Classical Electrodynamics
EPFL

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Abstract

This a summary of the third year physics bachelor course on Classical Electrodynamics.

Contents

Introduction

These notes are a summary of the main results discussed in the course of Classical Electrodynamics at EPFL. The notes should be regarded as a skeleton of the course but most of the detailed arguments connecting the equations are missing. These are discussed in the live lectures. The lectures notes of the 2015 course by Riccardo Rattazzi (in french) and the following books are useful complementary reading for this course:

- *Modern electrodynamics*, Andrew Zangwill, Cambridge University Press 2013. ISBN-13: 978-0521896979
- *Classical electrodynamics*, John David Jackson, 1999. ISBN:978-0-471-30932-1
- *The Feynman lectures on Physics*, Feynman, Leighton, Sands, 1995. ISBN:2-10-004504-0
- *Théorie des champs*, L. Landau, E. Lifshitz (traduit du russe par Sergueï Medvédev),1999. ISBN:5-03-000641-9

Chapter 1

Maxwell equations and their physical consequences

1.1 Lecture 1 - Maxwell Equations

This course requires some mathematical background. In appendix A, we summarize some of the relevant concepts. This lecture makes heavy use of vector calculus.

The electromagnetic fields are determined in terms of the charge and current distributions by Maxwell equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{MI}$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{MII}$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{MIII}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0} \tag{MIV}$$

where

- $\mathbf{E}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are the electric and magnetic fields.
- $\rho(\mathbf{x})$ is the charge density and $\mathbf{J}(\mathbf{x})$ is the current density.
- $\mu_0 = 4\pi \times 10^{-7}$ N.A^{-2} is the magnetic permeability of the vacuum.
- $\varepsilon_0 = 8.85 \cdot 10^{-12} \, \frac{\text{A}\cdot\text{s}}{\text{V}\cdot\text{m}}$ is the electric permittivity of the vacuum.

The electromagnetic fields produce a Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \,, \tag{1.1}$$

on a particle of charge q and moving with velocity $\mathbf{v}$.

Some important consequences of Maxwell equations are:

1. **Coulomb force** [1]
   Consider electrostatics ($\mathbf{B} = 0$ and $\mathbf{J} = 0$),
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}\,, \qquad \nabla \wedge \mathbf{E} = \mathbf{0}\,. \tag{1.2}$$
   Using spherical symmetry and Gauss theorem we can easily derive the electric field produced by a particle of charge Q at the origin,
$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r^3}\mathbf{r}\,. \tag{1.3}$$
   Using the Lorentz force we derive Coulomb formula for the electric force between two charged particles
$$\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0 r^3}\mathbf{r}\,. \tag{1.4}$$
   Using linearity of Maxwell equations we can generalize to many point like charges,
$$\rho(\mathbf{r}) = \sum_i q_i \delta^{(3)}(\mathbf{r} - \mathbf{r}_i) \qquad \Rightarrow \qquad \mathbf{E}(\mathbf{r}) = \sum_i \frac{q_i}{4\pi\varepsilon_0} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}\,. \tag{1.5}$$

2. **Ampère's force law** [2]
   Consider magnetostatics ($\mathbf{E} = 0$ and $\rho = 0$),
$$\nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J}\,. \tag{1.6}$$
   Using Stokes theorem, we can easily conclude that the magnetic field produced by a current I along the z-axis is given by
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi} \frac{\mathbf{e}_z \wedge \mathbf{r}}{|\mathbf{e}_z \wedge \mathbf{r}|^2}\,. \tag{1.7}$$
   Using the Lorentz force, we derive Ampère's law for the force per unit length between two parallel wires at distance d,
$$\frac{\delta F}{\delta \ell} = -\frac{\mu_0 I I'}{2\pi d}\,. \tag{1.8}$$
   The force is attractive if the currents flow in the same direction.

3. **Faraday's law of induction** [3]
   Using Stokes theorem, the equation (MIII) can be written as
$$\mathcal{E} \equiv \oint_{\partial S} \mathbf{dl} \cdot \mathbf{E} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{d}\sigma = -\frac{\partial}{\partial t} \Phi_S\,, \tag{1.9}$$

[1] In Paris in 1785, Coulomb measured the force between charged objects, just a few years before the French revolution in 1789.

[2] In 1825 working in Paris. Ampère was originally from Lyon. He was an autodidact because his father was an admirer of Rousseau and his critique of formal schooling.

[3] Discovered in London in 1831, the same year James Clerk Maxwell was born. Faraday used this principle to develop electrical generators.

where Φ_S is the magnetic flux through the surface S. The time variation of the magnetic flux through the surface S generates an electromotive force $\mathcal{E}$ around the boundary ∂S. When ∂S corresponds to a metal wire (circuit) the electromotive force generates a current that produces a magnetic field that opposes the original variation of the magnetic flux (Lenz's law).

4. **Charge conservation**
   Combining the time derivative of (MI) with the divergence of (MII) we obtain the charge conservation equation [4]

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0\,. \tag{1.10}$$

   The integral form of this equation says that the variation of the total charge in a region V is due to charge flux through the boundary of the region:

$$\frac{\partial}{\partial t} Q = \frac{\partial}{\partial t} \int_V d^3x\, \rho(\mathbf{x}) = -\oint_{\partial V} \mathbf{J} \cdot \mathbf{dS}\,. \tag{1.11}$$

5. **Electromagnetic waves** [5]
   Consider the electromagnetic field in the vacuum ($\rho = 0$ and $\mathbf{J} = 0$). Taking the curl of (MIII) and the time derivative of (MII) we find the wave equation

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \triangle \mathbf{E}\,. \tag{1.12}$$

   One concludes that the speed of light is given by

$$c^2 = \frac{1}{\mu_0 \varepsilon_0}\,. \tag{1.13}$$

   The simple ansatz $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 f(\mathbf{e}_z \cdot \mathbf{r} - c\,t)$ representing a wave moving at the speed c solves the equation above.

6. The vacuum permittivity ε_0 is a useless concept. By changing units:

$$\rho, \mathbf{J} \to \sqrt{\varepsilon_0}\rho, \sqrt{\varepsilon_0}\mathbf{J}$$
$$\mathbf{E}, \mathbf{B} \to \frac{1}{\sqrt{\varepsilon_0}}\mathbf{E}, \frac{1}{\sqrt{\varepsilon_0}}\mathbf{B}$$

   one can eliminate ε_0 from Maxwell equations without changing the Lorentz force. This is the basis of the Gaussian system of units where the speed of light is the only constant appearing in Maxwell equations.

[4] Maxwell added (guessed) the last term to (MII) precisely for this consistency reason.

[5] Maxwell presented this argument in 1865 at the Royal Society in London. He obtained 3.1×10^8 m/s for the speed of light. This was not far from the measurements 2.4×10^7 m/s by Roemer in 1676 using the eclipses of Jupiter's moon Io or 3.15×10^8 m/s by Fizeau in 1848 using a rotating cogwheel.

1.1.1 Integral form of Maxwell equations

$$\oint_{\partial V} \mathbf{E} \cdot \mathbf{dS} = \frac{1}{\varepsilon_0} \int_V d^3x \rho(\mathbf{x}) = \frac{Q}{\varepsilon_0} \tag{1.14}$$

$$\underbrace{\oint_{\partial S} \mathbf{dl} \cdot \mathbf{B} = \mu_0 \int_S \mathbf{d}\boldsymbol{\sigma} \cdot \mathbf{J}}_{\text{Ampère's law}} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{d}\boldsymbol{\sigma} \cdot \mathbf{E} \tag{1.15}$$

$$\oint_{\partial S} \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt} \int_S \mathbf{d}\boldsymbol{\sigma} \cdot \mathbf{B} \tag{1.16}$$

$$\oint_S \mathbf{d}\boldsymbol{\sigma} \cdot \mathbf{B} = 0 \tag{1.17}$$

The first equation is *Gauss law*: the total charge (divided by ε_0) inside a region V is equal to the flux of $\mathbf{E}$ through the boundary ∂V of that region. The third equation is *Faraday's law of induction* and the fourth equation encodes the absence of magnetic monopoles.

1.2 Lecture 2 - Solving Maxwell Equations - statics

This lecture uses Fourier transforms and the Dirac δ-function.

1.2.1 Electromagnetic potentials

Equations (MIII) and (MIV) imply that we can write

$$\begin{cases} \mathbf{B} &= \nabla \wedge \mathbf{A} \\ \mathbf{E} &= -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \end{cases} \tag{1.18}$$

where Φ and $\mathbf{A}$ are the scalar and vector potentials. This is easy to show in Fourier space. Then, equations (MI) and (MII) become

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \triangle\Phi = \frac{1}{\varepsilon_0}\rho + \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) \tag{1.19}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \triangle\mathbf{A} = \mu_0\mathbf{J} - \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) \tag{1.20}$$

1.2.2 Gauge invariance

Φ and $\mathbf{A}$ are not uniquely fixed by the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. In fact, the *gauge transformation*

$$\begin{cases} \mathbf{A} &\longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\alpha \\ \Phi &\longrightarrow \Phi' = \Phi - \frac{\partial\alpha}{\partial t} \end{cases} \tag{1.21}$$

leaves $\mathbf{E}$ and $\mathbf{B}$ invariant. We can use this freedom to simplify equations (1.19) and (1.20). In particular, we can work in Lorenz gauge: [6]

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0\,. \tag{1.22}$$

The Lorenz gauge is Lorentz invariant, *i.e.* it is invariant under changes of inertial reference frame. In Lorenz gauge, equations (1.19) and (1.20) simplify to

$$\Box\Phi \equiv \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = \frac{\rho}{\varepsilon_0} \tag{1.23}$$

$$\Box\mathbf{A} \equiv \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J} \tag{1.24}$$

These are decoupled wave equations for Φ and $\mathbf{A}$. The symbol $\Box$ denotes the D'Alembertian operator.

[6] This gauge is named after the danish physicist Ludvig Lorenz (1829–1891). Not the same person as the dutch physicist Hendrik Antoon Lorentz (1853–1928) whose name is used in Lorentz transformations and the Lorentz force.

1.2.3 Electrostatics

In this case, we have $\mathbf{A} = \mathbf{B} = 0$ and $\mathbf{E} = -\nabla\Phi$, with Φ obeying *Poisson equation*, [7]

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}\,. \tag{1.25}$$

If all charges are explicitly known and $\lim_{|\mathbf{x}|\to\infty}\Phi(\mathbf{x}) = 0$ then the general solution is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,. \tag{1.26}$$

This corresponds to summing the Coulomb potential produced by each infinitesimal charge $dq' = d^3x'\ \rho(\mathbf{x}')$.

Green Function Method

In many systems (like conductors), not all charges are known but we are given non-trivial boundary conditions for Φ. The general solution of these problems can be found with the use of a *Green function*, [8] defined by

$$-\nabla^2_{\mathbf{x}}G(\mathbf{x},\mathbf{x}') = \delta^3(\mathbf{x}-\mathbf{x}')\,. \tag{1.27}$$

Some important comments:

- $G(\mathbf{x},\mathbf{x}')$ depends on the boundary conditions.
- If $\nabla^2_{\mathbf{x}}F(\mathbf{x},\mathbf{x}') = 0$ then $G(\mathbf{x},\mathbf{x}') + F(\mathbf{x},\mathbf{x}')$ is also a Green function. This freedom is related to the choice of boundary conditions.

Let us now consider general boundary conditions for some region V. The boundary of V may contain disjoint surfaces $\partial V = \cup_i S_i$. The goal is to determine the potential Φ given the charge density ρ inside V and boundary conditions on ∂V. The solution is unique for both Dirichlet (Φ is given at the boundary) and Newmann (the normal component of the electric field $\mathbf{n}\cdot\mathbf{E} = -\mathbf{n}\cdot\nabla\Phi$ is given at the boundary) boundary conditions. The basic equation is

$$\phi(\mathbf{x}) = \frac{1}{\varepsilon_0}\int_V d^3x'\,G(\mathbf{x}',\mathbf{x})\rho(\mathbf{x}') + \int_{\partial V}\mathbf{d}\sigma'\cdot\left[G(\mathbf{x}',\mathbf{x})\nabla\phi(\mathbf{x}') - \phi(\mathbf{x}')\nabla_{\mathbf{x}'}G(\mathbf{x}',\mathbf{x})\right] \tag{1.28}$$

[7] Poisson published this equation in 1813 in the context of the gravitational potential. Poisson had an interesting role in the development of the "wave theory of light". In 1818 the *Academie Royale des Sciencies de l'Institut de France* proposed a prize to explain diffraction of light. Fresnel (who was 30 years old) presented his thesis based on wave interference and the Young double slit experiment (1799). However, Poisson was a strong supporter of the "particle theory of light" and he decided to study Fresnel's work carefully to be able to prove him wrong. He concluded that Fresnel's theory predicted the existence of a bright spot on the axis of the shadow of a circular object. Poisson thought this was obviously absurd. Fortunately, the president of the jury, François Arago (who later became prime-minister of France), decided to do the experiment carefully and he was able to observe the predicted bright spot. Fresnel won the prize and the "particle theory of light" was dead until the beginning of the 20th century (Planck).

