac{1}{2} \pi^{\frac{d}{2}} \Gamma\left(\frac{\sum \Delta_i - d}{2}\right) \prod_{i=1}^{n} \frac{\sqrt{\mathcal{C}_{\Delta_i}}}{\Gamma(\Delta_i)} \,. \tag{5.24}$$

Exercise 5.3.1 *Check the last statement. Start by using the integral representation of the bulk to boundary propagators and performing the integral over AdS using Poincare coordinates as explained in exercise 3.2.3. This turns (5.23) into*

$$\lambda\, \pi^{\frac{d}{2}} \Gamma\left(\frac{\sum \Delta_i - d}{2}\right) \int_0^\infty e^{-\sum_{i<j} s_i s_j P_{ij}} \prod_{i=1}^{n} \frac{\sqrt{\mathcal{C}_{\Delta_i}}}{\Gamma(\Delta_i)} s_i^{\Delta_i - 1} ds_i \,. \tag{5.25}$$

Next, use the Mellin representation ($c > 0$)

$$e^{-s_i s_j P_{ij}} = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma_{ij}}{2\pi i} \Gamma(\gamma_{ij}) (s_i s_j P_{ij})^{-\gamma_{ij}} \tag{5.26}$$

for $n(n-3)/2$ exponential factors. A good choice is to keep n factors, corresponding to the exponential

$$e^{-s_1 \sum_{i=2}^{n} s_i P_{1i} - s_2 s_3 P_{23}} \,. \tag{5.27}$$

The integrals over $s_4, \dots, s_n$ can be easily done in terms of Γ-functions. Finally, do the integrals over s_1, s_2, s_3 using the same type of change of variables as in exercise 3.2.3.

This result can be easily generalized to interaction vertices with derivatives. For example, the vertex $\lambda(\nabla_\alpha \phi_1 \nabla^\alpha \phi_2)\phi_3 \dots \phi_n$ gives rise to

$$\begin{aligned} \langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n)\rangle = \lambda \int_{AdS} dX \prod_{i=3}^{n} \frac{\sqrt{\mathcal{C}_{\Delta_i}}}{(-2P_i \cdot X)^{\Delta_i}} \times \\ \times (\eta^{AB} + X^A X^B) \frac{\partial}{\partial X^A} \frac{\sqrt{\mathcal{C}_{\Delta_1}}}{(-2P_1 \cdot X)^{\Delta_1}} \frac{\partial}{\partial X^B} \frac{\sqrt{\mathcal{C}_{\Delta_2}}}{(-2P_2 \cdot X)^{\Delta_2}} \,. \end{aligned} \tag{5.28}$$

Here we have used the fact that covariant derivatives in AdS can be computed as partial derivatives in the embedding space projected to the AdS sub-manifold.[4] This gives

$$\lambda \Delta_1 \Delta_2 \left(-2P_{12}\, D_{\Delta_1+1,\Delta_2+1,\Delta_3,\dots,\Delta_n} + D_{\Delta_1,\Delta_2,\Delta_3,\dots,\Delta_n}\right) \prod_{i=1}^{n} \sqrt{\mathcal{C}_{\Delta_i}} \tag{5.29}$$

[3] We are using CFT operators $\mathcal{O}_i$ normalized to have unit two point function.

[4] See appendix F.1 of [84] for a derivation of this statement in the analogous case of a sphere embedded in Euclidean space.

where we introduced the D-function [35]

$$D_{\Delta_1,\dots,\Delta_n} \equiv \int_{AdS} dX \prod_{i=1}^{n} \frac{1}{(-2P_i \cdot X)^{\Delta_i}} \,. \tag{5.30}$$

More generally, it is clear that the contact Witten diagram associated with a generic vertex $\lambda\nabla\dots\nabla\phi_1\nabla\dots\nabla\phi_2\dots\nabla\dots\nabla\phi_n$ with all derivatives contracted among different fields, gives rise to a linear combination of terms of the form

$$D_{\Delta_1+\Lambda_1,\dots,\Delta_n+\Lambda_n} \prod_{i<j}^{n} P_{ij}^{\lambda_{ij}} \tag{5.31}$$

where λ_{ij} are non-negative integers and $\Lambda_i = \sum_{j\neq i} \lambda_{ij}$. As we will see in the next exercise, the Mellin amplitude of (5.31) is a polynomial in the Mellin variables. Therefore, the Mellin amplitude associated to contact Witten diagrams is polynomial. The absence of poles in the Mellin amplitude means that the conformal block decomposition of the contact diagram only contains multi-trace operators, in agreement with previous results [85, 86].

