MATHEMATICAL TRIVIUM

Legend:
⏲: hardworking, i.e. long exercise
☢: hard exercise
✿: creative exercise

LINEAR ALGEBRA

1. Describe the figures that can be obtained as intersection of the cone $z^2 = x^2 + y^2$ with the plane $z - ax = 1$, $0 < a < \infty$, in $\mathbb{R}^3$.

2. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

(a) Find A^T, A^2, A^3, A^{-1}, $\mathrm{Tr}A$, $\det A$.

(b) Find the eigenvalues and the eigenvectors of A.

(c) Write the transformation that reduces A to diagonal form.

3. Find if the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 13 \\ 15 \end{bmatrix}$ are linear independent or not.

4. Let A be a non-degenerate matrix, and B is obtained from A by interchange two of its lines. How are $\det A$ and $\det B$ related?

5. Let A and B be non-degenerate $n \times n$ matrices, $c \in \mathbb{C}$. Write the relations between their determinants, if
(a) $B = A^T$, (b) $B = A^{-1}$, (c) $B = c \cdot A$.

6. ✿Let A be a non-degenerate matrix. Show that $\log \det A = \mathrm{Tr} \log A$.

7. Let $A = E + \epsilon B$, where E is the identity matrix and $\epsilon \ll 1$. Expand $\det A$ to the first order in ϵ.

8. Let A, B, C, D be non-degenerate matrices of the same dimension. Show that

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\left(AD - BD^{-1}CD\right). \tag{1}$$

9. Calculate $\det A$, if

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}. \tag{2}$$

10. Calculate the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \tag{3}$$

11. Consider the Vandermonde matrix

$$V(x_1, ..., x_n) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \dots & x_n^{n-1} \end{bmatrix}. \tag{4}$$

(a) Calculate $\det V(x_1, ..., x_n)$.

(b) Show that $\det V(x_1, ..., x_n) = 0$ if and only if $x_i = x_j$ for some $i \neq j$.

12. Consider the Wronskian

$$W_x(y_1, ..., y_n) = \det \begin{bmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ y_1' & y_2' & y_3' & \dots & y_n' \\ y_1'' & y_2'' & y_3'' & \dots & y_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \dots & y_n^{(n-1)} \end{bmatrix}, \tag{5}$$

where $y_1, ..., y_n$ are C^{n-1} functions of x.

(a) Let y_1 and y_2 be two solutions of the differential equation $y'' - ay' - by = 0$, where a and b are some known functions of x. Find an expression for the Wronskian $W_x(y_1, y_2)$ depending on a and b. Then, show that if one of the solutions, say, y_1, is known, then another can be found from the first order equation $y_1' - \frac{y_2'}{y_2} y_1 + \frac{W_x(y_1, y_2)}{y_2} = 0$.

(b) Show that under change of variable $x \to t(x)$ the Wronskian transforms as follows,

$$W_x(y_1, ..., y_n) = \left(\frac{dt}{dx}\right)^{\frac{n(n-1)}{2}} W_t(y_1, ..., y_n). \tag{6}$$

(c) Show that $W_x(yy_1, ..., yy_n) = y^n W_x(y_1, ..., y_n)$, where y is some C^{n-1} function.

13. Consider the matrices $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$, $B = \begin{bmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{bmatrix}$.

(a) Calculate A^{-1}, B^{-1}.

(b) Give a geometrical interpretation of the action of A and B on vectors in $\mathbb{R}^2$.

(c) What can one say about the eigenvectors of A and B?

14. Let A be a real matrix with $(\det A)^2 = 1$. Give a conclusion about orthogonality of A.

15. Let A be $n \times n$ real orthogonal matrix. Calculate the number of independent elements of A.

16. Construct the matrix that performs a reflection in $\mathbb{R}^3$

(a) across the origin O,

(b) across the axis Oz,

(c) across the plane xOy.

17. Consider the reflection maps in $\mathbb{R}^n$. The reflection across the origin transforms every vector $\vec{x} \in \mathbb{R}^n$ to $-\vec{x}$. The reflection across one of the basis axis Oi inverts all coordinates of $\vec{x}$ except x_i. Similarly, one can define the reflections across the planes in $\mathbb{R}^3$ and higher dimensional hyperplanes in $\mathbb{R}^n$, $n > 3$. Some of these reflections are equivalent to rotations around the origin O, others are not.

(a) Observe that in $\mathbb{R}^2$ the central reflection is equivalent to the rotation by an angle π around O. Show that no reflections across a line crossing O can be achieved by any rotation.

(b) In $\mathbb{R}^3$, find if one can achieve by some rotation around O 1) the reflection across O, 2) the reflection across an arbitrary line crossing O, 3) the reflection across an arbitrary plane crossing O.

(c) In $\mathbb{R}^n$, formulate and prove a general statement about the existence of rotations that perform the reflection across a given m-dimensional plane, $m = 0, 1, ..., n-1$, crossing the origin (where $m = 1$ corresponds to the line, and $m = 0$ - to the point).

18. ⌚Consider the rotation by an angle θ around a line determined by a unit radius-vector $\vec{n}$ with components n_x, n_y, n_z.

(a) Deduce the rotation matrix that performs this rotation.

(b) Relate the trace of this matrix to the angle θ.

19. Consider the Pauli matrices, $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(a) Calculate $\sigma_i^\dagger$, $i = 1, 2, 3$.

(b) Decompose a matrix $A = \begin{bmatrix} a & b + ic \\ b - ic & -a \end{bmatrix}$, $a, b, c \in \mathbb{R}$, into a sum of σ_i.

(c) Prove that σ_i, $i = 1, 2, 3$, constitute a basis in the space of 2×2 hermitian matrices with zero trace.

(d) Prove that $\{\sigma_i, i = 1, 2, 3;\ 1\}$ constitute a basis in the space of 2×2 hermitian matrices.

(e) Calculate the eigenvalues and the eigenvectors of σ_i, $i = 1, 2, 3$.

20. Let σ_i, $i = 1, 2, 3$ be the Pauli matrices.

(a) Calculate $[\sigma_i, \sigma_j] \equiv \sigma_i \sigma_j - \sigma_j \sigma_i$, $i, j = 1, 2, 3$.

(b) Calculate $\{\sigma_i, \sigma_j\} \equiv \sigma_i\sigma_j + \sigma_j\sigma_i$, $i, j = 1, 2, 3$.

(c) Calculate $[\sigma_i, [\sigma_j, \sigma_k]] + [\sigma_j, [\sigma_k, \sigma_i]] + [\sigma_k, [\sigma_i, \sigma_j]]$.

21. ✿Consider the linear transformations A, B of the vector space $\mathbb{R}^n$. Prove that $[A, B] \neq cE$, where E is the identity matrix, $c \in \mathbb{R}$.

22. Consider the transformations A, B of the vector space $\mathbb{R}^n$ sharing the same eigenvectors. Is this sufficient for their commutator to be zero? Is this necessary for their commutator to be zero?

23. Consider the basis $\vec{e}_i$, $i = 1, 2, 3$, in the vector space $\mathbb{R}^3$. In this basis, write the matrix of the transformation that

(a) stretches all directions by a factor of λ.

(b) stretches each direction along $\vec{e}_i$ by a factor of λ_i, $i = 1, 2, 3$.

(c) stretches the direction determined by a unit vector $\vec{n}$ with components n_x, n_y, n_z by a factor of λ.

Find the eigenvectors of these transformations.

24. Show that the translation $\vec{x} \to \vec{x} + \vec{a}$, where $\vec{x}, \vec{a} \in \mathbb{R}^n$, is not a linear transformation in $\mathbb{R}^n$.

25. A linear transformation A writes in some basis in $\mathbb{R}^n$ as follows, $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 8 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

(a) Write the transition matrix to the basis composed of the eigenvectors of A.

(b) Write the transformation A in this basis.

(c) Determine the invariant subspaces of A.

26. ✿Describe the linear transformations whose invariant subspaces are

(a) 3-dimensional sphere $x_1^2 + x_2^2 + x_3^2 = R^2$ in $\mathbb{R}^3$,

(b) 4-dimensional cone $x_1^2 - x_2^2 - x_3^2 - x_4^2 = 0$ in $\mathbb{R}^4$,

(c) n-dimensional hyperboloid $x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2 = R^2$ in $\mathbb{R}^n$.

27. For the vectors $A = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \sqrt{2} \\ 2 \end{bmatrix}$ find an orthogonal transformation that maps A to B.

28. Consider the vectors $\vec{a}_1$ and $\vec{a}_2$ in $\mathbb{R}^2$ with the lengths $|\vec{a}_1| = 2$, $|\vec{a}_2| = 6$ and the angle between them $\phi = \frac{\pi}{6}$. Construct the orthonormal basis $\vec{e}_1, \vec{e}_2$ such that $\vec{e}_1 || \vec{a}_1$, and write the components of $\vec{a}_1$, $\vec{a}_2$ in this basis.

29. Consider the infinite dimensional vector space with the orthonormal basis $\vec{e}_i$, $i = 1, 2, ...$ Construct the continuous family of linear transformations $U(\lambda)$, $0 \leqslant \lambda \leqslant 1$, acting in this space such that $U(1) = E$ and $U(0)\vec{e}_i = \vec{e}_{i+1}$, $i = 1, 2, ...$

30. Let U be $n \times n$ unitary matrix. Calculate the number of independent elements of U.

31. Define the matrix exponential e^A of a matrix A as follows: $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Consider the following matrices

$$M_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad M_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

(a) Write $\mathrm{Tr} M_i$, M_i^T, $i = x, y, z$.