[8] George Green (1793-1841) worked in his father's wind mill and only attended one year of school when he was 8 years old. He was so bored with his job that he joined the subscription of Nottingham's library and learned a lot of Physics and Mathematics just from reading books. He published his essay in 1828 (with his own money). After that, he went to the University of Cambridge at the age of 39.

This equation follows from applying Gauss theorem to the vector field $\Phi\nabla G - G\nabla\Phi$. The best strategy to find the potential given some boundary conditions is to choose $G(\mathbf{x}', \mathbf{x}) = 0$ for $\mathbf{x}' \in S_i$ if Φ obeys Dirichlet b.c. at the surface S_i, and $\mathbf{n}\cdot\nabla_{\mathbf{x}'}G(\mathbf{x}', \mathbf{x}) = 0$ for $\mathbf{x}' \in S_j$ if Φ obeys Newmann b.c. at the surface S_j. [9] [10]

Applications: Determine the electrostatic potential created by conducting surfaces at different potentials. Compute surface charge distributions on conducting surfaces. Compute forces between bodies at different potentials.

There are several methods to construct Green functions, like the *image charge method* or expansions in eigenfunctions of the laplacian operator. The choice of method depends on the specific problem.

Image charge method

The basic idea is better explained with the example of a grounded plane at $z = 0$ in Cartesian coordinates. Suppose we are given the charge distribution $\rho(\mathbf{x})$ in the region V above the plane. Then the potential above the plane is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'_R|} \tag{1.30}$$

where $\mathbf{x}'_R = (x', y', -z')$ is the reflection of the point $\mathbf{x}' = (x', y', z')$ on the plane $z = 0$. This is the correct (and unique) solution because it satisfies Poisson equation (1.36) above the plane and the boundary condition $\Phi = 0$ on the grounded plane at $z = 0$. This idea can also be used to determine the Green function for this geometry,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \pm \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'_R|}\,, \tag{1.31}$$

where the sign $\pm$ corresponds to Neumann or Dirichlet type boundary conditions.

In practice, the image charge method is only useful for highly symmetric geometries like planes, spheres or cylinders.

Expansion in eigenfunctions

We would like to find the Green function G obeying [11]

$$\left[-\nabla^2_{\mathbf{x}} + U(\mathbf{x})\right] G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x}-\mathbf{x}')\,, \tag{1.32}$$

in a region V and Dirichlet or Neumann boundary conditions at the boundary ∂V. Firstly, we find the eigenfunctions and eigenvalues for this problem:

$$\left[-\nabla^2 + U(\mathbf{x})\right] \psi_n(\mathbf{x}) = \lambda_n\psi_n(\mathbf{x})\,, \tag{1.33}$$

[9] If Φ obeys Newmann b.c. over the entire boundary then the best we can do is to demand

$$\mathbf{n}\cdot\nabla_{\mathbf{x}'}G(\mathbf{x}, \mathbf{x}')\big|_{\partial V} = -\frac{1}{\text{Area}(\partial V)}\,. \tag{1.29}$$

[10] The Dirichlet Green function is symmetric $G(\mathbf{x}', \mathbf{x}) = G(\mathbf{x}, \mathbf{x}')$. The Newmann Green function can be chosen symmetric but it is not automatic.

[11] We added the term U to Poisson equation because the method is equally simple for this more general equation.

with $\psi_n(\mathbf{x})$ obeying the boundary conditions. For real $U(\mathbf{x})$, the eigenvalues are real and the eigenfunctions form a basis that can be chosen to obey

$$\int_V d^3x\, \psi_n^*(\mathbf{x})\psi_m(\mathbf{x}) = \delta_{nm}\,. \tag{1.34}$$

Then, we can write [12]

$$\delta^3(\mathbf{x}-\mathbf{x}') = \sum_n \psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})\,, \qquad G(\mathbf{x},\mathbf{x}') = \sum_n \frac{1}{\lambda_n}\psi_n^*(\mathbf{x}')\psi_n(\mathbf{x})\,. \tag{1.35}$$

1.2.4 Magnetostatics

In this case, we have $\mathbf{E} = 0$ and $\mathbf{B} = \nabla \times \mathbf{A}$, with $\mathbf{A}$ obeying

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}\,, \qquad \nabla\cdot\mathbf{A} = 0\,. \tag{1.36}$$

The first equation is just the *Poisson equation* for each component of $\mathbf{A}$ and can therefore be solved with the same methods discussed above for electrostatics. Notice that the two equations are compatible because $\nabla\cdot\mathbf{J} = 0$ in magnetostatics.

1.2.5 Conductors

The simplest model of a conductor is given by *Ohm's law*:

$$\mathbf{J} = \sigma\mathbf{E}\,, \tag{1.37}$$

where σ is the conductivity. Notice that this implies an energy loss per unit volume $\mathbf{J}\cdot\mathbf{E} = \sigma\mathbf{E}^2$ by *Joule heating*.

In electrostatics we have $\mathbf{J} = \mathbf{0}$ and therefore $\mathbf{E} = \mathbf{0}$ inside the conductor. This is achieved by an accumulation of free charge ρ_s at the surface of the conductor. In this situation there is an outward pressure at the surface of the conductor given by

$$p = \frac{1}{2}\rho_s|\mathbf{E}| = \frac{\rho_s^2}{2\varepsilon_0}\,. \tag{1.38}$$

[12] You can easily check that the first equation gives rise to the right formula $f(\mathbf{x}) = \int d^3x'\delta^3(\mathbf{x}-\mathbf{x}')f(\mathbf{x}')$.

1.3 Lecture 3 - Solving Maxwell equations - dynamics

This lecture uses complex analysis.

1.3.1 Electrodynamics

In the general case, we have the wave equations (with sources)

$$\Box\Phi = \frac{\rho}{\varepsilon_0} \tag{1.39}$$

$$\Box\mathbf{A} = \mu_0\mathbf{J} \tag{1.40}$$

The associated Green function satisfies

$$\Box_{(\mathbf{x},t)}G(\mathbf{x},t;\mathbf{x}',t') = \delta^3(\mathbf{x}-\mathbf{x}')\delta(t-t') \tag{1.41}$$

and

$$\Phi(\mathbf{x},t) = \frac{1}{\varepsilon_0}\int d^3x'dt'\, G(\mathbf{x},t;\mathbf{x}',t')\rho(\mathbf{x}',t')\,. \tag{1.42}$$

Again, the Green function is not unique and depends on the boundary conditions. In empty $\mathbb{R}^3$ there are two standard choices:

1. *Retarded Green function*

$$G_{\mathrm{R}}(\mathbf{x},t;\mathbf{x}',t') = \int \frac{d\omega d^3k}{(2\pi)^4}\,\frac{e^{-i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2-\frac{1}{c^2}(\omega+i\epsilon)^2} = \frac{\delta(t-t'-\frac{r}{c})}{4\pi r} \tag{1.43}$$

2. *Advanced Green function*

$$G_{\mathrm{A}}(\mathbf{x},t;\mathbf{x}',t') = \int \frac{d\omega d^3k}{(2\pi)^4}\,\frac{e^{-i\omega(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2-\frac{1}{c^2}(\omega-i\epsilon)^2} = \frac{\delta(t-t'+\frac{r}{c})}{4\pi r} \tag{1.44}$$

where $r = |\mathbf{x}-\mathbf{x}'|$ and $\epsilon = 0^+$ is an infinitesimal positive number.

1.3.2 Retarded potentials

Using the retarded Green function, we can write the general solution as follows

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho\left(\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} + \Phi_0 \tag{1.45}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}\left(\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)}{|\mathbf{x}-\mathbf{x}'|} + \mathbf{A}_0 \tag{1.46}$$

where $\Phi_0, \mathbf{A}_0$ are solutions to the homogeneous wave equation (no sources) in the Lorenz gauge,

$$\begin{aligned} \Box\Phi_0 &= 0 \\ \Box\mathbf{A}_0 &= \mathbf{0} \\ \nabla\cdot\mathbf{A}_0 + \frac{1}{c^2}\frac{\partial\Phi_0}{\partial t} &= 0\,. \end{aligned} \tag{1.47}$$

The solution without waves coming in from the past corresponds to $\Phi_0 = 0$ and $\mathbf{A}_0 = \mathbf{0}$.

1.4 Lecture 4 - Electromagnetic energy and electromagnetic waves

1.4.1 Electrostatic potential energy

The potential energy of a set of point like charges is given by

$$U = \sum_{\langle i,j \rangle} \frac{q_i q_j}{4\pi\varepsilon_0 |\mathbf{r}_i - \mathbf{r}_j|}\,, \tag{1.48}$$

where the sum runs over all pairs of charges. Taking the continuum limit, we derive the potential energy of a continuous charge distribution

$$U = \frac{1}{2}\int d^3x d^3y \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{4\pi\varepsilon_0|\mathbf{x}-\mathbf{y}|} = \frac{1}{2}\int d^3x \rho(\mathbf{x})\Phi(\mathbf{x})\,. \tag{1.49}$$

Using Maxwell equation (MI) and integrating by parts, we obtain the equivalent formula

$$U = \int d^3x\, u(\mathbf{x})\,, \qquad\qquad u = \frac{\varepsilon_0}{2}|\mathbf{E}|^2\,. \tag{1.50}$$

1.4.2 Poynting vector

Conservation of energy implies the following equation

$$\int_V d^3x\, \mathbf{J}\cdot\mathbf{E} + \frac{\partial}{\partial t}\int_V d^3x\, u + \int_{\partial V} \mathbf{d}\sigma\cdot\mathbf{S} = 0\,, \tag{1.51}$$

where the first term represents work done on the charges inside V, the second term represents the time variation of the electromagnetic energy inside V and the third term represents flow of electromagnetic energy through the boundary of V. Using the differential form of this equation and Maxwell equations one can derive that

$$u = \frac{\varepsilon_0}{2}|\mathbf{E}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2\,, \qquad\qquad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\wedge\mathbf{B}\,. \tag{1.52}$$

Actually, the form of the Poynting vector $\mathbf{S}$ is not uniquely fixed by (1.51) but this is the standard (simplest) form.

1.4.3 Plane waves

We would like to find solutions of the homogeneous wave equations (1.47). We will start by considering plane waves because any solution can be written as a linear combination of plane waves.

We consider the ansatz

$$\Phi(\mathbf{x},t) = \widehat{\Phi}(\mathbf{k})e^{-i\varphi} + \text{c.c.} \tag{1.53}$$

$$\mathbf{A}(\mathbf{x},t) = \widehat{\mathbf{A}}(\mathbf{k})e^{-i\varphi} + \text{c.c.} \tag{1.54}$$

where $\varphi \equiv \omega_k t - \mathbf{k}\cdot\mathbf{x}$. This automatically solves the wave equation if $\omega_k = c|\mathbf{k}|$. This means that the phase velocity is the speed of light,

$$v_{\text{phase}} = \frac{\Delta x}{\Delta t} = \frac{\omega_k}{|\mathbf{k}|} = c\,. \tag{1.55}$$

Imposing Lorenz gauge, we find

$$\widehat{\Phi} = \frac{c^2}{\omega_k}\mathbf{k}\cdot\widehat{\mathbf{A}} = \frac{c\mathbf{k}\cdot\widehat{\mathbf{A}}}{|\mathbf{k}|}\,. \tag{1.56}$$

The electromagnetic fields are then given by

$$\mathbf{E}(\mathbf{x},t) = \widehat{\mathbf{E}}(\mathbf{k})e^{-i\varphi} + \text{c.c.} \tag{1.57}$$

$$\mathbf{B}(\mathbf{x},t) = \widehat{\mathbf{B}}(\mathbf{k})e^{-i\varphi} + \text{c.c.} \tag{1.58}$$

with $\widehat{\mathbf{E}} = ic|\mathbf{k}|\widehat{\mathbf{A}}_\perp$, $\widehat{\mathbf{B}}_\alpha = i\epsilon_{\alpha\beta\gamma}\mathbf{k}_\beta(\widehat{\mathbf{A}}_\perp)_\gamma$ and $\widehat{\mathbf{A}}_\perp = \widehat{\mathbf{A}} - \frac{\mathbf{k}\cdot\widehat{\mathbf{A}}}{\mathbf{k}^2}\mathbf{k}$. We conclude that

$$\mathbf{E}\cdot\mathbf{k} = \mathbf{B}\cdot\mathbf{k} = \mathbf{E}\cdot\mathbf{B} = 0\,, \qquad \mathbf{B} = \frac{1}{c}\mathbf{n}\wedge\mathbf{E}\,, \qquad \mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}\,. \tag{1.59}$$

1.4.4 Mean values

The electromagnetic fields of a wave oscillate in time. Therefore, it is convenient to consider time averages

$$\langle f\rangle \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T dt f(t)\,. \tag{1.60}$$

For the plane electromagnetic wave discussed above, we find

$$\langle u\rangle = \varepsilon_0\left\langle \mathbf{E}^2\right\rangle = 2\varepsilon_0\widehat{\mathbf{E}}\cdot\widehat{\mathbf{E}}^*\,, \qquad \langle S\rangle = \langle u\rangle\, c\,\mathbf{n} = \langle u\rangle\,\mathbf{v}\,. \tag{1.61}$$

1.5 Lecture 5 - Liénard-Wiechert potentials

The goal of this lecture is to determine the EM fields produced by a charged particle performing arbitrary motion in empty space. We denote the particle's trajectory with $\mathbf{x}_0(t)$ and its velocity with $\mathbf{v}(t) = \dot{\mathbf{x}}_0(t)$. The associated charge and current densities are:

$$\begin{cases} \rho(\mathbf{x}, t) & = q\delta^3(\mathbf{x} - \mathbf{x}_0(t)) \\ \mathbf{J}(\mathbf{x}, t) & = q\mathbf{v}(t)\delta^3(\mathbf{x} - \mathbf{x}_0(t)) \end{cases} \tag{1.62}$$

One can easily check that this is consistent with charge conservation $\dot{\rho} + \nabla \cdot \mathbf{J} = 0$.