Exercise 5.3.2 *If the vertex $\lambda\nabla\dots\nabla\phi_1\nabla\dots\nabla\phi_2\dots\nabla\dots\nabla\phi_n$ has $2N = 2\sum_{i<j}\alpha_{ij}$ derivatives with α_{ij} contractions of derivatives acting on ϕ_i and ϕ_j, show that the contact Witten diagram is given by*

$$\lambda \left(\prod_{i=1}^{n} \sqrt{\mathcal{C}_{\Delta_i}} \right) D_{\Delta_1+\Lambda_1,\dots,\Delta_n+\Lambda_n} \prod_{i<j}^{n} (-2P_{ij})^{\alpha_{ij}} + \dots \tag{5.32}$$

where $\Lambda_i = \sum_{j\neq i}\alpha_{ij}$ and the $\dots$ represent similar terms with less P_{ij} factors. Hint: use the trick of writing covariant derivatives in AdS as partial derivatives in the embedding space projected to the AdS sub-manifold.

The Mellin representation of the D-functions is very simple. As we saw in exercise 5.3.1, the Mellin amplitude associated to $D_{\Delta_1,\dots,\Delta_n}$ is simply

$$\frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{\sum\Delta_i - d}{2}\right)}{2\prod_{i=1}^{n}\Gamma(\Delta_i)} \,. \tag{5.33}$$

Show that the Mellin amplitude associated to the correlation function (5.32) is given by the polynomial

$$\lambda \left(\prod_{i=1}^{n} \sqrt{\mathcal{C}_{\Delta_i}} \right) \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{\sum\Delta_i+2N-d}{2}\right)}{2\prod_{i=1}^{n}\Gamma(\Delta_i+\Lambda_i)} \prod_{i<j}^{n} (-2\gamma_{ij})^{\alpha_{ij}} + \dots \tag{5.34}$$

where the $\dots$ represent terms of lower degree in γ_{ij}. Hint: this follows easily from shifting the integration variables in the Mellin representation (5.1).

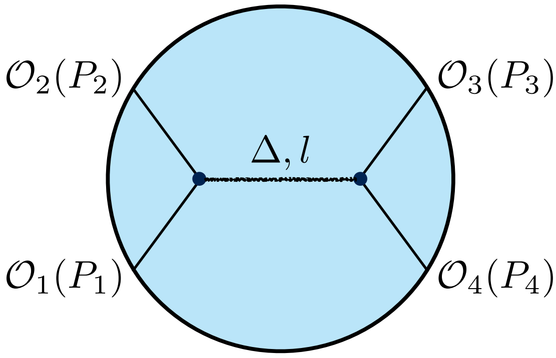


Figure 5.3 Witten diagram describing the exchange of a bulk field dual to an operator of dimension Δ and spin l.

Consider now the Witten diagram shown in figure 5.3 describing the exchange of a bulk field dual to a single-trace boundary operator $\mathcal{O}$ of dimension Δ and spin l. The conformal block decomposition of this diagram in the (12)(34) channel contains the single-trace operator $\mathcal{O}$ plus double-trace operators schematically of the form $\mathcal{O}_1(\partial^2)^n\partial_{\mu_1\dots\mu_j}\mathcal{O}_2$ and $\mathcal{O}_3(\partial^2)^n\partial_{\mu_1\dots\mu_j}\mathcal{O}_4$. Moreover, the OPE in the crossed channels only contains double-trace operators. This means that the Mellin amplitude is of the form

$$M = C_{12\mathcal{O}}C_{34\mathcal{O}}\sum_{m=0}^{\infty}\frac{Q_{l,m}(t)}{s-\Delta+l-2m} + R(s,t) \tag{5.35}$$

where the OPE coefficients $C_{12\mathcal{O}}$ and $C_{34\mathcal{O}}$ are proportional to the bulk cubic couplings and $R(s,t)$ is an analytic function. The residues are proportional to degree l Mack polynomials $Q_{l,m}(t)$ which are entirely fixed by conformal symmetry as we saw in 5.2.1. If we choose minimal coupling between the spin l bulk field and the external scalars, then $R(s,t)$ is a polynomial of degree $\leq l-1$. This is particularly simple in the case of a scalar exchange ($l=0$). Then the residues are independent of t and $R=0$ [87]. Notice that this simple looking Mellin amplitude gives rise to a rather involved function of the cross-ratios in position space. This example illustrates clearly the advantage of using the Mellin reprsentation to describe Witten diagrams.

The Mellin amplitude of a general tree-level scalar Witten diagrams was determined in [88, 89, 90, 91]. The final result can be summarized in the following Feynman rules:

- Associate a momentum p_j to every line (propagator) in the Witten diagram. External lines have incoming momentum p_i satisfying $-p_i^2=\Delta_i$. Momentum is conserved at every vertex of the diagram.

- Assign an integer m_j to every line. External lines have $m_i=0$.