(b) Calculate $e^{\theta M_x}$, $e^{\theta M_y}$, $e^{\theta M_z}$, where $\theta \in [0, 2\pi)$.

(c) Give a geometrical interpretation of the transformations $e^{\theta M_i}$, $i = x, y, z$, acting on the vectors in $\mathbb{R}^3$.

32. Consider the matrices A, B such that $[A, B] = 0$. Show that in this case $e^A e^B = e^B e^A = e^{A+B}$.

33. Prove that $e^A = \left(e^{A/N}\right)^N$, $N \in \mathbb{R}$.

34. Consider the matrices A, B and let $\lambda \ll 1$ be a small parameter. Expand the expression $e^{-\lambda B} A e^{\lambda B}$

(a) to the first order in λ.

(b) to any order in λ.

35. A linear transformation is given by the matrix e^A in some basis in $\mathbb{R}^n$. Consider the change of the basis determined by the transition matrix U. Find the matrix of the transformation in the new basis.

36. ✿"Trotter product formula". Consider $n \times n$ complex matrices A, B. Prove that $e^{A+B} = \lim_{N \to \infty} \left(e^{A/N} e^{B/N}\right)^N$, $N \in \mathbb{R}$.

37. Write the following matrix A as a product $A = SO$ of a symmetric matrix S and an orthogonal matrix O.
(a) $A = \begin{bmatrix} 5 & 0 \\ 4 & 3 \end{bmatrix}$, (b) $A = \begin{bmatrix} 0 & 0 \\ 4 & -3 \end{bmatrix}$.
In which case is such decomposition unique?

38. Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbb{R}$, $\det A = 1$. It transforms the basis vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\vec{g}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$, $\vec{g}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$. One can obtain a useful decomposition of A by making an inverse transformation in three steps.

(a) Construct the rotation matrix R^{-1} that sends $\vec{g}_1$ to $R^{-1}(\vec{g}_1) \| \vec{e}_1$. How does it act on $\vec{g}_2$?

(b) Construct the diagonal matrix P^{-1} such that $\det P^{-1} = 1$ and $P^{-1}(R^{-1}(\vec{g}_1)) = \vec{e}_1$. Show that the components of $P^{-1}(R^{-1}(\vec{g}_2))$ are $\begin{bmatrix} x \\ 1 \end{bmatrix}$, $x \in \mathbb{R}$.

(c) Apply a shear transformation T^{-1} that leaves $\vec{e}_1$ invariant and sends $P^{-1}(R^{-1}(\vec{g}_2))$ to $\vec{e}_2$. Then, we have $T^{-1}P^{-1}R^{-1}A = E$, or $A = RPT$.

(d) Show that this decomposition is unique.

39. Solve the system of linear equations

$$\begin{cases} y + 3z = -1 \\ 2x + 3y + 5z = 3 \\ 3x + 5y + 7z = 6 \end{cases} \tag{8}$$

40. Solve the following systems of linear equations

$$\text{(a)} \begin{cases} x + y + z = 0 \\ x + 2y + 3z = 0 \\ 2x + 3y + 4z = 0 \end{cases} \qquad \text{(b)} \begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ 3x_1 + 2x_2 + x_3 - x_5 = 0 \end{cases}$$

41. Check if the system of linear equations has a solution and solve

$$\begin{cases} x + 2y + 3z = -4 \\ 2x + 3y + 4z = 1 \\ 3x + 4y + 5z = 6 \end{cases} \tag{9}$$

42. Check if the following points belong to the same plane: $(6,1,2)$, $(2,3,1)$, $(3,4,1)$, $(6,2,2)$.

43. Consider two lines formed by intersection of the planes,

$$l_1 = \begin{cases} 3x + 2y + 5z - 1 = 0 \\ -x + 2y + 3z - 1 = 0 \end{cases}, \quad l_2 = \begin{cases} 4x - 6y + 7z + 2 = 0 \\ 5x + 3y - 8z - 3 = 0. \end{cases} \tag{10}$$

Find out if these lines are

(a) parallel to each other,

(b) crossing each other at some point.

44. Consider the quadratic form $f(\vec{x}) = 2x_1^2 + 4x_1x_2 + 3x_2^2 + 4x_2x_3 + 5x_3^2$, where x_i, $i = 1,2,3$ are components of $\vec{x}$ in $\mathbb{R}^3$.

(a) Reduce $f(\vec{x})$ to the canonical form. Calculate its rang and signature.

(b) Write the transition matrix to the basis where $f(\vec{x})$ has the canonical form.

45. Find if the quadratic form $f(\vec{x}) = 9x_1^2 + 6x_1x_2 + 6x_2^2 + 8x_2x_3 + 4x_3^2$ is positive definite or not.

46. Reduce the following quadratic forms to the canonical form and determine the rank and the signature:

$$\text{(a) } x_1^2 + 2\sum_{i=2}^{n} x_i^2 - 2\sum_{i=1}^{n-1} x_i x_{i+1}, \quad \text{(b) } \sum_{i=1}^{n} x_i^2 + \sum_{1\leqslant i<j\leqslant n} x_i x_j, \quad \text{(c) } \sum_{1\leqslant i<j\leqslant n} x_i x_j.$$

47. Consider the quadratic forms $f(\vec{x}) = x_1^2+2x_1x_2+3x_2^2$ and $g(\vec{x}) = 4x_1^2+16x_1x_2+6x_2^2$ in $\mathbb{R}^2$.

(a) Check if at least one of these forms is sign definite.

(b) Find the change of coordinates that reduces $f(\vec{x})$ and $g(\vec{x})$ to the diagonal form.

REAL ANALYSIS

1. Find $f'(x)$, if $f(x) = \log \frac{a}{x}$, $\cos \arcsin x$, $\frac{x^2+1}{x^3+1}$.

2. Find $f'(x)$, if $f(x) = x^x$.

3. Find $f^{(n)}(0)$, if $f(x) = e^{-\frac{1}{x^2}}$, $x \in \mathbb{R}$, $n > 0$.

4. Find $f^{(100)}(0)$, if $f(x) = (x^{100} + x)e^{100x}$.

5. ⌚Find $\sin^{100} 0.1$ with 10% accuracy.

6. Find $\frac{d}{dx} f(g(x))|_{x=0}$, where $f(u) = \cosh u$, $g(u) = \sqrt{1-u^2}$.

7. Find the minimal value of the function $f(x) = \frac{\lambda}{4}(x^2 - v^2)^2 - \epsilon x$, $\lambda, \epsilon > 0$, to the leading order in $\epsilon \ll 1$.

8. ✿"Huygens problem". Consider the ball with mass M moving with velocity V towards another ball with mass m that stays at rest. After the central collision, the second ball acquires the velocity

$$v = \frac{2M}{m+M} V. \tag{11}$$

This expression can be obtained using the momentum and energy conservation laws for the two-body system. One can observe that $V \leqslant v \leqslant 2V$ as far as $0 \leqslant m \leqslant M$. One may ask under what conditions the limit $v \leqslant 2V$ can be broken to make v arbitrary large. A possible solution is to insert a chain of balls staying at rest with intermediate masses $m_1, ..., m_n$ such that $m < m_1 < ... < m_n < M$ between the two original bodies, and to transfer the kinetic energy of the moving ball to the ball with mass m through a sequence of intermediate central collisions.

(a) Applying Eq.(11) to the sequence of central collisions between the balls, deduce how one should choose the masses $m_1, ..., m_n$ to yield the maximal velocity of the ball with mass m.

(b) Assuming $m \ll M$, investigate the limit $n \to \infty$.

9. Find $\lim\limits_{x\to 0} \frac{\sin x}{x}$.

10. ☢Find $\lim\limits_{x\to 0} \frac{\sin \tan x - \tan \sin x}{\arcsin \arctan x - \arctan \arcsin x}$.

11. Calculate $\int \frac{dx}{\tan x}$.

12. Calculate $\int \log x dx$.

13. Calculate $\int \frac{x^5+2}{x^2-1} dx$.

14. Calculate $\int e^{2x}\sin x dx$.

15. Calculate $\int \frac{dx}{\sin x}$.

16. Find $\int_0^{\infty} x^n e^{-x} dx$, $n \in \mathbb{Z}$, $n > 0$.

17. Find $\int_0^{\infty} e^{-x^2} dx$.

18. Find $\int_0^{\pi^2} \cos\sqrt{x} dx$.

19. Find $\int_0^{x_0} \sqrt{1-\frac{x^2}{x_0^2}} dx$.

20. Find $\int_{-\infty}^{\infty} e^{-x^{-4}-x^4}\sin^5 x dx$.

21. Find $\int_{\sqrt{2}}^{\infty} \frac{dx}{x+x^{\sqrt{2}}}$.

22. Show that $\int_0^1 \frac{dx}{(ax+b(1-x))^2} = \frac{1}{ab}$, $a, b \in \mathbb{R}$.

23. For what values of α the following integrals are convergent?

(a) $\int_0^1 \frac{dx}{x^\alpha}$, (b) $\int_1^{\infty} \frac{dx}{x^\alpha}$.

24. For what values of α and β the following integrals are convergent?

(a) $\int_0^1 \frac{dx}{x^\alpha \log^\beta x}$, (b) $\int_1^{\infty} \frac{dx}{x^\alpha \log^\beta x}$.

25. Is the integral $\int_0^{\phi} \frac{d\psi}{\sqrt{\sin^2\frac{\phi_0}{2}-\sin^2\frac{\psi}{2}}}$ convergent as $\phi \to \phi_0$?

26. Show that the cosine integral $\mathrm{Ci}x \equiv -\int_x^{\infty} \frac{\cos t}{t} dt$ for large enough x can be approximated as $\mathrm{Ci}x \approx \frac{\sin x}{x}$. Find the region of x for which the relative error of this approximation is less than 10^{-4}.