Using the retarded potentials we find

$$\begin{aligned} \Phi(\mathbf{x}, t) &= \frac{1}{4\pi\varepsilon_0} \int d^3x' dt' \, \frac{\rho(\mathbf{x}', t')\delta(t - t' - |\mathbf{x}' - \mathbf{x}|/c)}{|\mathbf{x}' - \mathbf{x}|} \\ &= \frac{q}{4\pi\varepsilon_0} \int dt' \, \frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}_0(t')|/c)}{|\mathbf{x} - \mathbf{x}_0(t')|} \end{aligned}$$

To compute the last integral, we define $f(t') = t - t' - |\mathbf{x} - \mathbf{x}_0(t')|/c$ and notice that

$$\frac{df}{dt'} = -1 + \frac{1}{c}\mathbf{n} \cdot \mathbf{v}(t') < 0 \,,$$

where we used $\mathbf{R} = \mathbf{x} - \mathbf{x}_0(t')$ and $\mathbf{n} = \mathbf{R}/|\mathbf{R}|$.

This leads to

$$\Phi(\mathbf{x}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|R - \mathbf{R} \cdot \boldsymbol{\beta}|} \tag{1.63}$$

$$\mathbf{A}(\mathbf{x}, t) = \frac{q}{4\pi\varepsilon_0 c} \frac{\boldsymbol{\beta}}{|R - \mathbf{R} \cdot \boldsymbol{\beta}|} \tag{1.64}$$

where the $\mathbf{x}$ and t dependence is implicit via (Landau-Lifschitz notation)

$$t' = t - \frac{R}{c} \,, \qquad \mathbf{R} = \mathbf{x} - \mathbf{x}_0(t') \,, \qquad R = |\mathbf{R}| \,, \qquad \boldsymbol{\beta} = \frac{\mathbf{v}(t')}{c} \,. \tag{1.65}$$

The EM fields are simply given by $\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \wedge \mathbf{A}$. However, the derivatives must be taken carefully taking into account the implicit dependence on $\mathbf{x}$ and t. For example,

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c}\nabla_{x_0} R \cdot \frac{\partial \mathbf{x}_0}{\partial t'}\frac{\partial t'}{\partial t} = 1 + \mathbf{n} \cdot \boldsymbol{\beta}\frac{\partial t'}{\partial t} \quad \Leftrightarrow \quad \frac{\partial t'}{\partial t} = \frac{1}{1 - \mathbf{n} \cdot \boldsymbol{\beta}} \tag{1.66}$$

The final result reads

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{1}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^3} \left((\mathbf{n} - \boldsymbol{\beta}) \left(1 - \beta^2\right) \frac{1}{R^2} + \mathbf{n} \wedge \left((\mathbf{n} - \boldsymbol{\beta}) \wedge \dot{\boldsymbol{\beta}} \right) \frac{1}{Rc} \right) \tag{1.67}$$

where $\mathbf{n} = \mathbf{R}/|\mathbf{R}|$ and

$$\mathbf{B} = \frac{1}{c}\mathbf{n} \wedge \mathbf{E} \tag{1.68}$$

The first term in (1.67) decays like $1/R^2$ and it is the (Lorentz contracted) Coulomb field of the particle. The second term decays as $1/R$ and it is proportional to the acceleration $\dot{\boldsymbol{\beta}}$ of the particle. This term represents EM radiation.

1.5.1 Radiation from a moving charge

The energy flux carried by EM radiation is obtained from the Poynting vector

$$\frac{d\mathcal{E}}{d\Omega dt} = \lim_{R\to\infty} R^2\,\mathbf{S}\cdot\mathbf{n} = \frac{1}{c\mu_0}\lim_{R\to\infty} R^2\,\mathbf{E}^2 = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\left(\mathbf{n}\wedge\left((\mathbf{n}-\boldsymbol{\beta})\wedge\dot{\boldsymbol{\beta}}\right)\right)^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^6} \tag{1.69}$$

Non-relativistic limit

Using $|\boldsymbol{\beta}| \ll 1$ we find

$$\frac{d\mathcal{E}}{d\Omega dt} = \frac{q^2}{16\pi^2\varepsilon_0 c}\left(\mathbf{n}\wedge\dot{\boldsymbol{\beta}}\right)^2 \tag{1.70}$$

and the total power is given by Larmor formula

$$\frac{d\mathcal{E}}{dt} = \int d\Omega\frac{d\mathcal{E}}{d\Omega dt} = \frac{q^2\dot{\mathbf{v}}^2}{16\pi^2\varepsilon_0 c^3}\int d\Omega\sin^2\theta = \frac{q^2\dot{\mathbf{v}}^2}{6\pi\varepsilon_0 c^3}\,. \tag{1.71}$$

Relativistic case

In this case, it is often more useful to know the radiated energy per unit time of the particle instead of per unit time of the distant observer. Using (1.66) this gives

$$\frac{d\mathcal{E}}{d\Omega dt'} = \frac{q^2}{16\pi^2\varepsilon_0 c}\frac{\left|\mathbf{n}\wedge\left((\mathbf{n}-\boldsymbol{\beta})\wedge\dot{\boldsymbol{\beta}}\right)\right|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} \tag{1.72}$$

Rectilinear motion

When the particle moves in a straight line ($\boldsymbol{\beta} \parallel \dot{\boldsymbol{\beta}}$) we can write

$$\frac{d\mathcal{E}}{d\Omega dt'} = \frac{q^2\dot{v}^2}{16\pi^2\varepsilon_0 c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{1.73}$$

where θ is the angle between the observation direction $\mathbf{n}$ and the acceleration $\dot{\boldsymbol{\beta}}$. The total power emitted is given by

$$\frac{d\mathcal{E}}{dt'} = \frac{q^2\dot{v}^2}{16\pi^2\varepsilon_0 c^3}2\pi\int_0^{2\pi}d\theta\,\frac{\sin^3\theta}{(1-\beta\cos\theta)^5} = \frac{q^2\dot{v}^2}{6\pi\varepsilon_0 c^3}\gamma^6\,, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}\,, \tag{1.74}$$

and it diverges when the velocity approaches the speed of light ($\beta \to 1$). Notice that this reduces to Larmor formula in the non-relativistic limit.

In the ultra-relativistic limit $\beta \to 1$, the radiation is emitted mostly at small angles relative to the line of motion of the particle. In this limit, we can approximate (1.73) by

$$\frac{d\mathcal{E}}{dt' d\Omega} = \frac{2}{\pi^2}\frac{q^2\dot{v}^2}{\varepsilon_0 c^3}\gamma^8\frac{(\theta\gamma)^2}{(1+(\theta\gamma)^2)^5} \tag{1.75}$$

which has a maximum for $\theta \simeq 1/(2\gamma)$ has depicted in figure 1.1.

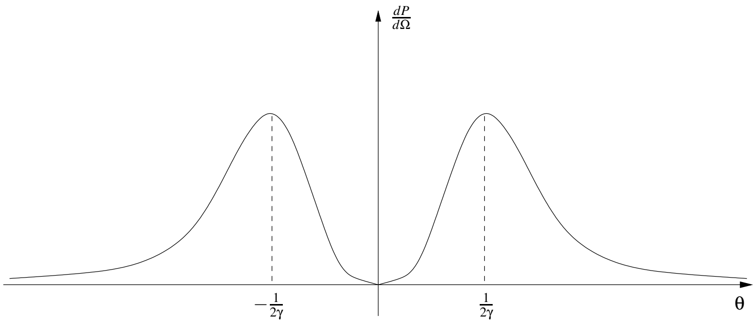

Figure 1.1 Angular distribution of the power emitted by a linearly accelerating charged particle.

1.6 Lecture 6 - Multipole expansion for static fields

The basic idea behind the multipole expansion is that any object can be described as a point if seen from very far away.

1.6.1 Electrostatics

Consider the potential produced by a charge distribution (see figure 1.2)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{1.76}$$

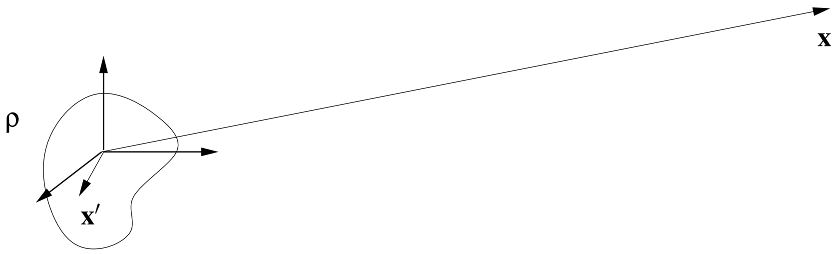

Figure 1.2 Object with charge density ρ. For $|\mathbf{x}|$ much larger than the size of the object we can use the multipole expansion.

We are interested in the potential at large distances. Therefore $x' \equiv |\mathbf{x}|' \ll x \equiv |\mathbf{x}|$ and we can Taylor expand

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} x'_{i_1} \dots x'_{i_n} \frac{\partial}{\partial x'_{i_1}} \cdots \frac{\partial}{\partial x'_{i_n}} \left. \frac{1}{|\mathbf{x}-\mathbf{x}'|} \right|_{\mathbf{x}'=0} \tag{1.77}$$

where the indices $i_1, i_2, \dots$ take values from 1 to 3 and are summed over using Einstein convention. Let us define the tensor

$$T_{i_1 \dots i_n}(\mathbf{x}) \equiv |\mathbf{x}|^{2n+1} \frac{\partial}{\partial x'_{i_1}} \cdots \frac{\partial}{\partial x'_{i_n}} \left. \frac{1}{|\mathbf{x}-\mathbf{x}'|} \right|_{\mathbf{x}'=0} \tag{1.78}$$

such that it is

- homogeneous

$$T_{i_1 \dots i_n}(\lambda \mathbf{x}) = \lambda^n T_{i_1 \dots i_n}(\mathbf{x}) \tag{1.79}$$

- symmetric

$$T_{i_1 \dots i_n}(\mathbf{x}) = T_{(i_1 \dots i_n)}(\mathbf{x}) \tag{1.80}$$

- traceless

$$\delta_{i_1 i_2} T_{i_1 \dots i_n}(\mathbf{x}) = 0 \tag{1.81}$$

In fact, we can write

$$T_{i_1 \dots i_n} = (2n-1)!! \, (x_{i_1} \dots x_{i_n}) - A_{i_1 \dots i_n} \tag{1.82}$$

where all terms in $A_{i_1 \dots i_n}$ contain at least one Kronecker-delta. The first multipoles are given in table 1.1.

n	name	analytic expression
0	monopole	$Q \;= \int d^3x \, \rho(\mathbf{x})$
1	dipole	$Q_i \;= \int d^3x \, \rho(\mathbf{x}) x_i$
2	quadripole	$Q_{ij} \;= \int d^3x \, \rho(\mathbf{x}) \left(3x_i x_j - x^2 \delta_{ij}\right)$
3	octopole	$Q_{ijk} = \dots$

Table 1.1 First terms in the multipole expansion.