- Every internal line (bulk-to-bulk propagator) contributes a factor[5]

$$\frac{S_{m_j}^{\Delta_j}}{p_j^2 + \Delta_j + 2m_j} \tag{5.37}$$

where Δ_j is the dimension of the propagating scalar field.

- Every vertex, $g\phi_1 \dots \phi_k$ joining k lines, contributes a factor[6]

$$g\, V_{m_1 \dots m_k}^{\Delta_1 \dots \Delta_k} \tag{5.38}$$

- Sum over all integers m_j associated with internal lines. Each sum runs from 0 to ∞.

- Multiply by

$$\mathcal{N} = \frac{\pi^{\frac{d}{2}}}{2} \prod_{i=1}^{n} \frac{\sqrt{\mathcal{C}_{\Delta_i}}}{\Gamma(\Delta_i)} \tag{5.39}$$

to get the n-point Mellin amplitude in our normalization of the external operators.

As an example, the Witten diagram in figure 5.4 gives rise to the following Mellin amplitude

$$\mathcal{N} \sum_{m_6=0}^{\infty} \sum_{m_7=0}^{\infty} V_{0\,0\,m_6}^{\Delta_1\Delta_2\Delta_6} \frac{S_{m_6}^{\Delta_6}}{p_6^2 + \Delta_6 + 2m_6} V_{m_6\,0\,m_7}^{\Delta_6\Delta_3\Delta_7} \frac{S_{m_7}^{\Delta_7}}{p_7^2 + \Delta_7 + 2m_7} V_{m_7\,0\,0}^{\Delta_7\Delta_4\Delta_5}$$

where $p_6^2 = (p_1 + p_2)^2 = 2\gamma_{12} - \Delta_1 - \Delta_2$ and $p_7^2 = (p_4 + p_5)^2 = 2\gamma_{45} - \Delta_4 - \Delta_5$. These Feynman rules suggest that we should think of the Mellin amplitude as an amputated amplitude because the bulk to boundary propagators do not contribute. In the case of scalar tree level diagrams (with non-derivative interaction vertices), the only dependence in the Mellin variables γ_{ij} comes from the bulk-to-bulk propagators. It is not known how to generalize these Feynman rules for loop diagrams or tree-level diagrams involving fields with spin. There are partial results in literature [89, 83] but nothing systematic. Mellin amplitudes are also useful in the context of weakly coupled CFTs. The associated Feynman rules for tree level diagrams were given in [92].

Exercise 5.3.3 *Consider the residue of the Mellin amplitude at the first pole ($m = 0$) associated to a bulk-to-bulk propagator. Show that the Feynman rules above are compatible with the factorization property (5.20) of this residue.* ***Extra:*** *check the factorization formula (5.21) for the satellite poles with $m > 0$.*

[5] The propagator numerator is given by

$$S_m^{\Delta} = \frac{\Gamma\left(\Delta - \frac{d}{2} + 1 + m\right)}{2(m!)\Gamma^2\left(\Delta - \frac{d}{2} + 1\right)}\,. \tag{5.36}$$

[6] The vertex factor is given by

$$V_{m_1 \dots m_k}^{\Delta_1 \dots \Delta_k} = \sum_{n_1=0}^{m_1} \cdots \sum_{n_k=0}^{m_k} \Gamma\left(\frac{\sum_j (\Delta_j + 2n_j) - d}{2}\right) \prod_{j=1}^{k} \frac{(-m_j)_{n_j}}{n_j!\left(\Delta_j - \frac{d}{2} + 1\right)_{n_j}}\,.$$

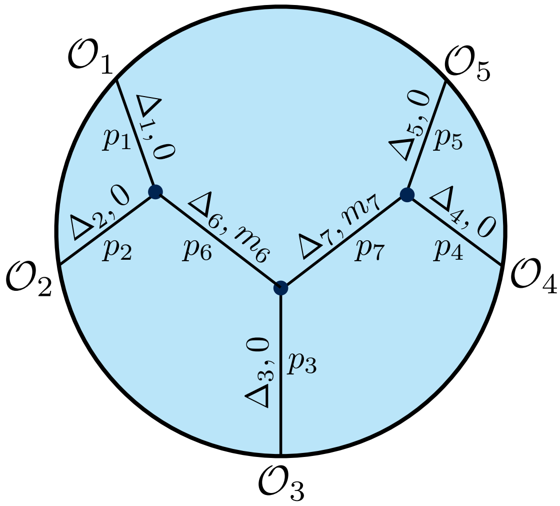


Figure 5.4 A tree level scalar Witten diagram contributing to a 5-point function. The auxiliary momenta p_i is conserved at each vertex, i.e. $p_6 = p_1 + p_2$ and $p_7 = p_4 + p_5$.

