

HW Puzzles: Week 5

March 19, 2026

Puzzle 1. *In each of three different universities there are n students. It is known that in each university every student is a friend of at least $n + 1$ students in the two other universities together. Show that there must be three students, one from each university, that are friends of each other.*

Puzzle 2. *We are given a three dimensional $n \times n \times n$ matrix. Each of the entries in the matrix is either 0 or 1. It is known that for every position in this matrix there is at least one entry in the union of the three "lines" (row, line, column) meeting there that is equal to 1. Show that there are at least $n^2/2$ entries in the matrix that are equal to 1.*

Puzzle 3. *Show that in any collection of segments in the plane one can find two of them, say a and b , such that for every line ℓ and every segment c in our collection it is true that the orthogonal projection of c on ℓ has length at most the sum of the lengths of the orthogonal projections on ℓ of a and of b .*

Hint: Look for two segments a and b with some maximum property.

Puzzle 4. *Consider the 3-dimensional version of the game peg-solitaire that was presented in class. We start with an arrangement of $n \times n \times n$ balls arranged in a grid. At every step we can take one ball and skip over its neighbor (in the direction of one of the axes) to an empty spot while removing the neighbor from the game. For which values of n it is possible to remain with only one ball at the end?*

Puzzle 5. *For red and blue points arranged in a circle we can add a red point somewhere and change the color of its two neighboring points. Or we can remove a red point and change the color of both of its two (ex-) neighbors (we must make sure that the number of points remaining is at least 2). Show that with these rules we can never start with two red points and end up with two blue points.*