The properties of $T_{i_1 \dots i_n}$ allows us to use

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x'_{i_1} \dots x'_{i_n} T_{i_1 \dots i_n}(\mathbf{x})}{|\mathbf{x}|^{2n+1}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x_{i_1} \dots x_{i_n}}{|\mathbf{x}|^{2n+1}} T_{i_1 \dots i_n}(\mathbf{x}') \tag{1.83}$$

in (1.76) and obtain

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{1}{n!} Q_{i_1 \dots i_n} \frac{x_{i_1} \dots x_{i_n}}{x^{2n+1}} = \sum_{n \geq 0} \Phi^{(n)} \tag{1.84}$$

where the multipoles are given by

$$Q_{i_1 \dots i_n} = \int d^3x' \rho(\mathbf{x}') T_{i_1 \dots i_n}(\mathbf{x}') \, . \tag{1.85}$$

For an object of size a and total charge q, the multipoles scale like $Q_{i_1 \dots i_n} \sim q \, a^n$ and therefore higher multipoles decay faster with the distance,

$$\Phi^{(n)} \sim \frac{q}{\varepsilon_0 x} \left(\frac{a}{x}\right)^n \, . \tag{1.86}$$

The associated electric field can be easily computed

$$\mathbf{E}^{(0)} = \frac{Q}{4\pi\varepsilon_0} \frac{\mathbf{r}}{r^3} \sim \frac{1}{r^2}$$

$$\mathbf{E}^{(1)} = \frac{1}{4\pi\varepsilon_0} \frac{3\mathbf{r}\left(\mathbf{Q}^{(1)} \cdot \mathbf{r}\right) - r^2 \mathbf{Q}^{(1)}}{r^5} \sim \frac{1}{r^3}$$

1.6.2 Magnetostatics

We can perform a similar expansion of the vector potential produced by a localized distribution of currents

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{1.87}$$

$$= \frac{\mu_0}{4\pi}\sum_{n\geq 0}\frac{1}{n!}\frac{x_{i_1}\dots x_{i_n}}{x^{2n+1}}\int d^3x'\,\mathbf{J}(\mathbf{x}')T_{i_1\dots i_n}(\mathbf{x}') = \sum_{n\geq 0}\mathbf{A}^{(n)}\,. \tag{1.88}$$

Using charge conservation and integration by parts one can show that $\mathbf{A}^{(0)} = 0$ and

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\left(\frac{\mathbf{m}\wedge\mathbf{x}}{|\mathbf{x}|^3} + \underbrace{\mathcal{O}(|\mathbf{x}|^{-3})}_{\text{quadrupole}}\right) \tag{1.89}$$

where the magnetic moment $\mathbf{m}$ is the integrated magnetization $\mathbf{M}$,

$$\mathbf{m} = \int d^3x\,\mathbf{M}(\mathbf{x})\,, \qquad \mathbf{M}(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}\wedge\mathbf{J}(\mathbf{x})\,. \tag{1.90}$$

1.7 Lecture 7 - Multipole expansion for EM radiation

1.7.1 Radiation fields

Consider the retarded potentials created by a localized source,

$$\Phi(\mathbf{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \tag{1.91}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \tag{1.92}$$

We are interested in the long distance behaviour of the fields that corresponds to the radiation emitted. For $x = |\mathbf{x}| \gg \max(a,\lambda)$, where a is the typical size of the source and λ is the wavelength of the emitted radiation, we can write

$$\begin{aligned}\Phi(\mathbf{x},t) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{x}\int d^3x'\,\rho(\mathbf{x}',t-x/c+\mathbf{n}\cdot\mathbf{x}'/c) + O(1/x^2)\\ \mathbf{A}(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\frac{1}{x}\int d^3x'\,\mathbf{J}(\mathbf{x}',t-x/c+\mathbf{n}\cdot\mathbf{x}'/c) + O(1/x^2)\end{aligned} \tag{1.93}$$

where $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$ is the propagation direction. This gives

$$\mathbf{B} = \frac{1}{c}\dot{\mathbf{A}}\wedge\mathbf{n} + O(1/x^2)\,, \qquad \mathbf{E} = c\,\mathbf{B}\wedge\mathbf{n} + O(1/x^2)\,. \tag{1.94}$$

This is sufficient to calculate the energy carried by the radiation per unit time per unit solid angle,

$$\frac{d\mathcal{E}}{dtd\Omega} = \lim_{x\to\infty} x^2\,\mathbf{n}\cdot\mathbf{S}(\mathbf{x},t) = \varepsilon_0 c\lim_{x\to\infty} x^2|\mathbf{n}\wedge\dot{\mathbf{A}}(\mathbf{x},t)|^2\,. \tag{1.95}$$

1.7.2 Slow source - dipole radiation

When the characteristic size a of the source is much smaller than the wave length λ of the radiation emitted we say the source is slow. This is equivalent to the statement that the period of the radiation is much shorter than the light crossing time a/c of the source. In this case, there are 3 different regions of interest:

- Quasi-static region: $|\mathbf{x}| \ll \lambda$
  Here we can write

  $$\Phi(\mathbf{x},t) \approx \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} \tag{1.96}$$

  $$\mathbf{A}(\mathbf{x},t) \approx \frac{\mu_0}{4\pi}\int d^3x'\,\frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|} \tag{1.97}$$

  so that t is just a parameter. We can use the multipole expansion of the static case if $|\mathbf{x}| \gg a$.

- Intermediate or induction region: $|\mathbf{x}| \sim \lambda$
  Here the multipole expansion is not useful but (1.93) can be used because $|\mathbf{x}| \gg a$.

- Radiation region: $|\mathbf{x}| \gg \lambda$
  Here we only care about the $O(1/x)$ part of the fields. As we shall see the multipole expansion is very useful in this region.

The multipole expansion for the radiation fields is obtained from

$$\mathbf{A} = \frac{\mu_0}{4\pi x} \sum_{n\geq 0} \frac{1}{n!} \int d^3x' \left(\frac{\mathbf{n}\cdot\mathbf{x}'}{c}\right)^n \partial_t^n \mathbf{J}(\mathbf{x}', t - x/c) = \sum_{n\geq 0} \mathbf{A}^{(n)}\,. \tag{1.98}$$

Notice that each term scales like $\mathbf{A}^{(n)} \sim (a/\lambda)^n$. We shall focus on the first term

$$\mathbf{A}^{(0)} = \frac{\mu_0}{4\pi x} \int d^3x' \, \mathbf{J}(\mathbf{x}', t - x/c) = \frac{\mu_0}{4\pi x} \dot{\mathbf{d}}(t - x/c)\,, \tag{1.99}$$

where we have introduced the dipole $\mathbf{d}(t) = \int d^3x \, \mathbf{x}\rho(\mathbf{x}, t)$. This leads to

$$\frac{d\mathcal{E}}{dtd\Omega} = \frac{1}{16\pi^2\varepsilon_0 c^3} |\ddot{\mathbf{d}}|^2 \sin^2(\theta)\,, \tag{1.100}$$

where θ is the angle between $\ddot{\mathbf{d}}$ and $\mathbf{n}$. Integrating over the solid angle, we find *Larmor formula*:

$$\frac{d\mathcal{E}}{dt} = \frac{1}{6\pi\varepsilon_0 c^3} |\ddot{\mathbf{d}}|^2\,. \tag{1.101}$$

1.8 Lecture 8 - Electromagnetic fields in medium

1.8.1 Macroscopic Maxwell equations

The Maxwell equations

$$\nabla \cdot \mathbf{e} = \frac{\eta}{\varepsilon_0} \tag{1.102}$$

$$\frac{1}{\varepsilon_0 \mu_0} \nabla \wedge \mathbf{b} - \frac{\partial \mathbf{e}}{\partial t} = \frac{\mathbf{j}}{\varepsilon_0} \tag{1.103}$$

$$\nabla \wedge \mathbf{e} + \frac{\partial \mathbf{b}}{\partial t} = \mathbf{0} \tag{1.104}$$

$$\nabla \cdot \mathbf{b} = 0 \tag{1.105}$$

govern the dynamics of the *microscopic* fields $\mathbf{e}$ and $\mathbf{b}$ in terms of the *microscopic* charge and current density, η and $\mathbf{j}$. Inside a real material the fields $\mathbf{e}$ and $\mathbf{b}$ are very complicated and vary significantly at the atomic scale $a \sim 10^{-10} m$. For most purposes we are not interested in these details. We are only interested in the fields averaged over regions containing many atoms. Therefore, we introduce the *macroscopic* fields:

$$\begin{cases} \mathbf{E}(\mathbf{z}) = \displaystyle\int d^3x \ \mathbf{e}(\mathbf{x}) f(\mathbf{x} - \mathbf{z}) = \langle \mathbf{e}(\mathbf{z}) \rangle \\ \mathbf{B}(\mathbf{z}) = \displaystyle\int d^3x \ \mathbf{b}(\mathbf{x}) f(\mathbf{x} - \mathbf{z}) = \langle \mathbf{b}(\mathbf{z}) \rangle \ , \end{cases} \tag{1.106}$$

where $f(\mathbf{x})$ is goes to zero for $|\mathbf{x}| \geq R$, with R some distance scale much larger than the atomic scale a and much smaller than the system under consideration. We choose $\int d^3x \, f(\mathbf{x}) = 1$ so that it corresponds to spatial averaging.

Using the fact that derivatives commute with spatial averaging we conclude that

$$\nabla \cdot \mathbf{E} = \frac{\langle \eta \rangle}{\varepsilon_0} \tag{1.107}$$

$$\frac{1}{\varepsilon_0 \mu_0} \nabla \wedge \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \frac{\langle \mathbf{j} \rangle}{\varepsilon_0} \tag{1.108}$$

$$\nabla \wedge \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{1.109}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.110}$$

In a material, the total charge density η is a sum of the charge density η_n of every particle or molecule,

$$\langle \eta(\mathbf{z}) \rangle = \sum_n \int d^3x \, \eta_n(\mathbf{x} - \mathbf{x}_n) f(\mathbf{x} - \mathbf{z}) \, ,$$

where $\mathbf{x}_n$ is the center of the n-th molecule. Taylor expanding $f(\mathbf{x} - \mathbf{z})$ at $\mathbf{x} = \mathbf{x}_n$, we find

$$\langle \eta(\mathbf{z}) \rangle = \rho(\mathbf{z}) - \nabla \cdot \mathbf{P}(\mathbf{z}) + \dots$$

where ρ is the free charge density and $\mathbf{P}$ is the polarization (or density of dipoles),

$$\rho(\mathbf{z}) = \sum_n q_n f(\mathbf{x_n} - \mathbf{z}), \qquad q_n = \int d^3x\, \eta_n(\mathbf{x}) \tag{1.111}$$

$$\mathbf{P}(\mathbf{z}) = \sum_n \mathbf{d}_n f(\mathbf{x}_n - \mathbf{z}), \qquad \mathbf{d}_n = \int d^3x\, \mathbf{x}\eta_n(\mathbf{x}) \tag{1.112}$$

and the dots stand for quadrupole (and higher) terms which are usually negligible.

Treating the current density $\mathbf{j}$ in a similar fashion we find

$$\begin{cases} \langle\eta\rangle = \rho - \nabla\cdot\mathbf{P} \\ \langle\mathbf{j}\rangle = \mathbf{J} + \dfrac{\partial\mathbf{P}}{\partial t} + \nabla\wedge\mathbf{M}\ , \end{cases} \tag{1.113}$$

where $\mathbf{J}$ is the free current density and the magnetization is given by

$$\mathbf{M}(\mathbf{z}) = \sum_n \mathbf{m}_n f(\mathbf{x}_n - \mathbf{z}), \qquad \mathbf{m}_n = \frac{1}{2}\int d^3x\, \mathbf{x}\wedge\mathbf{j}_n(\mathbf{x})\,. \tag{1.114}$$

Notice that conservation of the microscopic charge implies conservation of the free charge.

$$0 = \frac{\partial}{\partial t}\langle\eta\rangle + \nabla\cdot\langle\mathbf{j}\rangle = \frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{J}\,. \tag{1.115}$$

It is convenient to introduce

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \tag{1.116}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{1.117}$$

to obtain the *macroscopic* Maxwell equations

$$\nabla\cdot\mathbf{D} = \rho \tag{1.118}$$

$$\nabla\wedge\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{J} \tag{1.119}$$

$$\nabla\wedge\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = \mathbf{0} \tag{1.120}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{1.121}$$

1.8.2 Linear medium

In order to solve the macroscopic Maxwell equations we need to know how the polarization and magnetization of the material change under the influence of electromagnetic fields. Equivalently, we need a relation between $\mathbf{D}, \mathbf{H}$ and $\mathbf{E}, \mathbf{B}$. In many situations, a local and instantaneous linear relation is a good approximation

$$D_i = \varepsilon_{ij}E_j \tag{1.122}$$

$$B_i = \mu_{ij}H_j \tag{1.123}$$

where the permittivity tensor ε and the permeability tensor μ are characteristics of the material. If the material is isotropic, then

$$\varepsilon_{ij} = \varepsilon\delta_{ij} \,, \qquad \mu_{ij} = \mu\delta_{ij} \,, \tag{1.124}$$

which implies

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E} \,, \qquad \mathbf{M} = \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\mathbf{B} \,. \tag{1.125}$$

We can think of a *dielectric* as a collection of small dipoles in thermal equilibrium. This leads to

$$\varepsilon = \varepsilon_0 \left(1 + \frac{1}{3\epsilon_0}\frac{d^2 n}{k_B T}\right) > \varepsilon_0 \tag{1.126}$$

where T is the temperature, n is the number of dipoles per unit volume, d is the magnitude of each molecular dipole and we have assumed weak field $|\mathbf{E}| \ll k_B T/d$.

