HW Puzzles: Week 4

March 12, 2026

Puzzle 1. *Prove the puzzle about the light bulbs we saw in class without using linear algebra. Reminder: we have a graph and there is a light bulb at every vertex. We start when the bulbs are all off. At every time we press a vertex we change the status of the light bulb at this vertex and at the neighboring vertices. Show we can always turn on all the light bulbs.*

Hint: Induction.

Puzzle 2. *We want to cover the complete graph K_n by edge-disjoint complete subgraphs that are smaller than K_n. Show we need at least n such subgraphs.*

Puzzle 3. *Let B be a 3-dimensional box with dimensions $x \times y \times z$. Assume that B can be partitioned into smaller boxes, each of which with the property that at least one of its facets (two dimensional sides) is a square. Show there exist three integers a, b, c, not all equal to 0, such that $ax + by + cz = 0$.*

Puzzle 4. *Let S be the unit sphere in $\mathbb{R}^3$ and let $\mathcal{A}$ be a finite collection of spherical caps of the same radius $r < 1$ on S. The spherical caps can overlap. Show that we can find a finite set P of points on S such that every spherical cap in $\mathcal{A}$ contains an odd number of points from P. You may assume each spherical cap contains its boundary.*

Can we replace 'an odd number of points' by 'one point' for every family $\mathcal{A}$?

Puzzle 5. *Let $\mathcal{A}$ be a finite collection of closed circular discs in the plane. This time the discs can have distinct radii. Show that we can find a finite set P of points on S such that every disc in $\mathcal{A}$ contains an odd number of points from P.*

What happens if the discs in $\mathcal{A}$ are spherical caps on the unit sphere in $\mathbb{R}^3$, similar to the previous puzzle, but possibly with distinct radii this time? Can we always find such a set P?