27. Calculate $J(y) = \int_0^{\infty} e^{-ax}\frac{\sin xy}{x} dx$, $a > 0$.

28. Calculate the Fourier image $\tilde{f}(k)$ of the function $f(x)$, if $f(x)$ is given by

(a) $f(x) = \begin{cases} f_0, |x| \leqslant x_0, \\ 0, |x| > x_0 \end{cases}$

(b) $f(x) = \begin{cases} e^{-ax}, & x \geqslant 0, \quad a > 0, \\ 0, & x < 0 \end{cases}$

29. Calculate the inverse Fourier image $f(x)$ of the function $\tilde{f}(k) = -\frac{1}{\pi}\frac{k}{a^2+k^2}$.

30. Find a function $f(x)$ such that its Fourier image $\tilde{f}(p) = cf(p)$, where c is a constant.

31. (a) The functions $f(x)$ and $g(x)$ are related by $f(x) = g(ax)$. How are their Fourier images related?

(b) The functions $f(x)$ and $g(x)$ are related by $f(x) = g(x - x_0)$. How are their Fourier images related?

32. Calculate the Fourier image $\tilde{f}(\vec{k})$ of the function $f(\vec{r}) = \alpha \frac{e^{-\mu r}}{r}$, where $r = |\vec{r}|$, $\vec{r} \in \mathbb{R}^3$, and α and μ are constants.

33. Calculate the Fourier image $\tilde{f}(\vec{k})$ of the function $f(\vec{r}) = \begin{cases} f_0, & r \leqslant r_0, \\ 0, & r > r_0 \end{cases}$ in $\mathbb{R}^3$.

34. Calculate the Fourier image $\tilde{f}(\vec{k})$ of the function $f(\vec{r}) = f_0 \left(\frac{1}{r} + \frac{1}{a}\right) e^{-\frac{2r}{a}}$ in $\mathbb{R}^3$.

35. For what values of α the series $\sum_{n=1}^{N} \frac{1}{n^\alpha}$ diverges as $N \to \infty$?

36. Compute the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

37. Compute the sum $\sum_{n=1}^{\infty} \frac{x^{n-1}}{n(n+1)}$ for all possible values of x.

38. Consider the function $J(a) = \sum_{n=0}^{\infty} n e^{-an}$ defined for positive a .

(a) Write an explicit expression for $J(a)$.

(b) Expand $J(a)$ around the point $a = 0$ up to $O(a^0)$.

39. Compute the series $J(a) = \sum_{n=-\infty}^{\infty} |2n+1| e^{-a|2n+1|}$ for $a > 0$.

40. Compute the following series:

(a) $\sum_{k=0}^{n} r^k e^{ik\phi}$, (b) $\sum_{k=0}^{n} r^k \cos k\phi$, (c) $\sum_{k=0}^{n} r^k \sin k\phi$.

41. ✿Consider the rubber cord with one end attached to a tree and another - to a car moving away from the tree with the velocity $V = 10m/s$. At some point an ant appears on the fixed end of the cord and starts running along it with the velocity $v = 10cm/s$. Will it reach the car? Assume the cord can be stretched up to arbitrary length.

42. Find all differentiable solutions of the equation $f(x)f(y) = f(x+y)$. Find all continuous solutions of the same equation.

43. ✿☢Give an example of the $C^\infty(\mathbb{R})$ function $f(x)$ with the following properties:

$$\begin{array}{ll} 1) & f^{(n)}(0) = 1 \text{ for some } n \geqslant 0, \\ 2) & f^{(i)}(0) = 0, \quad i \geqslant 0, \quad i \neq n, \\ 3) & \displaystyle\int_0^\infty |f(x)|dx < \infty. \end{array} \tag{12}$$

44. ✿Show that $\displaystyle\int_0^\pi \log(1 - 2a\cos x + a^2)dx = 0$ for $|a| < 1$.

45. ✿Consider the integral

$$I_n = \int_{-\infty}^{\infty} \frac{dx}{x} \sin(x) \prod_{k=1}^{n} \cos\left(\frac{x}{2k+1}\right).$$

Show that $I_n = \pi$ for $n = 1, 2, 3, 4, 5, 6$. Compute I_7.

VECTOR CALCULUS

1. Compute or show that the following limits do not exist:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^2}\sin(x^2+y^2)$

(b) ✿$\lim_{(x,y)\to(0,0)} x\ln(xy)$

2. Compute the Jacobian matrix of $f(x,y,z)=(y^2, xz-y, z+xy)$.

3. Find a vector along which the derivative of $f(x,y)=x(y^2+xy)$ at $(x,y)=(1,2)$ is nule.

4. Determine the tangent plane to the paraboloid

$$P=\{(x,y,z)\in\mathbb{R}^3 : z=1-x^2-y^2\} \tag{13}$$

at the point $(0,1,0)$.

5. Determine and classify all stationary points of $f(x,y)=\frac{y^2}{2}+xy+x^4$.

6. ⌚Invert the integration order of $\int_0^{2\pi}\int_{-1}^{\sin y} dx\,dy$.

7. Consider the region $D=\{(x,y)\in\mathbb{R}^2 : 0<2x<y<3-x^2\}$ and compute its area and centroid.

8. ⌚Compute the inertia moment around the Ox axis of the solid

$$S=\{(x,y,z)\in\mathbb{R}^3 : y^2+z^2\leq 1;\ 0\leq x\leq (y^2+z^2)^{\frac{1}{4}};\ y\geq 0;\ z\geq 0\} \tag{14}$$

with mass density given by $\rho(x,y,z)=x(y^2+z^2)$.

9. Find the surface area of a body formed by rotation of a curve $y=\sin x$ around x-axis in $\mathbb{R}^3$, $x\in[0,\pi]$.

10. Given the region $V_t=\{(x,y,z)\in\mathbb{R}^3 : 1\leq x^2+y^2\leq t;\ 0\leq z\leq 1;\ y>0\}$ and $F:[1,+\infty)\to\mathbb{R}$ the function given by

$$F(t)=\int_{V_t}\frac{e^{t(x^2+y^2)}}{x^2+y^2}dx\,dy\,dz, \tag{15}$$

compute $F'(4)$.

11. Find Jacobi matrix and Jacobian for the change of coordinates in $\mathbb{R}^3$ from Cartesian coordinates to

(a) spherical coordinates,

(b) cylindrical coordinates.

In what regions of $\mathbb{R}^3$ are these coordinate changes regular?

12. Consider the vector field $\vec{A}=\vec{\nabla}\log\frac{1}{r}$, where $r=\sqrt{x^2+y^2+z^2}$. Write the components of this field

(a) in Cartesian coordinates (x, y, z),

(b) in cylindrical coordinates (r, ϕ, z),

(c) in spherical coordinates (r, θ, ϕ).

13. ☢If (x, y, z) parametrize a surface embedded in $\mathbb{R}^3$ then show that

$$\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = -1. \tag{16}$$

14. Find local minima and local maxima of the function $f(x, y, z) = x^2 + y^2 + z^2$ on a surface defined by equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $0 < a < b < c$.

15. Determine the dimensions of a rectangular box with fixed volume V that minimize its area.

16. Show that maximizing the entropy $S = -\sum_i p_i \ln p_i$ with the constraints $\sum_i p_i = 1$ and $E = \sum_i p_i \epsilon_i$ yields the Boltzmann distribution.

17. A body moves in (xy)-plane along the trajectory $y = \log \cos x$. Find the path length of the body when $x \in [0, \frac{\pi}{6}]$.

18. Consider the field

$$\mathbf{F}(x, y, z) = \left(\frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2}, 2z\right). \tag{17}$$

(a) Compute the work done by $\mathbf{F}$ along the line $\{(\cos t, \sin t, t), 0 \leq t \leq 2\pi\}$.

(b) Compute the work done by $\mathbf{F}$ along the line defined by $y^2 + z^2 = 1$ and $x = y^2 - z^2$.

19. Consider the vector field $\vec{A} = \vec{\nabla}\frac{1}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) Find $\mathbf{div}\vec{A}$, $\triangle\vec{A}$, $\mathbf{curl}\vec{A}$.

(b) How do the values of $\mathbf{div}\vec{A}$ and $\triangle\vec{A}$ depend on the choice of coordinate system?

20. Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 + z^2;\ 0 < z < 1\}, \tag{18}$$

oriented with unit vector n such that $n_z > 0$. Let $F(x, y, z) = \big(2xyz, z^2 - zy^2, z(1 - z)\big)$. Use the divergence theorem to compute the flux of F through S according to n.

21. ✿Use the divergence theorem to compute the volume of a cone in a quick way.

22. Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1;\ z < 0\}, \tag{19}$$

oriented with unit vector n such that $n_z > 0$. Let $F(x, y, z) = \big(xz, yz, 1 - z^2\big)$. Compute the flux of F through S according to n using

(a) the divergence theorem.

(b) Stokes' theorem.