A similar calculation for a *paramagnet* yields

$$\mu = \frac{\mu_0}{1 - \frac{nm^2\mu_0}{3kT}} > \mu_0$$

where T is the temperature, n is the number of magnetic dipoles per unit volume, m is the magnitude of each magnetic dipole and we have assumed weak field $|\mathbf{B}| \ll k_B T/m$.

1.8.3 Electromagnetic energy

Conservation of energy leads to

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} + \mathbf{E} \cdot \langle \mathbf{j} \rangle = 0 \,, \tag{1.127}$$

where the energy density u and the Poynting vector $\mathbf{S}$ are given by (1.52). Notice that using (1.107)-(1.110), we can derive this formula in exactly the same way as we did in vacuum. In a continuous medium, it is convenient to split the work done on the *free* charges from the work done on the bound charges encoded in $\mathbf{P}$ and $\mathbf{M}$. Using (1.113) and (1.109) we can write

$$\mathbf{E} \cdot \langle \mathbf{j} \rangle = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{M} \wedge \mathbf{E}) \,, \tag{1.128}$$

which allows us to rewrite the energy conservation equation as

$$\frac{\partial u_{\text{eff}}}{\partial t} + \nabla \cdot \mathbf{S}_{\text{eff}} + \mathbf{E} \cdot \mathbf{J} = 0 \,, \tag{1.129}$$

with

$$\mathbf{S}_{\text{eff}} = \mathbf{S} + \mathbf{M} \wedge \mathbf{E} = \mathbf{E} \wedge \mathbf{H} \tag{1.130}$$

$$du_{\text{eff}} = du + \mathbf{E} \cdot d\mathbf{P} - \mathbf{M} \cdot d\mathbf{B} \,. \tag{1.131}$$

In a linear medium we can easily integrate the last equation to find

$$u_{\text{eff}} = u + \frac{1}{2}\mathbf{E} \cdot \mathbf{P} - \frac{1}{2}\mathbf{M} \cdot \mathbf{B} = \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}\right) \,. \tag{1.132}$$

1.8.4 Interface conditions

Sometimes one needs to determine the electromagnetic fields in a system with more than one medium. This requires that we know how the fields change at the interface between two media. Such interfaces conditions can be easily derived by considering the macroscopic Maxwell equations in integral form

$$\int_{\partial V} \mathbf{d}\sigma \cdot \mathbf{D} = \int_V \rho(\mathbf{x}) d^3x \tag{1.133}$$

$$\int_{\partial V} \mathbf{d}\sigma \cdot \mathbf{B} = 0 \tag{1.134}$$

$$\int_{\partial S} \boldsymbol{d\ell} \cdot \mathbf{E} = -\frac{d}{dt} \int_S \mathbf{d}\sigma \cdot \mathbf{B} \tag{1.135}$$

$$\int_{\partial S} \boldsymbol{d\ell} \cdot \mathbf{H} = \int_S \mathbf{d}\sigma \cdot \mathbf{J} + \frac{\partial}{\partial t} \int_S \mathbf{d}\sigma \cdot \mathbf{D} \tag{1.136}$$

The idea is to apply these equations to a small volume V and a small surface S at the interface as shown in figure 1.3.

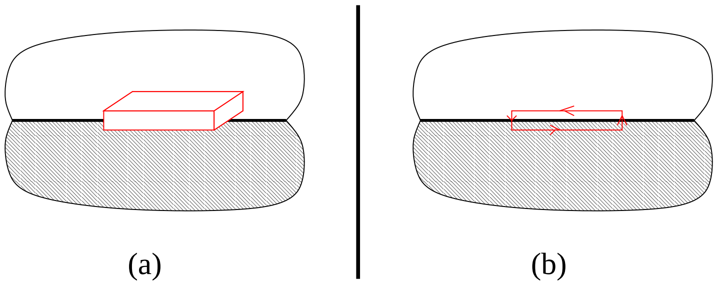

Figure 1.3 (a) Small volume V at the interface. (b) Small surface S at the interface.

This leads to the following interface conditions:

$$(\mathbf{D}_2 - \mathbf{D}_1)_\perp = \rho_s \, \mathbf{n} \tag{1.137}$$

$$(\mathbf{B}_2 - \mathbf{B}_1)_\perp = 0 \tag{1.138}$$

$$(\mathbf{E}_2 - \mathbf{E}_1)_{||} = 0 \tag{1.139}$$

$$(\mathbf{H}_2 - \mathbf{H}_1)_{||} = \mathbf{J}_s \wedge \mathbf{n} \tag{1.140}$$

where we have used the unit vector $\mathbf{n}$ normal to the interface and pointing from medium 1 to medium 2. The symbols $\perp$ and $||$ stand for the components of a vector normal and parallel to the interface. Finally, ρ_s is the free surface charge density at the interface and $\mathbf{J}_s$ is the free surface current at the interface.

1.9 Lecture 9 - EM waves in continuous media

We consider an isotropic linear medium,

$$\mathbf{D} = \epsilon \mathbf{E}\,, \qquad \mathbf{B} = \mu \mathbf{H}\,. \tag{1.141}$$

In the absence of free charges ($\rho = 0$ and $\mathbf{J} = \mathbf{0}$), the macroscopic Maxwell equations imply the wave equation

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - \frac{1}{\epsilon\mu}\Delta \mathbf{B} = 0\,. \tag{1.142}$$

We conclude that the phase velocity of light in a medium is given by

$$v^2 = \frac{1}{\epsilon\mu} = c^2 \frac{\epsilon_0 \mu_0}{\epsilon\mu} = c^2 \frac{1}{n^2}\,, \tag{1.143}$$

where we have introduced the index of refraction n.

1.9.1 Plane waves

Similarly to the vacuum case, we can write

$$\mathbf{E} = \mathbf{E}_0 \exp[i\mathbf{k}\cdot\mathbf{x} - i\omega t] + \text{c.c.} \tag{1.144}$$

where $\mathbf{k}$ is the wave vector and $\omega = v|\mathbf{k}|$ is the frequency. Using Maxwell equations we conclude that the magnetic field is given by

$$\mathbf{B} = \frac{1}{\omega}\mathbf{k} \times \mathbf{E}\,. \tag{1.145}$$

1.9.2 Wave reflection

Consider the reflection and refraction of a wave through a simple interface with no free charges ($\sigma = 0$ and $\mathbf{K} = \mathbf{0}$). We shall use the following notation (see figure 1.4)

i) incident wave : $E_0,\ \mathbf{k},\ \omega,$

ii) transmitted wave : $E_0',\ \mathbf{k}',\ \omega',$

iii) reflected wave : $E_0'',\ \mathbf{k}'',\ \omega''.$

Continuity in time implies that $\omega = \omega' = \omega''$. Continuity along the interface implies that

$$(\mathbf{k})_{\parallel} = \left(\mathbf{k}'\right)_{\parallel} = \left(\mathbf{k}''\right)_{\parallel}\,, \tag{1.146}$$

which means that the three vectors $\mathbf{k},\ \mathbf{k}',\ \mathbf{k}''$ lay in the same plane. This also implies that the reflection angle is equal to the incident angle $i'' = i$, that $k'' \equiv |\mathbf{k}''| = |\mathbf{k}| \equiv k$ and *Snell's law*:

$$\frac{\sin r}{\sin i} = \frac{n_1}{n_2}\,. \tag{1.147}$$

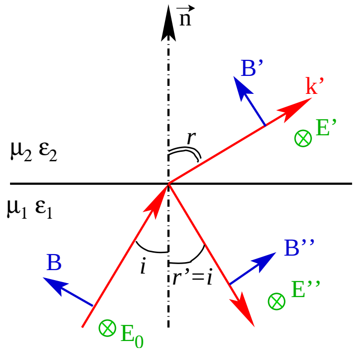

Figure 1.4 Wave reflection and transmission at the interface between two media.

Notice that his equation has no solution for $n_1 > n_2$ if

$$i > i_0 = \arcsin\left(\frac{n_2}{n_1}\right) .$$

This corresponds to the phenomenon of *total reflection* for $i \geq i_0$.

Using the interface conditions (1.137)-(1.140) we can determine the amplitude and polarization of the transmitted and reflected waves in terms of the incident wave. For simplicity, we consider the case where the electric field of the incident wave is parallel to the interface (TE polarization). In this case, we find the following *Fresnel relations*

$$\frac{E_0'}{E_0} = \frac{2n_1 \cos i}{n_1 \cos i + \frac{\mu_1}{\mu_2} n_2 \cos r}$$
$$\frac{E_0''}{E_0} = \frac{n_1 \cos i - \frac{\mu_1}{\mu_2} n_2 \cos r}{n_1 \cos i + \frac{\mu_1}{\mu_2} n_2 \cos r}$$

One can check that this is compatible with energy conservation, which implies continuity of the normal component of the Poynting vector,

$$\frac{kE_0^2}{\mu_1} \cos i - \frac{k''(E_0'')^2}{\mu_1} \cos i'' = \frac{k'(E_0')^2}{\mu_2} \cos r \,. \tag{1.148}$$

Chapter 2

Special Relativity

The symmetries of Newtonian classical mechanics are different from the symmetries of Maxwell equations. This fact puts Galileo's Principle of Relativity at stake. It suggests that one might be able to measure absolute velocities (relative to the *luminiferous ether*) by performing electromagnetic experiments. However, the experiments of Michelson-Morley showed no such effect. The solution was proposed by Einstein in 1905: one should correct Newtonian mechanics so that it is Lorentz invariant like Maxwell equations.

2.1 Lecture 10 - Symmetries of Newton and Maxwell equations

2.1.1 Symmetries of Newtonian mechanics

Consider a system with n particles with pairwise interactions. The lagrangian reads

$$L = \frac{1}{2}\sum_{I=1}^{n} m_I \dot{\mathbf{x}}_I^2 - \underbrace{\sum_{I\neq J} U_{IJ}(|\mathbf{x}_I - \mathbf{x}_J|)}_{V} \tag{2.1}$$

and the equations of motion are

$$m_I \ddot{\mathbf{x}}_I = -\nabla_{\mathbf{x}_I} V = -\sum_{J\neq I} U'_{IJ}(|\mathbf{x}_I - \mathbf{x}_J|)\frac{\mathbf{x}_I - \mathbf{x}_J}{|\mathbf{x}_I - \mathbf{x}_J|}\,, \qquad \forall I \in 1,\ldots,n \tag{2.2}$$

These equations are invariant under the transformations

$$\begin{cases} x_I^i & \longrightarrow x_I'^i = R^{ij} x_I^j + a^i + v^i t \\ t & \longrightarrow t' = t + t_0 \end{cases} \tag{2.3}$$

where t_0 and a^i represent time and space translations, v^i is a Galilean boost and the tensor R^{ij} implements a spatial rotation (or reflection). The group of rotations (and reflections) in $\mathbb{R}^3$ is defined by [1]

$$O(3) = \left\{ R \in \mathrm{GL}(3,\mathbb{R}) : R^t R = \mathbb{I} = R R^t \right\} \tag{2.4}$$

[1] The notation $GL(3,\mathbb{R})$ stands for the group of 3×3 real matrices.

The matrix condition $R^t R = \mathbb{I} = RR^t$ is equivalent to the statement $R^{ij}R^{ik} = \delta^{jk}$, which means that rotations preserve the Euclidean inner product

$$a \cdot b \to a' \cdot b' = a'^i b'^i = R^{ij}R^{ik}a^j b^k = \delta^{jk}a^j b^k = a \cdot b\,. \tag{2.5}$$

A general rotation is parametrized by 3 angles (Euler angles). Therefore, the group of symmetries of Newtonian mechanics has 10 parameters (4 translations, 3 rotations and 3 Galilean boosts).