5.3.2 Flat space limit of AdS

If we consider a scattering process where all length scales are much smaller than the AdS radius R then the curvature effects should be negligible. Consider a relativistic invariant theory in flat spacetime with a characteristic length scale ℓ_s (this scale could come from a mass or from a dimensionful coupling). Then, a scattering amplitude $\mathcal{T}_n$ of n massless scalar particles in this theory will depend on ℓ_s and on the relativistic invariants $k_i \cdot k_j$, where k_i are the momenta of the external particles. On the other hand, this theory in AdS will give rise to Mellin amplitudes that depend on the dimensionless parameter $\theta = R/\ell_s$ and the Mellin variables γ_{ij}. We claim that these two quantities are related by

$$\frac{\mathcal{T}_n(\ell_s, k_i)}{\ell_s^{n\frac{d-1}{2}-d-1}} = \lim_{\theta\to\infty} \frac{1}{\mathcal{N}} \int_\Gamma \frac{d\alpha}{2\pi i} \alpha^{\frac{d-\sum \Delta_i}{2}} e^{\alpha} \frac{M_n\left(\theta, \gamma_{ij} = \frac{\theta^2}{2\alpha}\ell_s^2\, k_i \cdot k_j\right)}{\theta^{n\frac{1-d}{2}+d+1}} \tag{5.40}$$

where the contour Γ runs parallel to the imaginary axis and passes to the right of the branch point at $\alpha = 0$ and to the left of all poles of M_n. The powers of ℓ_s where introduced to make both sides of the equation dimensionless and the constant $\mathcal{N}$ was given in (5.39). The external particles are massless in flat space but in AdS they can have any scaling dimension Δ_i of order 1. We expect this equation to hold when both sides of the equation are well defined. In case the flat space scattering amplitude $\mathcal{T}_n$ is IR divergent, we expect that the limit $\theta \to \infty$ of the Mellin amplitude will not be finite.[7]

Exercise 5.3.4 *Consider the vertex $\lambda\nabla\ldots\nabla\phi_1\ldots\nabla\ldots\nabla\phi_n$ discussed in exercise 5.3.2 in $d+1$ spacetime dimensions. Start by writing the coupling constant λ as a power $(\ell_s)^q$ of a characteristic length scale ℓ_s and determine the value of q. Then, use the Mellin*

[7]It might be useful to think of large θ as an IR regulator for the scattering amplitude.

amplitude (5.34) *in the flat space limit formula* (5.40) *and obtain the expected n-particle scattering amplitude*

$$\mathcal{T}_n = \lambda \prod_{i<j} (-k_i \cdot k_j)^{\alpha_{ij}} \ . \tag{5.41}$$

The last exercise can be seen as a derivation of the flat space limit formula (5.40). The point is that a generic Feynman diagram can be written as a (infinite) sum of contact diagrams with any number of derivatives. This corresponds to integrating out the internal particles and replacing there effect by contact vertices among the external particles. Since formula (5.40) works for any contact diagram it should work in general. This has been tested in several explicit examples, including 1-loop diagrams [87, 88, 93]. In addition, the same formula was derived in [90] using a wave-packet construction where the scattering region was limited to a small flat region of AdS.

In principle, formula (5.40) provides a non-perturbative definition of string theory scattering amplitudes in terms of SYM correlation functions. However, we do not know how to directly compute SYM correlators at strong coupling. In practice, what we can do is to use formula (5.40) in the opposite direction, *i.e.* we can use known string scattering amplitudes in flat space to obtain information about the strong coupling expansion of SYM correlators [87, 94]. If the external particles are massive in flat space then formula (5.40) is not adequate. This case was studied in [95].

5.4 Open questions

The study of Mellin amplitudes is still very incomplete. Firstly, it is important to understand in what conditions do we have a well defined analytic Mellin amplitude. For example, in free CFTs the Mellin representation requires some form of regularization. This might be a technical detail but it would be useful to understand in general the status of the Mellin representation. Another important question is the asymptotic behavior of the Mellin amplitude when the Mellin variables are large. In the case of the four-point Mellin amplitude discussed in 5.2.1, the limit of large s with fixed t is called the Regge limit in analogy with flat space scattering amplitudes. In [96], we studied this limit using Regge theory techniques and making some reasonable assumptions about the large spin behaviour of the conformal partial amplitudes. Proving these assumptions is an important open question. The bound on chaos [97] is another possible approach to the Regge limit of Mellin amplitudes. Notice that if we can tame the asymptotic behaviour of $M(s,t)$ when $s \to \infty$, then we can write a dispersion relation that expresses $M(s,t)$ in terms of its poles in s, which are given by (5.17). This could provide a reformulation of the conformal bootstrap approach.

In the holographic context, it would be interesting to establish more general Feynman rules for Mellin amplitudes associated to Witten diagrams involving loops and particles with spin. It would also be useful to generalize more modern approaches to scattering amplitudes, like BCFW [98] or CHY [99], to Mellin amplitudes.

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