23. ✿☢If $B_R = \{\mathbf{r} \in \mathbb{R}^3 : ||\mathbf{r}|| < R\}$ and $f(\mathbf{r})$ and $g(\mathbf{r})$ are two scalar fields that are zero if $||\mathbf{r}|| > 1$, show that

$$\int_{B_\pi} f\nabla^2 g = \int_{B_\pi} g\nabla^2 f. \tag{20}$$

24. ✿☢Let S be an orientable surface with normal unit vector $\mathbf{n}$, g a scalar function which diverges at ∂S and $\mathbf{N}$ a vector field which, for any point, is a normal unit vector to the equipotential surface of g where that point belongs. Show that

$$\int_S \frac{\nabla \times \mathbf{N}}{g} \cdot \mathbf{n}\, dS = 0. \tag{21}$$

25. ✿☢Use Leibniz's rule and Stokes' theorem to show that Faraday's law of induction,

$$\int_{\partial\Sigma(t)} \big(\mathbf{E}(t,\mathbf{r}) + \mathbf{v}(t,\mathbf{r}) \times \mathbf{B}(t,\mathbf{r})\big) \cdot \mathbf{dl} = -\frac{d}{dt}\int_{\Sigma(t)} \mathbf{B}(t,\mathbf{r}) \cdot \mathbf{n}(t)\, dS, \tag{22}$$

implies the Maxwell-Faraday equation

$$\nabla \times \mathbf{E}(t,\mathbf{r}) = -\frac{\partial \mathbf{B}}{\partial t}(t,\mathbf{r}), \tag{23}$$

where $\Sigma(t)$ is an orientable surface that changes in time and $\mathbf{v}(t,\mathbf{r}) = \frac{d\mathbf{r}}{dt}$ for $\mathbf{r} \in \partial\Sigma$ is the velocity of the boundary.

COMPLEX ANALYSIS

1. Write the following complex numbers in polar form:

$$3i, \qquad 1-i, \qquad 2+i, \qquad -4, \qquad 2-i\sqrt{3}, \qquad \frac{2-i}{1+4i}, \qquad |3+i|.$$

2. Write the following complex numbers in cartesian form

$$e^{i\theta}, \qquad 3e^{i\pi/4}, \qquad \frac{1}{2}e^{i\pi}, \qquad 2e^{2i\pi/3}, \qquad e^{-3i\pi/4}.$$

3. Find the (possibly complex) roots of these polynomials

$$z^2+3z+12=0, \qquad z^4+5z^2+4=0, \qquad z^6=1, \qquad z^3=-1.$$

4. Compute the Laurent series of $f(z) = \dfrac{e^z}{(z-1)^2}$ around $z_0 = 1$; give the region of convergence.

5. ⌚Compute the Laurent series of $f(z) = \dfrac{1}{(z-3)^3}$ around $z_0 = i$; give the region of convergence.

6. Compute the Laurent series of $f(z) = \dfrac{z}{(z+1)(z-1)}$ around $z_0 = 1$; give the region of convergence.

7. Compute the Laurent series of $f(z) = \sin\dfrac{z}{1-z}$ around $z_0 = 1$; give the region of convergence.

8. ⌚Find the Laurent series of $f(z) = \frac{1}{(z+1)(z+2)}$ such that it converges in the regions

- (a) $|z| < 1$;
- (b) $1 < |z| < 2$;
- (c) $|z| > 2$;
- (d) $0 < |z+1| < 1$.

9. Find an analytic map that sends the unit disk ($|z| < 1$) onto the left half plane ($\mathrm{Re}(z) < 0$)

10. Take a disk of radius R with a branch cut on the negative real axis; what does $\log z$ map this onto? Where is the origin mapped onto?

11. The function $f(z) = \sqrt{z}$ has branch points at the origin and at infinity. Characterize the branch cut corresponding to the parametrization $z = \rho e^{i\theta}$ with $\theta \in (-\frac{\pi}{4}, \frac{15\pi}{4})$.

12. ✿Take the function $f(z) = \sqrt{z}$ with branch cut $z(t) = te^{it}$ with $t \geq 0$. Compute the discontinuity of $f(z)$ at $z = i\frac{\pi}{2}$.

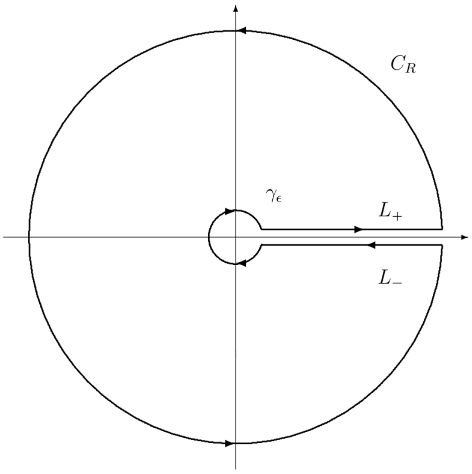

Figure 1: Contour for problem 23

13. Determine the branch points of $f(z) = \log(1+z) - \log(1-z)$. Suggest two branch cut configurations, one where $f(z)$ is analytic on the imaginary axis, and one where it is not.

14. What is the image of $\{z|\mathrm{Re}(z) > \mathrm{Im}(z) > 0\}$ under the mapping e^{z^2}?

15. Compute $\displaystyle\int_0^\infty \frac{1}{1+x^2}\mathrm{d}x$;

16. Compute $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^6}\mathrm{d}x$;

17. Compute $\displaystyle\int_{-\infty}^\infty \frac{e^{i\alpha x}}{x^2+m^2}\mathrm{d}x$ with $\alpha, m \in \mathbb{R}$ (hint: consider the two cases of positive or negative α separately);

18. Compute $\displaystyle\int_{-\infty}^\infty \frac{e^{i\alpha x}}{(3-ix)(1+ix)}\mathrm{d}x$;

19. Compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\mathrm{d}x$;

20. Compute $\displaystyle\int_{-\infty}^\infty \frac{x^2}{x^4 - 2x^2\cos 2\theta + 1}\mathrm{d}x$;

21. Compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}\mathrm{d}x$ with $|a| < 1$;

22. Compute $\displaystyle\int_0^{\pi} (\cos\theta)^{2n}\mathrm{d}\theta$;

23. (a) Compute the integral $I_1 = \displaystyle\int_C \frac{\mathrm{d}z}{(z+i)\sqrt{z}}$, where the contour C is shown in figure 1, $C = L_+ \cup C_R \cup L_- \cup \gamma_\epsilon$, and we send the radius of γ_ϵ to zero and the radius of C_R to infinity. Note that, because of the square root, $z = 0$ is a branch point. We choose to have a branch cut on the positive real axis.

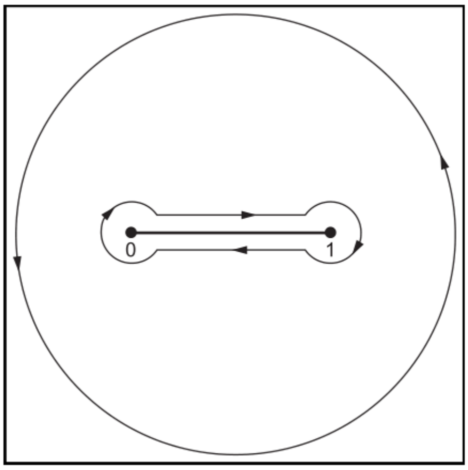

Figure 2: Integration contour for exercise 25.

(b) Find a relation between the integral I_1 and $I_2 = \displaystyle\int_0^\infty \frac{1}{(x+i)\sqrt{x}}\mathrm{d}x$, then use the result from the previous point to find the value of I_2.

24. Repeat exercise 23 this time with $I_1 = \displaystyle\int_C \frac{z^p}{z^2+1}\mathrm{d}z$ and $I_2 = \displaystyle\int_0^\infty \frac{x^p}{x^2+1}\mathrm{d}x$ with $0 < p < 1$; the contour is the same as before.

25. ☢ Compute $\displaystyle\int_0^1 \frac{1}{(x^2-x^3)^{1/3}}$ (**hint:** use the contour in figure 2);

26. ⌚ Compute $\displaystyle\int_0^\infty \frac{\log x}{1+x^\alpha}\mathrm{d}x$ for $\alpha \in \mathbb{N}$, $\alpha > 1$ (**hint:** solve problem 37 first. Now, as a contour, use a circular wegde of the complex plane that makes a $2\pi/\alpha$ angle with the positive real axis);

27. ⌚Compute $\displaystyle\int_0^\infty \frac{\log^2 x}{1+x^2}\mathrm{d}x$ (**hint:** solve problem 26 first. Use the function $\frac{\log^3 x}{1+x^2}$ integrated over some smart choice of contour);

28. (a) Matsubara summation: in statistical mechanics, one often has to carry out summations over Matsubara frequencies. These frequencies appear when the system is put at finite temperature, and the summation can be tedious to carry out. We will consider the expectation value of the number of particles of a bosonic non-interacting gas. Consider the function $h(\omega_n) = -\dfrac{T}{i\omega_n - \xi}$. Here ω_n are called Matsubara frequencies. In this case (the bosonic one) they are given by $\omega_n = 2\pi n T$.

What we want to compute is $S \equiv \sum_n h(\omega_n)$. To do so, we introduce an auxiliary function $g(z) = \frac{\beta}{e^{\beta z}-1}$ (setting $k_B = 1$, $\beta = T^{-1}$).

- Where are the poles of h? Where are those of g?
- Consider now the function $g(z)h(-iz)$. Find a contour for which

$$\frac{1}{2\pi i}\oint \mathrm{d}z g(z)h(-iz) = S\,.$$

This contour encompasses an infinite number of poles.

- Since for large z the function decays fast enough the residue at infinity vanishes; inflate the contour and flip its orientation, so that it includes only a finite number of poles, in this case only one.
- Carry out the integration. You should find $-T\sum_n \frac{1}{i\omega_n - \xi} = \frac{1}{e^{\beta\xi}-1}$; as expected, this is the Bose distribution.

(b) Redo the previous exercise for fermions: the frequencies are $\omega_n = (2n+1)\pi T$. It's convenient to pick $g(z) = \frac{\beta}{e^{\beta z}+1}$. You should find

$$T\sum_n \frac{1}{i\omega_n - \xi} = \frac{1}{e^{\beta\xi}+1}$$

29. Use the saddle point method to approximate $f(t) = \int_{-\infty}^{\infty} \frac{e^{-t(z^2-1/4)}\cos tz}{1+z^2}\mathrm{d}z$ for $t \gg 1$.