2.1.2 Symmetries of Maxwell equations

To state the symmetries of Maxwell equations it is convenient to define a 4-vector

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{2.6}$$

which we denote by x^μ with $\mu = 0, 1, 2, 3$. Maxwell equations are invariant under the Poincaré transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{2.7}$$

where a^μ represents a translation and the matrix $\Lambda^\mu{}_\nu$ implements a general Lorentz transformation. The group of Lorentz transformations is

$$O(1,3) = \left\{\Lambda \in \mathrm{GL}(4,\mathbb{R}) : \Lambda G \Lambda^t = G\right\} \tag{2.8}$$

where $G = \mathrm{diag}(1,-1,-1,-1)$ is called the Minkowski metric. It is convenient to write the matrix condition in Einstein notation [2]

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta G^{\alpha\beta} = G^{\mu\nu}\,, \qquad \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta G_{\mu\nu} = G_{\alpha\beta}\,. \tag{2.9}$$

It is easy to see that the wave equation is invariant under Lorentz transformations. First, we consider the transformation of derivatives

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu} = \Lambda^\nu{}_\mu \partial'_\nu \tag{2.10}$$

Then, the invariance of the d'Alembertian follows

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = G^{\mu\nu}\partial_\mu\partial_\nu = G^{\mu\nu}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \partial'_\alpha \partial'_\beta = G^{\alpha\beta}\partial'_\alpha\partial'_\beta = \Box' \tag{2.11}$$

Lorentz transformations include spatial rotations [3] and Lorentz boosts. A Lorentz boost

$$\begin{cases} ct' &= \gamma(ct - \boldsymbol{\beta}\cdot\mathbf{x}) \approx ct \\ \mathbf{x}'_\parallel &= \gamma(\mathbf{x}_\parallel - \boldsymbol{\beta} ct) \approx \mathbf{x}_\parallel - \mathbf{v}t \\ \mathbf{x}'_\perp &= \mathbf{x}_\perp \end{cases} \tag{2.12}$$

[2] Notice that the matrix equation $\Lambda G \Lambda^t = G$ is equivalent to $\Lambda^t G^{-1}\Lambda = G^{-1}$. The matrices G^{-1} and G are identically but it is convenient to write G with indices up and G^{-1} with indices down.

[3] A rotation R corresponds to a Lorentz transformation with $\Lambda^0{}_0 = 1$, $\Lambda^0{}_i = \Lambda^i{}_0 = 0$ and $\Lambda^i{}_j = R^{ij}$ for $i, j \in \{1,2,3\}$.

is the coordinate transformation that corresponds to a reference frame moving with velocity $\mathbf{v} = c\boldsymbol{\beta}$. In this expression, $\mathbf{x}_{\parallel}/\mathbf{x}_{\perp}$ stands for the parallel/orthogonal component to the velocity $\mathbf{v}$ and $\gamma = 1/\sqrt{1-\beta^2}$. Sometimes it is convenient to parametrize the Lorentz boost by the rapidity χ defined by

$$\beta = \tanh\chi\,, \qquad \gamma = \cosh\chi\,. \tag{2.13}$$

Notice that Lorentz boosts reduce to Galilean boosts when $|\mathbf{v}| \ll c$. This is important because we should recover Newtonian mechanics in the limit of non-relativistic speeds.

In order to check Lorentz invariance of Maxwell equations it is convenient to introduce the 4-vectors

$$A^\mu \equiv \begin{pmatrix}\Phi \\ c\mathbf{A}\end{pmatrix} \qquad J^\mu \equiv \begin{pmatrix}c\rho \\ \mathbf{J}\end{pmatrix} \tag{2.14}$$

This allows us to write the charge conservation equation ($\partial_\mu J^\mu = 0$), the Lorenz gauge condition ($\partial_\mu A^\mu = 0$) and the equation of motion

$$\Box A^\mu = \frac{1}{c\varepsilon_0} J^\mu \tag{2.15}$$

in a manifestly Lorentz invariant way. Under Lorentz transformations, a 4-vector A^μ transforms as follows

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)\,, \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu\,. \tag{2.16}$$

In particular, under a Lorentz boost we have

$$\begin{cases} c\rho' &= \gamma(c\rho - \boldsymbol{\beta}\cdot\mathbf{J}) \\ \mathbf{J}'_\parallel &= \gamma(\mathbf{J}_\parallel - \boldsymbol{\beta}c\rho) \\ \mathbf{J}'_\perp &= \mathbf{J}_\perp \end{cases} \qquad \begin{cases} \Phi' &= \gamma(\Phi - c\boldsymbol{\beta}\cdot\mathbf{A}) \\ c\mathbf{A}'_\parallel &= \gamma(c\mathbf{A}_\parallel - \boldsymbol{\beta}\Phi) \\ c\mathbf{A}'_\perp &= c\mathbf{A}_\perp \end{cases} \tag{2.17}$$

The symmetry group of Maxwell equations has 10 parameters (4 translations, 3 rotations and 3 Lorentz boosts).

2.2 Lecture 11 - Physics of Lorentz transformations

Lorentz boosts implement the change from one reference frame to another one moving with constant velocity $\mathbf{v}$ with respect to the first. As one can see in equation (2.12) this transformation mixes time and space and leads to several counter-intuitive phenomena. Such phenomena is best understood using spacetime diagrams.

Addition of velocities: A particle moving with velocity $\mathbf{u}$ with respect to an observer $\mathcal{O}$ will move with velocity

$$\mathbf{u}'_\parallel = \frac{\mathbf{u}_\parallel - \mathbf{v}}{1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}}\,, \qquad \mathbf{u}'_\perp = \frac{\mathbf{u}_\perp}{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right)}\,, \qquad \gamma = \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}\,, \tag{2.18}$$

with respect to an observer $\mathcal{O}'$ moving with velocity $\mathbf{v}$ with respect to $\mathcal{O}$. Notice that $|\mathbf{u}| = c$ implies that $|\mathbf{u}'| = c$ which means that photons move at the speed of light in every inertial reference frame.

Time dilation: The time coordinate of a moving observer is dilated by a factor of γ relative to the time coordinate of an observer at rest. This has real physical consequences. For example, the lifetime of an unstable particle increases with the velocity of the particle.

Lorentz contraction: Lengths parallel to the direction of motion are contracted by a factor of γ. This also has physical consequences. For example, in high energy collisions of heavy nuclei, each nuclei is effectively a thin pancake due to the Lorentz contraction in the longitudinal direction.

2.3 Lecture 12 - Covariant formulation of electrodynamics

Vector notation makes rotational invariance manifest. For example, the equation $\mathbf{F} = m\mathbf{a}$ is an equality between vectors, therefore it is preserved by rotations because both sides of the equation transform in the same way.

Invariance under Lorentz transformations becomes manifest if we use 4-vector (and tensor) notation. In this notation, there are two types of indices: **contravariant** (up) and **covariant** (down). A contravariant tensor transforms as follows

$$T^{\mu\nu} \to T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} \tag{2.19}$$

For example, x^μ, A^μ and J^μ are contravariant 4-vectors. A covariant tensor transforms as follows [4]

$$V_{\mu\nu} \longrightarrow V'_{\mu\nu} \qquad \text{with} \qquad \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}V'_{\mu\nu} = V_{\alpha\beta} \tag{2.21}$$

As shown in equation (2.10), the derivative ∂_μ is a covariant 4-vector.

One can use the Minkowski metric G to lower or raise an index. For example,

$$A_\mu = G_{\mu\nu}A^\nu \qquad \Leftrightarrow \qquad A^\mu = G^{\mu\nu}A_\nu\,, \tag{2.22}$$

implies that A_μ is a covariant 4-vector if A^μ is a contravariant 4-vector. This follows easily from the identities (2.9).

Contracting one index up with one index down is a Lorentz invariant operation. For example,

$$A^\mu B_\mu = G_{\mu\nu}A^\mu B^\nu \to G_{\mu\nu}A'^\mu B'^\nu = G_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}A^\alpha B^\beta = G_{\alpha\beta}A^\alpha B^\beta = A^\alpha B_\alpha\,. \tag{2.23}$$

In particular, the **interval**

$$\Delta x^\mu \Delta x_\mu = (c\Delta t)^2 - |\Delta\mathbf{x}|^2 \tag{2.24}$$

is a Lorentz invariant.

In order to formulate Maxwell equations in covariant form it is useful to introduce the Maxwell tensor (or field strength)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.25}$$

[4] In general, a tensor can have both contravariant and covariant indices. In this case, the transformation rule is the obvious one

$$V^{\rho}_{\mu\nu} \longrightarrow V'^{\rho}_{\mu\nu} \qquad \text{with} \qquad \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}V'^{\rho}_{\mu\nu}(x') = \Lambda^{\rho}{}_{\sigma}V^{\sigma}_{\alpha\beta}(x) \tag{2.20}$$

Properties:

- $F_{\mu\nu}$ is a covariant tensor:

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F'_{\mu\nu}(x') = F_{\rho\sigma}(x) \tag{2.26}$$

- $F_{\mu\nu}$ is antisymmetric $F_{\mu\nu} = -F_{\nu\mu}$, and therefore it has 6 independent components.
- $F_{\mu\nu}$ is invariant under gauge transformations:

$$A_\mu \xrightarrow{\text{gauge}} A_\mu + \partial_\mu \alpha \tag{2.27}$$

$$F_{\mu\nu} \xrightarrow{\text{gauge}} F_{\mu\nu} + \partial_\nu\partial_\mu\alpha - \partial_\mu\partial_\nu\alpha = F_{\mu\nu} \tag{2.28}$$

In fact, $F_{\mu\nu}$ encodes the electromagnetic fields

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix} \tag{2.29}$$

or equivalently, $F_{0i} = \mathbf{E}^i$ and $F_{ij} = -c\varepsilon_{ijk}\mathbf{B}^k$.

We can now write Maxwell equations in a covariant form: [5]

$$\partial_\mu F^{\mu\nu} = \frac{1}{c\varepsilon_0} J^\nu \tag{2.30}$$

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma} = 0 \tag{2.31}$$

where

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases} \tag{2.32}$$

is the Levi-Civita tensor (totally antisymmetric tensor).

2.4 Lecture 13 - Covariant formulation of Newtonian mechanics

The goal of this lecture is to find a Lorentz invariant formulation of Newtonian mechanics. This should be formulated in the language of 4-vectors and it should reduce to Newtonian mechanics in the limit of low velocities $v \ll c$.

[5]Lorentz invariance of the second equation follows from the fact that the Levi-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$ is a Lorentz invariant tensor (like the Minkowski metric $G_{\mu\nu}$). Alternatively, one can write the second equation in the form $\partial_{[\mu}F_{\rho\sigma]} = 0$ where the square brackets denote total anti-symmetrisation.

2.4.1 Spacetime structure

We consider the **interval**

$$S_{12}^2 = (x_1^\mu - x_2^\mu)\left(x_{1\mu} - x_{2\mu}\right) = c^2(t_1 - t_2)^2 - (\mathbf{x}_1 - \mathbf{x}_2)^2 \tag{2.33}$$

between two events x_1^μ and x_2^μ. The interval can be of 3 types (see figure 2.1):

- $S_{12}^2 > 0$: **timelike** separation. In this case $c|t_1 - t_2| > |\mathbf{x}_1 - \mathbf{x}_2|$ and it is possible to go from one event to the other moving at a speed lower than c. We say the events are causally connected.

- $S_{21}^2 < 0$: **spacelike** separation. In this case $c|t_1 - t_2| < |\mathbf{x}_1 - \mathbf{x}_2|$ and it is not possible to connect the two events moving with a velocity lower than c. We say the events are causally disconnected.

- $S_{12}^2 = 0$: **lightlike** separation. In this case $c|t_1 - t_2| = |\mathbf{x}_1 - \mathbf{x}_2|$ and the two events can be connected by a light ray.

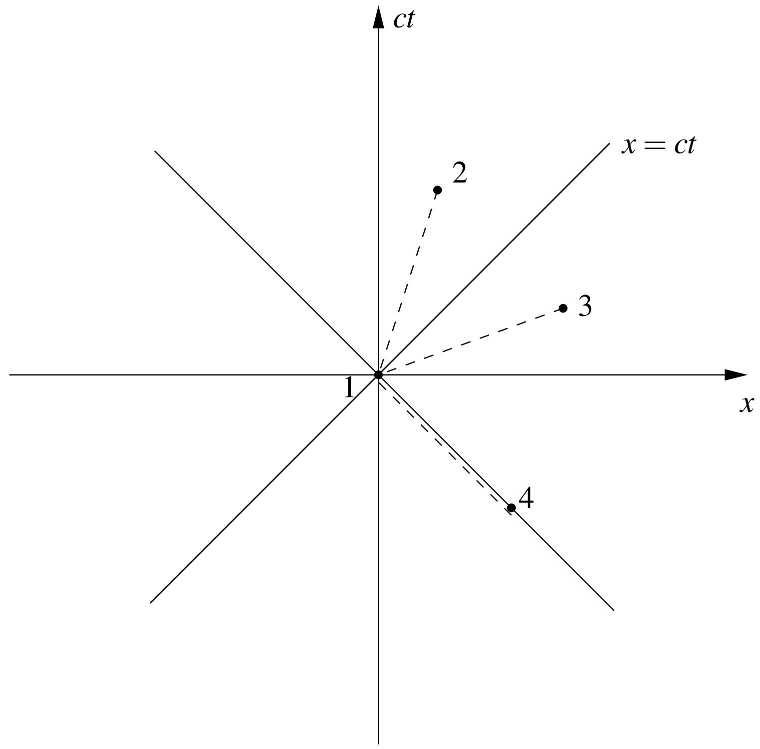

Figure 2.1 Light-cone and the different types of spacetime separations. S_{12} is timelike, S_{13} is spacelike and S_{14} is null or lightlike.

2.4.2 Proper time and 4-velocity

Two spacetime points along a particle trajectory are timelike separated. We define the infinitesimal proper time τ of the particle by

$$d\tau = \frac{1}{c}\sqrt{dx^\mu dx_\mu} = \sqrt{dt^2 - d\mathbf{x}^2/c^2} = \frac{dt}{\gamma}\,. \tag{2.34}$$

Notice that the proper time corresponds to the time measured in an inertial reference frame where the particle is instantaneously at rest. The proper time elapsed along a trajectory is then given by

$$\Delta\tau = \int_{\tau_1}^{\tau_2} d\tau = \int_{t_1}^{t_2} dt\sqrt{1-\frac{|\mathbf{v}(t)|^2}{c^2}}\,. \tag{2.35}$$

We introduce the 4-velocity

$$u^\mu = \frac{dx^\mu}{d\tau} \equiv \gamma \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix} \tag{2.36}$$

Notice that it is obviously a (contravariant) 4-vector because the infinitesimal proper time $d\tau$ is an invariant. The 4-velocity has constant norm

$$u^\mu u_\mu = \frac{dx^\mu dx_\mu}{d\tau^2} = c^2 \tag{2.37}$$

Similarly, we introduce the 4-acceleration:

$$a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} \tag{2.38}$$

Notice that $a^\mu u_\mu = 0$ which implies that a^μ is a spacelike 4-vector ($a_\mu a^\mu < 0$).

2.4.3 Lorentz force

We can know write the covariant form of Newton's 2nd law in the presence of electromagnetic forces:

$$m\frac{d^2x^\mu}{d\tau^2} = m\frac{du^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu \tag{2.39}$$

The spatial components of this equation can be written as

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{2.40}$$

where

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} = m\gamma\mathbf{v} \tag{2.41}$$

is the relativistic momentum of a particle. This shows that (2.39) reduces to the usual Lorentz force formula in the non-relativistic limit.

The time component of (2.39) can be written as

$$\frac{d\mathcal{E}}{dt} = \underbrace{q\mathbf{E}\cdot\mathbf{v}}_{\text{work/time}} = \frac{d\mathcal{W}}{dt} \tag{2.42}$$

where

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = m\gamma c^2 = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4)\,, \tag{2.43}$$

is the energy of the particle.

It is also useful to introduce the 4-momentum

$$p^\mu = mu^\mu \equiv \begin{pmatrix} \mathcal{E}/c \\ \mathbf{p} \end{pmatrix} \tag{2.44}$$

with

$$\mathcal{E} = \sqrt{m^2c^4 + \mathbf{p}^2c^2}\,, \qquad \mathbf{p} = m\gamma\mathbf{v}\,.$$

Notice that the invariant $p^\mu p_\mu = m^2c^2 > 0$ corresponds to the mass of the particle.

2.4.4 Simple applications

Constant electric field

Consider a charged particle moving on the xy plane in the presence of a constant electric field $\mathbf{E} = E\mathbf{e}_x$. It is easy to show that the momentum is given by

$$\begin{cases} p_x = qEt + p_x^0 \\ p_y = p_y^0 \end{cases} \tag{2.45}$$

and therefore the velocity is

$$\mathbf{v} = c^2\frac{\mathbf{p}}{\mathcal{E}} = \frac{\left[(qEt + p_x^0)\mathbf{e}_x + p_y^0\mathbf{e}_y\right]c^2}{\sqrt{m^2c^4 + (p_y^0)^2c^2 + (qEt + p_x^0)^2c^2}}$$

Notice that in the non-relativistic limit $c \to \infty$ the velocity $\mathbf{v} \to \mathbf{v}_0 + \frac{qt}{m}\mathbf{E}$. However, when $t \to \infty$ the velocity $\mathbf{v} \to c\,\mathbf{e}_x$ and never exceeds the speed of light.

One can also show that the trajectory on the plane is given by

$$x(y) = \frac{\mathcal{E}_0}{qE}\left(\cosh(\frac{qEy}{cp_y^0}) - 1\right)\,, \tag{2.46}$$

where we have chosen it to pass through the point $x = y = 0$.

Constant magnetic field

In this case, equation (2.42) tell us that the energy of the particle is constant and therefore its speed v is also constant. Equation (2.40) can be written as

$$\frac{d\mathbf{v}}{dt} = \omega_B \mathbf{v} \times \mathbf{e}_z \ , \qquad \omega_B = \frac{qB}{\gamma m} = \frac{qBc^2}{\mathcal{E}} \tag{2.47}$$

where we have assumed $\mathbf{B} = B\,\mathbf{e}_z$.

The solution is a spiral as depicted in figure 2.2. The trajectory is given by (with a specific initial condition)

$$\begin{cases} z = v_z t \\ x = -r_B \cos(\omega_B t) \\ y = r_B \sin(\omega_B t) \end{cases} , \qquad r_B = \frac{v_\perp}{\omega_B} \ . \tag{2.48}$$

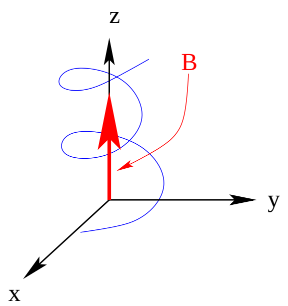

Figure 2.2 Particle motion in a constant magnetic field.

2.5 Lecture 14 - Radiation from a moving charge

We would like to find a relativistic generalization (and a covariant formulation) of Larmor formula. Let us denote by dP^μ the 4-momentum radiated by a moving particle during a piece dx^μ of its trajectory. It is not hard to check that the only covariant expression that reduces to Larmor formula in the non-relativistic limit and it is quadratic in the acceleration, is the following

$$dP^\mu = -\frac{q^2}{6\pi\varepsilon_0 c^5}\left(\frac{du^\nu}{d\tau}\frac{du_\nu}{d\tau}\right) dx^\mu \ . \tag{2.49}$$

It turns out that the power emitted is a Lorentz scalar

$$\frac{d\mathcal{E}}{dt} = c^2 \frac{dP^0}{dx^0} = -\frac{q^2}{6\pi\varepsilon_0 c^3}\frac{du^\nu}{d\tau}\frac{du_\nu}{d\tau} = \frac{q^2}{6\pi\varepsilon_0 c}\gamma^6\left[|\dot{\boldsymbol{\beta}}|^2 - |\boldsymbol{\beta}\wedge\dot{\boldsymbol{\beta}}|^2\right] \ . \tag{2.50}$$

Notice that this matches with (1.74) for the case of rectilinear motion $\boldsymbol{\beta}\wedge\dot{\boldsymbol{\beta}} = 0$.

We can use (2.39) to write

$$\frac{d\mathcal{E}}{dt} = -\frac{1}{6\pi\varepsilon_0 c}\frac{q^4}{m^2c^4}F^{\mu\nu}u_\nu F_{\mu\rho}u^\rho \,. \tag{2.51}$$

Notice that the emitted spatial momentum is proportional to the velocity ($dP^i \propto v^i$) with a positive coefficient. This implies that the radiation is emitted in the direction of motion and the particle "feels" a radiation reaction force opposed to the velocity.

2.5.1 Accelerators

Linear accelerator

In a linear accelerator, we have $\mathbf{E} \parallel \mathbf{v}$ (and $\mathbf{B} = 0$) which leads to

$$\frac{d\mathcal{E}}{dt} = \frac{q^4\mathbf{E}^2}{6\pi\varepsilon_0 m^2 c^3} \tag{2.52}$$

Synchrotron radiation

In a synchrotron, the magnetic field $\mathbf{B}$ is orthogonal to the velocity of the particle. This leads to

$$\frac{d\mathcal{E}}{dt} = \frac{q^4}{6\pi\varepsilon_0 m^2 c^3}\gamma^2 B^2 v^2 = \frac{q^2}{6\pi\varepsilon_0 c^3 r_B^2}\gamma^4 v^4 \tag{2.53}$$

where $r_B = \frac{v\gamma m}{q|\mathbf{B}|}$ is the radius of the circular trajectory in the plane orthogonal to $\mathbf{B}$.

2.5.2 Radiation Reaction

The radiated momentum affects the motion of the particle as follow

$$m\frac{du^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu + f^\mu_{\text{reaction}} \,. \tag{2.54}$$

Naively, one expects

$$f^\mu_{\text{reaction}} \stackrel{?}{=} -\frac{dP^\mu_{\text{rad}}}{d\tau} = -\frac{q^2}{6\pi\varepsilon_0 c^5}\frac{du^\nu}{d\tau}\frac{du_\nu}{d\tau}u^\mu \,, \tag{2.55}$$

however this is inconsistent because $u_\mu f^\mu_{\text{reaction}} \neq 0$. The correct expression is

$$f^\mu_{\text{reaction}} = -\frac{dP^\mu_{\text{rad}}}{d\tau} + \underbrace{\frac{q^2}{6\pi\varepsilon_0 c^3}\frac{d^2u^\mu}{d\tau^2}}_{\text{Schott term}} = \frac{q^2}{6\pi\varepsilon_0 c^3}\left(\delta^\mu_\nu - \frac{1}{c^2}u^\mu u_\nu\right)\frac{d^2u^\nu}{d\tau^2} \tag{2.56}$$

where the Schott term corresponds to the change in 4-momentum in the near EM field. Notice that it is a total derivative and therefore its integrated effect drops out for periodic motion or trajectories with non-zero acceleration just for a finite time interval. See for example arXiv:1909.00960 for more details.

In the non-relativistic limit, (2.54) reduces to

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + m\tau_c \ddot{\mathbf{v}}\,, \qquad \tau_c = \frac{q^2}{6\pi\varepsilon_0 mc^3}\,. \tag{2.57}$$

This equation leads to paradoxes. For example, for $\mathbf{E} = \mathbf{B} = 0$ there is a *self-accelerating* solution $\mathbf{v} = \mathbf{v}_0 e^{t/\tau_c}$. The solution is to interpret (2.54) as the first term in a derivative expansion, *i.e.* in the spirit of Effective Field Theory. In practice, this means that we should compute $\frac{d^2u^\nu}{d\tau^2}$ in (2.56) using $\frac{du^\nu}{d\tau}$ from the leading order equation of motion (2.39). This leads to

$$f^\mu_{\text{reaction}} = \frac{q^2}{6\pi\varepsilon_0 c^3}\left[\frac{q}{cm}u^\alpha u_\nu \partial_\alpha F^{\mu\nu} + \frac{q^2}{c^2m^2}F^{\nu\alpha}F_{\alpha\beta}u^\beta(\delta^\mu_\nu - u_\nu u^\mu/c^2)\right]\,. \tag{2.58}$$

Equation (2.54) with this form for f^μ_{reaction} is known as the Landau-Lifshitz equation.