30. The modified Bessel function of the second kind has the following integral representation: $K_\nu(x) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{x}{2}\left(s+\frac{1}{s}\right)\right)\frac{\mathrm{d}s}{s^{1-\nu}}$. Find the asymptotic expansion for $x \gg 1$.

31. ☢✿The Henkel function of the first kind has the following integral representation: $H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{0+i\epsilon}^{-\infty+i\epsilon} \exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right]\frac{\mathrm{d}z}{z^{\nu+1}}$. The $i\epsilon$ factor is present since we have a branch cut along the negative real axis. Find the asymptotic behavior as $x \gg 1$.

32. Find Stirling's approximation: consider $n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x}dx$ for $n \gg 1$.

33. Using Stirling's approximation, find the leading behavior of:

(a) $\binom{2N}{N}$ for large N.

(b) $S = \log Z(N,m)$ with $Z = \frac{N!}{(N-m)!m!}$ for large N.

34. Find an analytic continuation $g(z)$ of the function $f(z) = \int_0^\infty te^{-zt}dt$. $f(z)$ is defined only for $z > 0$. What is the domain of $g(z)$?

35. ⌚Riemann's zeta function is defined as $\zeta(z) = \sum_{n=1}^\infty n^{-z}$.

- For which values of z does this converge?
- Show that the zeta function admits the integral representation

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t$$

hint: the relation $\sum_{m=1}^\infty e^{-mt} = \frac{e^{-t}}{1-e^{-t}}$ might prove useful.

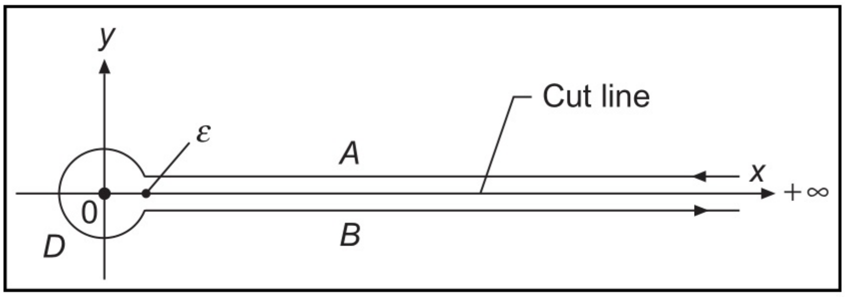

Figure 3: Contour for problem 35

- Now we take the contour of figure 3. Since we want to allow non integer values of z, there is a branch cut along the positive real axis. What is $\frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t$ along this contour?
- For $z < 0$ we can deform the contour by sending the radius of the circle D to infinity; the price to pay is that, to compute I, we have to evaluate an infinite number of poles, but this can be done. By comparing this result to what you did in the previous step, you should find that

$$\zeta(z) = \zeta(1-z)\frac{e^{3\pi iz/2} - e^{\pi iz/2}}{e^{2\pi iz} - 1}\frac{(2\pi)^z}{\Gamma(z)} .$$

- Using the formula you just found, show that $\zeta(-1) = -\frac{1}{12}$. Notice that this doesn't mean that $1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}$, since that $\zeta(z) = \sum_{n=1}^\infty n^{-z}$ is valid only for $z > 1$. On a side note, this is the reason why string theory (without supersymmetry) needs 26 spacetime dimensions.

36. ✿The Gamma function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$ is originally defined only for $t > 0$; it can however be analytically continued to negative values of z. Show that, as $z \to -n$, where $n \in \mathbb{N}_0$, $\Gamma(z)$ has poles; compute the order and the residue of these poles.

37. Solve the integral $\displaystyle\int_0^\infty \frac{1}{1+x^\alpha}dx$ for $\alpha \in \mathbb{N}, \alpha > 1$ by integrating the function $\displaystyle\frac{\log z}{1+z^\alpha}$ along the contour of figure 1.

VARIATIONAL PRINCIPLE

1. ☢"Fubini instanton" Consider the functional $S[f] = 2\pi^2 \int_0^\infty dx x^3 \left(\frac{1}{2} f'^2 + V(f)\right)$, where $V(f) = -\frac{\lambda}{4} f^4$, and $f \in C^2[0, \infty]$.

(a) Write the differential equation on f, whose solution is an extremum of $S[f]$.
(b) Find all solutions to this equation satisfying the boundary conditions $f(\infty) = f'(\infty) = 0$.
(c) Compute the value of $S[f]$ on these solutions.

2. Find an extremum of the functional $J[f] = \int_0^{\pi/2} dx(f'^2 - f^2)$ in a class of functions $f \in C^2[0, \pi/2]$ satisfying the boundary conditions $f(0) = 0$, $f(\pi/2) = 1$.

3. "Brachistochrone curve" On a vertical plane xOy consider the two points A and B. Find a curve $y = y(x)$ connecting these points and such that an ideal point-like body, that starts at rest at the point A and moves along this curve without friction under constant gravity, reaches the point B within the shortest time.

4. ⌚Consider the points A and B lying on the plane xOy. Find a curve $y = y(x)$ connecting these points and such that rotation of $y(x)$ around the axis Ox gives a surface in $\mathbb{R}^3$ with a minimal area.

5. Find an extremum of the functional $J[f] = \int_0^1 dx(360x^2 f - f''^2)$ in a class of functions $f \in C^3[0, 1]$ satisfying the boundary conditions $f(0) = 0$, $f'(0) = 1$, $f(1) = 0$, $f'(1) = 2.5$.

6. Consider the functional $S[x, y, z] = \int_{t_1}^{t_2} dt L(x, y, z, x', y', z')$, where

$$L = \frac{m}{2}(x'^2 + y'^2 + z'^2) - U(x, y, z), \tag{24}$$

and U is a C^1 function of x, y, z. Write the system of equations on $x(t), y(t), z(t)$ whose solution is an extremum of $S[x, y, z]$ in a class of $C^2([t_1, t_2])$ functions with fixed boundary conditions.

7. Consider the functional $J[f] = \iint\limits_D dxdy \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$, where D is a domain in (xy)-plane with the boundary ∂D. Write the equation for f, whose solution in a class of functions $f \in C^2(\bar{D})$, satisfying the boundary condition $f(x, y)|_{\partial D} = f_0(x, y)$, is an extremum of $J[f]$.

8. Consider the functional $J[f] = \iint\limits_D dxdy \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 2fg\right]$, where D is a domain in (xy)-plane with the boundary ∂D, and $g = g(x, y)$ is a continuous function in $\bar{D}$. Write the equation on f, whose solution In a class of functions $f \in C^2(\bar{D})$, satisfying the boundary condition $f(x, y)|_{\partial D} = f_0(x, y)$, is an extremum of $J[f]$.

9. Consider the biharmonic equation

$$\frac{\partial^4 f}{\partial x^4}+2\frac{\partial^4 f}{\partial x^2\partial y^2}+\frac{\partial^4 f}{\partial y^4}=0. \tag{25}$$

Find a functional $J[f]$ whose extrema in a class of $C^4(\bar{D})$ functions $f=f(x,y)$ with fixed boundary conditions, $f(x,y)|_{\partial D}=f_0(x,y)$, satisfy this equation.

10. Consider the following problem

$$\begin{cases} F_f-\dfrac{\partial}{\partial x}F_{f_x}-\dfrac{\partial}{\partial y}F_{f_y}=0, & (x,y)\in D \\ F_{f_x}n_x+F_{f_y}n_y+=g(s), & x\in\partial D, \end{cases} \tag{26}$$

where $f=f(x,y)\in C^2(\bar{D})$, $F=F[f,f_x,f_y](x,y)\in C^2(\mathbb{R}^3\times\bar{D})$, $g\in C^1(\partial D)$, $\partial/\partial n$ denotes a normal derivative on ∂D, $f_x\equiv\frac{\partial f}{\partial x}$ and $F_f\equiv\frac{\partial F}{\partial f}$.

(a) Show that the solutions to this equation are given by extrema of the functional

$$J[f]=\iint\limits_D dxdyF-\int_{\partial D}dsfg. \tag{27}$$

(b) How must the functional above be modified to give the mixed boundary conditions on the function f:

$$F_{f_x}n_x+F_{f_y}n_y+h(s)f=g(s), \quad h\in C^1(\partial D) \tag{28}$$

11. Let $f\in C^2(\bar{D})$ be an extremum of the functional $S[f]=\displaystyle\int_D dxL(f,f')$ in a class of functions satisfying the boundary condition $f|_{\partial D}=f_0$. What additional condition must be imposed on f to make it an extremum of $S[f]$ in a class of $C^2(\bar{D})$ functions with *all* possible boundary conditions?

12. Construct the functional $S[\psi,\psi^\dagger]$ whose variation with respect to $\psi=\psi(\vec{r},t)$ and $\psi^\dagger=\psi^\dagger(\vec{r},t)$ gives

(a) the Schrodinger equation $i\hbar\dfrac{\partial\psi}{\partial t}=\hat{H}\psi$ and its conjugated,

(b) the stationary Schrodinger equation $\hat{H}\psi=E\psi$ and its conjugated.

Here $\hat{H}$ is some hermitian operator.

13. Consider the functional $S[\psi,\psi^\dagger]=\displaystyle\int_{-\infty}^{\infty}dx\psi^\dagger(E-\hat{H})\psi$, where

$$\hat{H}=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega^2x^2}{2}. \tag{29}$$

(a) Find an extremum of $S[\psi,\psi^\dagger]$ in a class of $C^1(\mathbb{R})$ functions of the form $\psi(x)=\sqrt{\pi\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$.