2.6 Lecture 15 (extra) - Lagrangian formulations

The action for electrodynamics is given by

$$S = \int dt d^3x \mathcal{L}\,, \qquad \mathcal{L} = -\frac{\varepsilon_0}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\mu J^\mu\,. \tag{2.59}$$

The action is local and gauge invariant if J^μ is a conserved current, *i.e.* $\partial_\mu J^\mu = 0$. The term $\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = \partial_\mu(2\varepsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta})$ is also local and gauge invariant. However, including it in the lagrangian would not change the equations of motion because it is a total derivative. In principle, one could also add higher order terms like $(F_{\mu\nu}F^{\mu\nu})^2$ but these lead to non-linear e.o.m. (Euler-Heisenberg lagrangian). Varying the action with respect to the gauge field A_μ,

$$\delta S = \int dt d^3x\, \delta A_\nu \left[\varepsilon_0 \partial_\mu F^{\mu\nu} - \frac{1}{c}J^\nu\right]\,, \tag{2.60}$$

one derives Maxwell equations (2.30). In non-relativistic notation, the lagrangian density reads

$$\mathcal{L} = \frac{\varepsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 - \rho\Phi + \mathbf{J}\cdot\mathbf{A}\,. \tag{2.61}$$

The action for a charged particle is

$$S_{\text{part}} = -\int d\lambda \left[mc\sqrt{\dot{x}_\mu \dot{x}^\mu} + \frac{q}{c}\dot{x}^\mu A_\mu(x(\lambda))\right]\,, \tag{2.62}$$

where we parametrized the worldline of the particle $x^\mu(\lambda)$ by a generic parameter λ. Notice that the action is invariant under change of parametrization $\lambda \to f(\lambda)$. Varying this action with respect to the path $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$, one derives the e.o.m. (2.39). Choosing the parameter λ to be the time coordinate $t = x^0/c$, we find

$$S_{\text{part}} = \int dt \left[-mc^2\sqrt{1 - \frac{\mathbf{v}^2}{c^2}} - q\Phi + q\mathbf{v}\cdot\mathbf{A}\right]\,. \tag{2.63}$$

2.7 Extra topics

There are several interesting topics that could not be discussed for lack of time. I list here some of them to guide the students interested in knowing more:

- Radiation pressure - this can be computed using the electromagnetic stress-energy tensor discussed in the exercises.
- Dispersive media
- Kramers-Kronig relation
- Wave guides
- Cherenkov radiation - radiation produced by a charged particle moving faster than the speed of light in a medium.
- Thomson scattering - see exercises.
- Rayleigh scattering - see exercises.
- Superradiance - wave amplification when reflected by a rotating body.
- Transmission lines

Appendix A

Mathematics

In this appendix, we summarize some of the mathematical tools used in the main text.

A.1 Vector calculus

Gradient: is defined by the following equation

$$f(\mathbf{x}+\mathbf{dx}) = f(\mathbf{x}) + \mathbf{dx}\cdot\nabla f + O(\mathbf{dx}^2) \tag{A.1}$$

In cartesian coordinates, this gives

$$\nabla f = \sum_i \frac{\partial f}{\partial x_i}\mathbf{e}_i = \partial_i f\,\mathbf{e}_i \tag{A.2}$$

where $\mathbf{e}_i\,, i\!=\!1,2,3$ is an orthonormal basis of a Cartesian system of coordinates. In the last expression, we used *Einstein's convention*, where repeated indices are summer over. For example, $A_i = \alpha_i B_j C_j = \alpha_i \sum_j B_j C_j$.

The expression in other coordinate system follows easily from (A.1). For example, in spherical coordinates $\mathbf{dx} = dr\,\mathbf{e}_r + rd\theta\,\mathbf{e}_\theta + r\sin\theta d\varphi\,\mathbf{e}_\varphi$ implies that

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\mathbf{e}_\varphi \tag{A.3}$$

Divergence: is defined by

$$\nabla\cdot\mathbf{A}(\mathbf{x}) = \lim_{V\to\mathbf{x}} \frac{\oint_{\partial V}\mathbf{d}\boldsymbol{\sigma}\cdot\mathbf{A}}{\int_V d^3x} = \frac{\text{flux of } \mathbf{A}}{\text{volume}} \tag{A.4}$$

where the region V shrinks to zero volume around the point $\mathbf{x}$. By choosing a region V adapted to the coordinate system, one can easily derive explicit expressions for the divergence of a vector field. For example, in Cartesian coordinates a small cube ($dx \times dy \times dz$) leads to

$$\nabla\cdot\mathbf{A} = \sum_i \frac{\partial A_i}{\partial x_i} = \partial_i A_i \tag{A.5}$$

Curl: is defined by

$$\mathbf{n} \cdot (\nabla \wedge \mathbf{A}) = \lim_{S \to \mathbf{x}} \frac{\oint_{\partial S} \mathbf{dl} \cdot \mathbf{A}}{\int_S d\sigma} = \frac{\text{circulation of } \mathbf{A}}{\text{area}} \tag{A.6}$$

where $\mathbf{n}$ is the unit normal to the surface S, which is shrinking to the point $\mathbf{x}$. By choosing a small surface adapted to the coordinate system, one can easily derive explicit expressions for the curl. For example, in Cartesian coordinates, choosing a small square $dx \times dy$ gives us the component of the curl along $\mathbf{e}_z$. Similarly, one can find the full vector

$$\nabla \wedge \mathbf{A} = \nabla \times \mathbf{A} = \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) \mathbf{e}_1 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) \mathbf{e}_2 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \mathbf{e}_3 \tag{A.7}$$

$$= \varepsilon_{ijk}\, \mathbf{e}_i\, \partial_j A_k \tag{A.8}$$

Laplacian: can be defined by

$$\nabla^2 f = \nabla \cdot (\nabla f) \tag{A.9}$$

or (more geometrically) by

$$\nabla^2 f(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{6}{\epsilon^2} \left[\frac{\int_S d\sigma f}{\int_S d\sigma} - f(\mathbf{x}) \right] \tag{A.10}$$

where S is a sphere of radius ϵ centred at $\mathbf{x}$.

Gauss or divergence theorem:

$$\int_V d^3x\, \nabla \cdot \mathbf{A} = \oint_{\partial V} \mathbf{d\boldsymbol{\sigma}} \cdot \mathbf{A} \tag{A.11}$$

Stokes theorem:

$$\oint_{\partial S} \mathbf{dl} \cdot \mathbf{A} = \int_S \mathbf{d\boldsymbol{\sigma}} \cdot (\nabla \wedge \mathbf{A}) \tag{A.12}$$

A.2 Tensors

The *Einstein's convention*, where repeated indices are summer over, is very convenient to simplify expressions involving tensors.

In $\mathbb{R}^3$, tensors are objects with simple transformation properties under rotations,

$$T_{ij\ldots k} \to R_i^{i'} R_j^{j'} \ldots R_k^{k'}\, T_{i'j'\ldots k'} \tag{A.13}$$

where R is a 3×3 matrix implementing a rotation. A vector is a special case of a tensor with a single index. There are two important tensors that are invariant under rotations: the Kronecker-delta δ_{ij} and the Levi-Civita symbol ϵ_{ijk}.

$$\delta_{ij} = R_i^{i'} R_j^{j'}\, \delta_{i'j'} \Leftrightarrow R^t R = \mathbb{I} \Leftrightarrow R \in O(3) \tag{A.14}$$

where in the second equation we used matrix notation for R and $O(3)$ stands for the group of (real) orthogonal 3×3 matrices. The Levi-Civita symbol is defined by

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj}\,, \qquad \epsilon_{123} = 1 \tag{A.15}$$

and it is invariant under rotations [1]

$$\epsilon_{ijk} = R_i^{i'} R_j^{j'} R_k^{k'} \epsilon_{i'j'k'} \Leftrightarrow \det R = 1 \tag{A.16}$$

Often tensors have well defined properties under permutation of their indices. For example $\delta_{ij} = \delta_{ji}$ is symmetric while ϵ_{ijk} is totally anti-symmetric under the exchange of any pair of indices. The contraction of a symmetric tensor $S_{ij} = S_{ji}$ with and anti-symmetric tensor $A_{ij} = -A_{ji}$ vanishes,

$$S_{ij}A_{ij} = -S_{ji}A_{ji} = 0\,. \tag{A.17}$$

A tensor is said to be traceless if any contraction with δ_{ij} vanishes. For example, the following tensor is traceless

$$S_{ij} - \frac{1}{3}\delta_{ij}S_{kk} \tag{A.18}$$

The identity

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{A.19}$$

is very useful to simplify expressions with two (or more) cross-products. For example, it easily leads to:

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{A.20}$$

$$\nabla \wedge (\nabla \wedge \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \triangle \mathbf{A} \tag{A.21}$$

A.3 Fourier transform

In our conventions, the Fourier transform of a function $f(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$ is given by

$$\tilde{f}(\mathbf{k}) = \int_{\mathbb{R}^n} d^n x\, f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\,, \tag{A.22}$$

where $\mathbf{k} \cdot \mathbf{x}$ is the Euclidean scalar product. The inverse transform is then

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{A.23}$$

For functions of space and time (spacetime) we shall use the convention

$$\tilde{f}(\mathbf{k}, \omega) = \int_{\mathbb{R}^4} d^3x dt\, f(\mathbf{x}, t) e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}$$

$$f(\mathbf{x}, t) = \int_{\mathbb{R}^4} \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \tilde{f}(\mathbf{k}, \omega) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$$

As explained in chapter 2, this corresponds to using the inner product of Minkowski space.

[1] Notice that δ_{ij} is also invariant under reflections but ϵ_{ijk} is only invariant under rotations (it changes sign under reflections).

A.4 Transverse and longitudinal components of a vector field

A vector field $\mathbf{V}(\mathbf{x},t)$ can be decomposed as $\mathbf{V} = \mathbf{V}_\perp + \mathbf{V}_\parallel$, where

$$\mathbf{V}_\parallel \equiv \nabla f \qquad \Rightarrow \qquad \nabla \wedge \mathbf{V}_\parallel = 0 \tag{A.24}$$

is called the *longitudinal component*, and

$$\mathbf{V}_\perp \equiv \nabla \wedge \mathbf{U} \qquad \Rightarrow \qquad \nabla \cdot \mathbf{V}_\perp = 0 \tag{A.25}$$

is called the *transverse component.* This decomposition is simpler in Fourier space,

$$\mathbf{V}(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\, \tilde{\mathbf{V}}(\mathbf{k},t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{A.26}$$

where it corresponds to the decomposition $\tilde{\mathbf{V}}(\mathbf{k},t) = \tilde{\mathbf{V}}_\perp + \tilde{\mathbf{V}}_\parallel$ of the vector $\tilde{\mathbf{V}}$ into the parallel and transverse components to the vector $\mathbf{k}$. Clearly

$$\tilde{\mathbf{V}}_\parallel = \mathbf{k}\,\tilde{f}\,, \qquad \qquad \tilde{\mathbf{V}}_\perp = \mathbf{k} \wedge \tilde{\mathbf{U}}\,, \tag{A.27}$$

for some $\tilde{f}$ and some $\tilde{\mathbf{U}}$.

A.5 The Dirac δ–function

Consider a rapidly decreasing function $F(\mathbf{x})$ such that: [2]

$$\begin{aligned} &1) \quad F(\mathbf{x}) > 0 \qquad \forall x \\ &2) \quad \int d^n x F(\mathbf{x}) = 1 \end{aligned}$$

Then, we can define the Dirac δ–function as the limit

$$\delta^n(\mathbf{x}-\mathbf{x}_0) = \lim_{\varepsilon\to 0} \delta^n_\varepsilon(\mathbf{x}-\mathbf{x}_0)\,, \qquad \qquad \delta^n_\varepsilon(\mathbf{x}-\mathbf{x}_0) = \frac{1}{\varepsilon^n} F\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)\,. \tag{A.28}$$

Properties:

a)

$$\int d^n x\, f(\mathbf{x}) \delta^n(\mathbf{x}-\mathbf{x}_0) = f(\mathbf{x}_0) \tag{A.29}$$

b)

$$\int d^n x f(\mathbf{x}) \nabla \delta^n(\mathbf{x}-\mathbf{x}_0) = -\nabla_\mathbf{x} f(\mathbf{x}_0) \tag{A.30}$$

[2] More precisely, $F(\mathbf{x})$ should be a Schwartz function, which means it is C^∞ and all its derivatives decay faster than any power of $|\mathbf{x}|$ at large $|\mathbf{x}|$. For example, we can take a Gaussian $F(\mathbf{x}) = \pi^{-n/2} e^{-\mathbf{x}^2}$.

c)

$$\delta^n(\mathbf{g}(\mathbf{x})) = \sum_i \frac{\delta^n(\mathbf{x} - \mathbf{x}_i)}{\left|\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|} \tag{A.31}$$

where we assumed that $\mathbf{g}$ only has simple zeros at $\mathbf{x} = \mathbf{x}_i$ and $\left|\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|$ denotes the modulus of the determinant of the jacobian.

d) The δ–function has dimension $\frac{1}{L^n}$, where n is the dimension of space and L is a length scale.

e) Fourier representation

$$\delta^n(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{d^n k}{(2\pi)^n}\, e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{A.32}$$

A.6 Residue theorem

Let $U \in \mathbb{C}$ be a simply connected region $U \subset \mathbb{C}$ and $f(z)$ a holomorphic function on $U\backslash\{z_1, z_2, \ldots, z_n\}$. Then, for any closed curve $\gamma \subset U$ avoiding the singular points $z_1, z_2, \ldots, z_n$, we have

$$\oint_\gamma dz\, f(z) = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)\,. \tag{A.33}$$

If the singular point z_k is a pole of order q then the residue is given by

$$\mathrm{Res}(f, z_k) = \frac{1}{(q-1)!} \lim_{z\to z_k} \frac{d^{q-1}}{dz^{q-1}} (z - z_k)^q\, f(z) \tag{A.34}$$