(b) Find an eigenvalue of $\hat{H}$ corresponding to this extremum.

14. Consider the functional $S[\phi, \phi^*] = 4\pi \int_{-\infty}^{\infty} dt \int_0^{\infty} drr^2(\dot{\phi}\dot{\phi}^* - \phi'\phi'^* - U(\phi\phi^*))$, where $U \in C^1(\mathbb{R})$ and ϕ is a $C^2(\mathbb{R} \times [0, \infty))$ complex-valued function of t and x.

(a) Varying with respect to ϕ and ϕ^*, obtain the Euler-Lagrange equation and its conjugated. Substituting the *ansatz* $\phi(r, t) = f(r)e^{i\omega t}$, rewrite them as an equation on a function f of a single variable r.

(b) Substitute the ansatz $\phi(r, t) = f(r)e^{i\omega t}$ into the functional $S[\phi, \phi^*]$ first, and, varying with respect to f, obtain the Euler-Lagrange equation on f. Compare with the result of the previous point.

15. "Derrick's theorem" Consider the functional

$$E[\phi] = \int d^d x \left(\frac{1}{2} K_{ab}(\phi) \sum_{i=1}^{d} \partial_i \phi^a \partial_i \phi^b + V(\phi) \right), \tag{30}$$

where $\phi = \phi(x_1, ..., x_d) \in C^2(\mathbb{R}^d)$, $K_{ab}(\phi)$ is a positive-definite matrix for any ϕ, i.e.,

$$K_{ab}(\phi) \partial_i \phi^a \partial_j \phi^b \geqslant 0, \tag{31}$$

where the equality implies $\phi = 0$, and $V(\phi) \geqslant 0$, $V(\phi) = 0 \Rightarrow \phi = 0$.
Suppose $\phi_0(x)$ is a non-zero extremum of $E[\phi]$. Consider the configurations of the form $\phi_\lambda(x) = \phi_0(\lambda x)$, obtaining from $\phi_0(x)$ by stretching the coordinates by a factor of λ.

(a) Show that $E(\lambda) \equiv E[\phi_\lambda]$ must satisfy

$$\left. \frac{dE}{d\lambda} \right|_{\lambda=1} = 0. \tag{32}$$

(b) Using the notations

$$\Gamma = \int d^d x \frac{1}{2} K_{ab}(\phi_0) \sum_{i=1}^{d} \partial_i \phi_0^a \partial_i \phi_0^b, \quad \Pi = \int d^d x V(\phi_0), \tag{33}$$

show that the above relation implies

$$(2 - d)\Gamma - d\Pi = 0. \tag{34}$$

(c) Give a conclusion about the existence of ϕ_0, if

(a) $d > 2$ (b) $d = 2$ (c) $d = 1$

DIFFERENTIAL EQUATIONS

1. Solve $y'(x) = xe^{x^2 - 2\log y(x)}$

2. Solve $\begin{cases} x'(t) = -x(t) + 6y(t) \\ y'(t) = 2x(t) + 3y(t) \end{cases}$

3. Solve $(x+1)\dfrac{dy}{dx} = 2y + (x+1)^{5/2}$ with $y(0) = 3$

4. ✿Solve $y'(x) = f(x)y(x) + g(x)y^n(x)$

5. Solve $y'' - y' - 2y = 0$

6. Solve $y'' - 6y' + 9y = 0$ with $y(1) = 1$ and $y'(3) = 0$

7. Solve $y''' - 3y' + 2y = 0$

8. Solve $y'' - 3y' + 2y = \sin x$

9. Solve $y'' + 3y' + 2y = \tanh x$

10. Find $C(t)$ when

$$C'(t) = \alpha(a - C(t))(b - C(t))$$

if

(a) $a \neq b$
(b) $a = b$

and with $C(0) = 0$.

11. Solve

$$(xy^2 - y)\mathrm{d}x + x\mathrm{d}y = 0$$

12. Solve $y' - y = e^{3t}$ with $y(0) = 2$.

13. Solve $y'' - 3y' + 2y = e^{3t}$ with $y(0) = 1$, $y'(0) = 0$

14. ⌚Solve $y'' - 6y' + 15y = 2\sin 3t$ with $y(0) = -1$ and $y'(0) = 4$

15. Solve $y' + 2y = e^{-t}\theta(t)$, where θ is the Heaviside function, $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x > 0$.

16. Solve $y'' + 16y = \theta(\pi - t)$ with $y(0) = y'(0) = 0$ and where θ is the Heaviside function, $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x > 0$.

17. Consider a forced damped harmonic oscillator:

$$\ddot{y}(t) + 2k\dot{y}(t) + \Omega^2 y(t) = f(t) \tag{35}$$

the Green function $G(t)$ is defined such that

$$y(t) = \int dt' G(t - t') f(t'). \tag{36}$$

Find $G(t)$ when

(a) $\Omega > k > 0$ (oscillating system)

(b) $\Omega = k$ (critical damping)

(c) $k > \Omega > 0$ (overdamped system)

18. Find the charge distribution that gives the electrostatic potential

$$\varphi(x, y, z) = \frac{Z}{4\pi\varepsilon_0} \frac{e^{-ar}}{r} \tag{37}$$

19. ⌚Find the general solution of the equation :

$$x(2-x)\frac{d^2y}{dx^2} + 3(1-x)\frac{dy}{dx} - y = 0 \tag{38}$$

as a power series about $x = 1$.

20. Given the differential equation

$$\frac{d}{d\xi}\left(\xi \frac{du}{d\xi}\right) + \left(\frac{1}{2}E\xi + \alpha - \frac{m^2}{4\xi} - \frac{1}{4}F\xi^2\right) u = 0 \tag{39}$$

Find the first three terms of a series solution around $\xi = 0$ by using the largest solution of the indicial equation.

21. ⌚Consider Schrodinger equation for a quantum harmonic oscillator with small quartic perturbation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{x^2}{2} + \frac{gx^4}{4}\right)\psi(x) = E_0(g)\psi(x) \tag{40}$$

and make the ansatz

$$\psi(x) = e^{-x^2/2} \sum_{n=0}^{\infty} \left(\frac{g}{4}\right)^n B_n(x) \qquad \text{with} \quad B_0(x) = 1 \tag{41}$$

$$E_0(g) = \sum_{k=0}^{\infty} a_k \left(\frac{g}{4}\right)^k . \tag{42}$$

We already know that $a_0 = \frac{1}{2}$ from the unperturbed oscillator. We want to find the first two corrections a_1 and a_2.

(a) Find a recurrence relation for $B_k(x)$ and a_k

(b) Solve the relation by assuming $B_i(x) = \sum_{j=1}^{2i} x^{2j}(-1)^i B_{i,j}$.

(c) Considering different powers of x, find the following relations

$$a_n = (-1)^{n+1} B_{n,1} \tag{43}$$

$$2jB_{n,j} = (j+1)(2j+1)B_{n,j+1} + B_{n-1,j-2} - \sum_{k=1}^{n-1} B_{n-k,1}B_{k,j} \tag{44}$$

(d) Find a_1 and a_2. You can check that your result agrees with the usual perturbation theory.

22. Consider the equation

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0 \tag{45}$$

and show that if y_1 is a solution to this equation, then also $y_2 = y_1 \int^x ds \frac{e^{-\int^s dt P(t)}}{[y_1(s)]^2}$ is.

23. Verify that $\displaystyle\int_1^\infty \mathrm{d}t \frac{e^{-xt}}{\sqrt{t^2-1}}$ is a solution to the differential equation $y'' + \frac{1}{x}y' - y = 0$.

24. Bessel equation of order p is

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \tag{46}$$

Assuming $y = \sum_{m=0} a_m x^{r+m}$, find the two roots of the indicial equation (the two possible values of r). For both of them, solve the recurrence relation for the a_i. You should find, for the largest root, $J_p(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}$, and for the smallest $J_{-p}(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}$ (for the normalization, assume that $a_0 = \frac{1}{2^n n!}$). These are Bessel function of the first kind.

25. Prove the following properties of the Bessel functions of the first kind:

(a) $\dfrac{d}{dx}\left(x^\nu J_\nu(x)\right) = x^\nu J_{\nu-1}(x)$

(b) $\dfrac{d}{dx}\left(x^{-\nu} J_\nu(x)\right) = -x^{-\nu} J_{\nu+1}(x)$

(c) $\dfrac{d}{dx} J_\nu(x) = \dfrac{1}{2}\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)$

(d) $J_{\nu-1}(x) + J_{\nu+1}(x) = \dfrac{2\nu}{x} J_\nu(x)$

26. ✿The function $K_\nu(x)$ is the solution of the equation $x^2y'' + xy' - (x^2+\nu^2)y = 0$ that diverges when $x \to 0$. Find the asymptotic behavior for small x for $\nu \neq 0$. What happens for $\nu = 0$? For the sake of completeness, $K_\nu(x)$ is called modified Bessel function of the second kind.

27. Given the eigenvalue problem $\mathcal{L}(x)\psi(x) = \lambda\psi(x)$ with $\mathcal{L}(x) = p_0(x)\dfrac{d^2}{dx^2} + p_1(x)\dfrac{d}{dx} + p_2(x)$, $\mathcal{L}$ is self-adjoint if $p_0' = p_1$. Find a function $f(x)$ such that, by multiplying the following ODE, it makes them self adjoint:

(a) Laguerre's ODE: $xy'' + (1-x)y' + ay = 0$

(b) Hermite's ODE: $y'' - 2xy' + 2\alpha y = 0$

(c) Chabyshev's ODE: $(1-x^2)y'' - xy' + n^2y = 0$

28. ⌚Develop a series solution for Laguerre's ODE (given in problem 27).

29. Find the solutions of:

$$y'' + \lambda y = 0 \tag{47}$$

with $y'(0) = y'(\pi) = 0$;

30. Find the solutions of

$$x^2 y'' + 3xy' + \lambda y = 0 \tag{48}$$

with $x \in [1, e]$ such that $y(1) = y(e) = 0$.

31. Find the non-zero solutions of

$$(xu'(x))' = -\lambda \frac{u(x)}{x} \tag{49}$$

such that $u(1) = 0$ and $u'(e) = 0$. What values of λ are allowed?

32. Show that $\delta(x) = \lim_{\epsilon \to 0} \operatorname{Im} \frac{1}{\pi} \frac{1}{x - i\epsilon}$

33. Show that $\delta(\alpha x) = \frac{1}{\alpha}\delta(x)$, and more generally, $\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$ where x_i are the root of f, i.e. $f(x_i) = 0$.

34. Compute $\int_{-\infty}^{\infty} \delta''(x-2) \frac{1}{1+x^2} \mathrm{d}x$

35. Find the coefficients a, b, c so that

$$\delta''(x - x_0) \frac{1}{(1 + x - x_0)} = a\delta''(x - x_0) + b\delta'(x - x_0) + c\delta(x - x_0)\,.$$

36. Solve $y''(x) - 3y'(x) + 2y(x) = \delta(x - 1)$ with $y(0) = 0$ and $y(1) = 1$.

37. Solve $\left(-\frac{d^2}{dx^2} + m^2\right) G(x) = \delta(x)$

38. ☢Show that $\frac{d}{dx} \log x = \frac{1}{x} - i\pi\delta(x)$

39. Find the most general solution of $\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} + (x - 2y)f = 0$

40. Find the most general solution of $x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 0$

41. Find the most general solution of $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) f(x, y, z) = x - y$

42. Solve $\partial_x^2 u + 2\partial_x \partial_y u + \partial_y^2 u = 0$ with the boundary conditions $u(x, 0) = \sin x$ and $u(0, y) = y^2$.

43. Solve

$$(\nabla^2 + k^2)u(x, y, z) = 0 \tag{50}$$

so that $u = 0$ whenever any of the coordinates is equal to 0 or L. What are the allowed values of k^2?

44. A two dimensional rectangular slab (with the two sides of length a and b) has its edges fixed; at time $t = 0$ it has the profile

$$u(x, y, 0) = \sin \frac{2x\pi}{a} \sin \frac{3y\pi}{b}; \tag{51}$$

find $u(x, y, t)$ knowing that it satisfies the wave equation with a velocity v.

45. ⌚The surface of a sphere of radius R is kept at a constant temperature T_U for the upper hemisphere ($0 \le \theta < \pi/2$) and T_L for the lower hempisphere ($\pi/2 \le \theta \le \pi$). Find the stationary temperature distribution inside the sphere, at a distance r from the center, in an expansion in terms of $\frac{r}{R}$. Compute terms up to third order.

46. Consider an semi-infinite ($x \ge 0$) metal rod with conductivity κ. Find the heat distribution $u(x, t)$ if $u(x, 0) = u_0 \delta(x)$

47. ⌚Consider a metal rod of length L. One end is kept at temperature 0, and the other at temperature T_0. Find the temperature $T(x, t)$, knowing that $T(x, 0) = 0$.

48. ✿A string has endpoints which are fixed at $x = 0$ and $x = L$. At $t = 0$, the string is hit at $x = a$ so it starts vibrating:

$$y(x, 0) = 0 \qquad \partial_t y(x, 0) = L v_0 \delta(x - a) \tag{52}$$

where y is the amplitude of the oscillations (assume the amplitude to be small and the wave velocity to be v). Find $y(x, t)$.

49. ⌚☢Compute the stationary temperature distribution $u(\rho, \theta, z)$ of a semi-infinite cylinder of radius 1 when the curved surface is kept at temperature 0 if

(a) the flat surface is kept at a temperature u_0

(b) the flat surface is kept at a temperature $u_0 \rho \sin \theta$

50. Consider a spherically symmetric potential satisfying the Laplace equation in d dimensions ($d \ge 3$) and vanishing at infinity. Show that this potential can be written as $u(r) = \frac{a}{r^b}$, where a is a constant, and find the value of b.

51. ✿A conducting sphere of radius a with zero total charge is exposed to a uniform electric field $\vec{E}$ in the z direction. Find the electrostatic potential outside the sphere if we set the surface of the sphere to have potential zero.

PROBABILITY

1. A collection of stories in 5 volumes is placed on a bookshelf in a random order. What is the probability that the order is correct (direct or inverse)?

2. Find the least number of students in a group such that the probability that at least two students have the same birthday is not less than $\frac{1}{2}$.

3. What is more probable - to get at least one 1 in throwing four dice, or to get at least two 1 in 24 throws of two dice?

4. Among N tickets for N students there are n happy tickets. The students take the tickets one by one, each takes one random ticket. What is the probability that j's student gets a happy ticket, $1 \leqslant j \leqslant N$?

5. 5 people decide to have a party with presents. Everyone prepares one present and brings it to the party, where the presents are mixed. Then everyone takes one random present. What is the probability that nobody gets his own present?

6. Three faces of a tetrahedron are painted red (R), green (G) and blue (B), while the forth face is painted in all three colors (see the figure). Denote by $P(A)$ the probability that it falls on a face containing the color A, $A = R, G, B$.

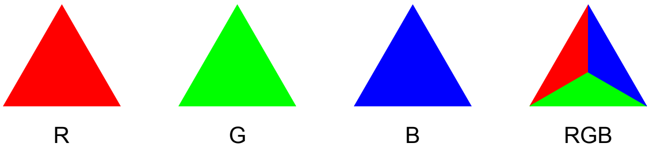

Figure 4: The faces of the tetrahedron

(a) Check if $P(A \cap B) = P(A)P(B)$ for all $A, B = R, G, B$.

(b) Check if $P(R \cap G \cap B) = P(R)P(G)P(B)$. Are the events R, G, B pairwise independent? Are they mutually independent?

7. Let a, b, c be three independent random variables distributed uniformly between 0 and 1. Find the probability that the roots of the equation $ax^2 + bx + c = 0$ are real.

8. Two persons agreed to meet at some place between 2 and 3 o'clock. Whoever arrives first, he waits 10 minutes, then leaves. What is the probability to fail the meeting? Assume that anyone can arrive at any time within the given interval.

9. "Buffon's needle" A needle of length l is thrown randomly on a plane lined with parallel lines with distance d between them, $l < d$. Find the probability that the needle will cross some line.

10. Consider the particle moving in a gas of other particles. Given that the last collision of the particle occurred at $t = 0$, the probability that the next collision will occur between t and $t + \Delta t$ equals $\lambda \Delta t + o(\Delta t)$, when $\Delta t \to 0$. Find the probability $P(t)$ that the time between the nearest collisions will exceed t.

11. ⏲☘✿In nuclear physics, the intensity of a particle source is measured with Geiger-Muller counters. A particle entering the counter generates a discharge in it that lasts time τ, during which the counter does not record any particles entering the counter. Find the probability that the counter will count all particles entering it during time t if the following conditions are fulfilled:

(a) the particles enter the counter independently;

(b) the probability that during the time interval from t to $t + \Delta t$, k particles entered the counter is given by

$$p_k(t, t + \Delta t) = \frac{(a\Delta t)^k e^{-a\Delta t}}{k!}, \tag{53}$$

where a is the rate.

12. ☘✿"Random walk" Let A, B, x be integers, $A \leqslant x \leqslant B$. Consider the particle that starts moving from the point x at time $t = 0$. At each step $\Delta t = 1$ the particle can move left or right from its recent position with the probabilities p and $q = 1 - p$ correspondingly. If at some step it reaches the points A or B, it stays there forever (see example figure below).

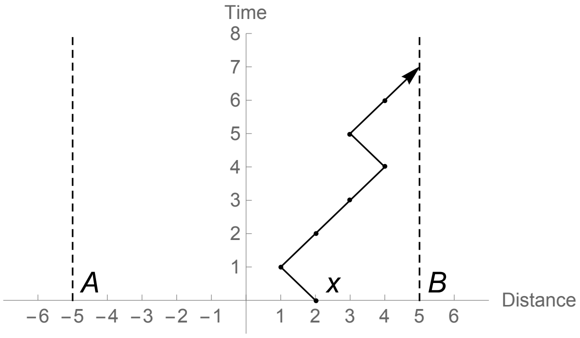

Figure 5: One possible particle's trajectory. Here $A = -B = 5$, $x = 2$, and the particle reaches the point B after 7 steps.

(a) Assuming that the total number of steps approaches infinity, compute the probabilities $\alpha(x)$, $\beta(x)$ to find the particle at the points A and B correspondingly, as functions of the initial position x of the particle.

(b) Find the mean time $m(x)$ of a random walk of the particle before it hits A or B. Assume that $m(x) < \infty$. Check that if $p = q = \dfrac{1}{2}$ and $A = -B$, then $m(0) = B^2$. Hence the mean time of random walk is given by a square of distance traveled.

13. ✿Prove that the equation

$$\int_{-\infty}^{\infty} \frac{x(t)dt}{(y-t)^2 + 1} = e^{-y^2} \tag{54}$$

has only negative solutions $x(t)$.

14. ✿Prove that if for some discrete random variable ξ, $\mathbf{E}\xi^2 = \mathbf{E}\xi^3 = \mathbf{E}\xi^4$, then ξ can only take values 0 or 1.

15. ⌚Consider two independent random variables ξ_1 and ξ_2 that are distributed according to the normal distribution with mean values a_1, a_2 and variances σ_1^2, σ_2^2 correspondingly. Find the distribution of a random variable $\xi_1 + \xi_2$. Repeat the exercise in case if $\xi_{1,2}$ are distributed according to the Poisson distribution with parameters λ_1 and λ_2.

16. Let ξ be a continuous random variable.

(a) Prove that if $\mathbf{E}e^{\lambda\xi}$ is finite then

$$P(\xi \geqslant x) \leqslant e^{-\lambda x}\mathbf{E}e^{\lambda\xi}, \quad \lambda > 0. \tag{55}$$

(b) Prove that if $\mathbf{E}|\xi|^m$ is finite then

$$P(|\xi| \geqslant x) \leqslant x^{-m}\mathbf{E}|\xi|^m, \quad x > 0, \ \ m > 0. \tag{56}$$

17. ⌚Consider the series ω of n experiments whose results are given by independent variables ξ_i, $i = 1, ..., n$, taking the values 1 (success) with the probability p, and 0 (fail) with the probability $q = 1-p$. Compose the sum $S_n(\omega) = \xi_1 + ... + \xi_n$. It is clear that for any *typical* series ω and for large n, $S_n(\omega)/n$ must be close enough to p. But what is the total amount of the typical series and how it behaves as n grows? Denote by $C(n,\epsilon)$ all typical series, or, more precisely,

$$C(n,\epsilon) = \left\{\omega | \ \left|\frac{S_n(\omega)}{n} - p\right| \leqslant \epsilon\right\}, \tag{57}$$

with some small ϵ.

(a) Show that if $\omega \in C(n,\epsilon)$, then the probability $p(\omega)$ of such series to realize is enclosed withing the region

$$e^{-n(H+\tilde{\epsilon})} \leqslant p(\omega) \leqslant e^{-n(H-\tilde{\epsilon})}, \tag{58}$$

where

$$\tilde{\epsilon} = \max\left\{\epsilon, \ \ \epsilon\left(-2\log(pq)\right)\right\}, \tag{59}$$

and the quantity

$$H = -p\log p - q\log q \tag{60}$$

is called *entropy*.

(b) Show that the total amount of the typical series lies in the region

$$e^{n(H-\tilde{\epsilon})} \leqslant N(C(n,\epsilon)) \leqslant e^{n(H+\tilde{\epsilon})}. \tag{61}$$

Hence the number of the typical series is exponentially large, while the probability of each of them is exponentially small.

18. Let ξ and η be two independent random variables taking values 1 and 0 with the probabilities p and $q = 1-p$ correspondingly. Find

(a) $\mathbf{E}(\xi + \eta|\eta)$, (b) $\mathbf{E}(\xi|\xi + \eta)$.

19. Let $\xi_1, ..., \xi_n, \tau$ be independent random variables, $\xi_1, ..., \xi_n$ have the same distribution, τ takes the values $1, ..., n$. Consider the sum of a random number of the random variables $S_\tau = \xi_1 + ... + \xi_\tau$. Show that

(a) $\mathbf{E}S_\tau = \mathbf{E}\tau \cdot \mathbf{E}\xi_1$,

(b) $\mathbf{E}(S_\tau|\tau) = \tau\mathbf{E}\xi_1$,

(c) $\mathbf{D}S_\tau = \mathbf{E}\tau \cdot \mathbf{D}\xi_1 + \mathbf{D}\tau \cdot (\mathbf{E}\xi_1)^2$,

(d) $\mathbf{D}(S_\tau|\tau) = \tau\mathbf{D}\xi_1$.

20. ⏲✿Find the expectation value of the area of the projection of a 3-dimensional randomly oriented cube with edge of length 1 onto a given plane.

21. ✿A reasonable way to estimate the number of birds in a large flock is to mark some of them. Suppose M birds were selected, marked and then released. Long time after, in a sample of n randomly selected birds X had the marker. What is the most probable total amount of birds in the population? Assume the flock was not mixed with other groups of birds.

22. Consider the sum a of 10^N real numbers a_k, $a = \sum\limits_{k=1}^{10^N} a_k$. Let $\tilde{a}_k$ be an approximation of a_k with precision 10^{-m}. Assume that the round-off errors $\delta_k = a_k - \tilde{a}_k$ are distributed uniformly within the interval $(-0.5 \cdot 10^{-m},\ 0.5 \cdot 10^{-m})$. Compose the sum $\tilde{a} = \sum\limits_{k=1}^{10^N} \tilde{a}_k$ and let $\delta = a - \tilde{a}$ be the total error of the approximation. For the given values of N and m find ϵ such that

$$P(|\delta| < \epsilon) > 0.99. \tag{62}$$

23. The dice is thrown 12000 times. Find the probability that the total number of 6's lies between 1800 and 2100.

24. "Monte-Carlo method" Consider the function $f(x_1, ..., x_n)$ defined in $V = \{-1 \leqslant x_i \leqslant 1,\ i = 1, ..., n\}$ and bounded from below and above, $|f(x_1, ..., x_n)| \leqslant C$. Define a random variable $\eta = f(\xi_1, ..., \xi_n)$, where ξ_i are distributed uniformly between -1 and 1.

(a) Show that $\mathbf{E}\eta = I$, where $I = \int\limits_V f(x_1, ..., x_n) d^n x$. Hence the random variables can be used to compute the high-dimensional integrals with a given precision.

(b) Consider the series of N random variables $\eta_i = f(\xi_{i1}, ...\xi_{in})$, $i = 1, ..., N$ where all ξ_{ij} are distributed uniformly in $[-1, 1]$. Then the quantity $\tilde{I} = \frac{1}{N}(\eta_1 + ... + \eta_N)$ for large N approaches I with high enough probability. Estimate how large N should be to ensure that

$$P(|\tilde{I} - I| < \Delta) \geqslant 1 - \alpha, \tag{63}$$

where Δ and α are given small numbers.

25. From a collection of 500 goods 70 were investigated, and in 14 of them various defects were revealed. Find the interval in which the fraction of the defected goods in the whole group lies with the probability 96%.

26. Here is the data about wheat yield from 8 identical wheat fields in conventional units:

26.5 26.2 35.9 30.1 32.3 29.3 26.1 25.0

There is a suspicion that the data about the third field is incorrect. Check if it should be dropped with 5% significance level.

27. In an experiment on the detection of cosmic rays a detector counts particles with different energies coming from different directions. The observed spectrum of the particles is shown in table below.

Energy, MeV	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
# of p.	15	71	75	68	39	17	10	4	1

At the significance level 0.05, test the hypothesis that the particle spectrum is distributed according to the Poisson distribution with parameter λ. Find the effective estimate for λ.

28. ✿A dice is turned randomly from one face to one of four adjoined faces. Suppose it fell 6 at $t = 0$. Find the probability P_n, $n \geqslant 1$, that after n such turns it will show 6 again. Find the limit $\lim\limits_{n\to\infty} P_n$.

29. An electron can occupy one of a countable number of energy levels in atom. The transition probabilities from i's to j's level per second are given by

$$P_{ij}(t = 1) = c_i e^{-\alpha|i-j|}, \tag{64}$$

where $\alpha > 0$ and c_i are some constants. Find

(a) $P_{ij}(t = 2)$,

(b) c_i.

30. One may naively argue that in one toss of two coins the probability to have two heads or tails equals 2/3. Indeed, if we use the scheme with three elementary events (it fell two heads, or two tails, or one head and one tail), the probability that the coins fall equally may seem to be 2/3. The correct scheme, however, must contain four different events (head-head, tail-tail, head-tail, tail-head) with the probabilities equally distributed among them, and it gives the correct answer 1/2. An experiment can be conducted to overcome the doubts about which scheme is more reasonable. Let the first hypothesis claim that the correct value is 2/3, and the second - that it is 1/2. Find how many coins tosses one should make to eliminate the first hypothesis with type I error 0.05 (probability to reject the second hypothesis when it is true) and type II error 0.05 (probability to accept the first hypothesis when it is false).

31. ✿It is known that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges while the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. One may ask about the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\xi_n}{n}$, where ξ_n are independent random variables taking the values $+1$ or -1 with the probabilities p and $q = 1 - p$ correspondingly.

(a) Show that if $p = q = \frac{1}{2}$, the series converges with the unit probability.

(b) Show that otherwise the series diverges with the unit probability.

32. ⌚*Normal numbers*

Let $x \in [0, 1)$. Consider the infinite decimal notation for x:

$$x = 0.a_1 a_2 ..., \quad a_i = 0, 1, ..., 9, \quad i \in \mathbb{N}, \tag{65}$$

where for the numbers with finite amount of decimal places we complete the records with infinite series of 0s. Now select *randomly* one x. What can one say about the typical distribution of digits $0, ..., 9$ in x? To answer the question, consider the following sequence of approximations:

$$\begin{aligned} &x_0 = 0 \\ &x_1 = 0.a_1 \\ &... \\ &x_n = 0.a_1 ... a_n \\ &... \end{aligned} \tag{66}$$

(a) Show that

$$P\left(\lim_{n \to \infty} \frac{1}{n} I_n(i) = \frac{1}{10} \right) = 1, \quad \forall i, \tag{67}$$

where $I_n(i)$ gives the number of i digit in x_n. The result implies that almost all numbers contain equal (and infinite) amount of all 10 digits. Such numbers are called *normal*.

(b) Check if the rational numbers are normal.

(c) Check if the following number is normal:

$$x = 0, 12345678910111213... \tag{68}$$

(all integers are written in ascending order).