HW Puzzles: Week 3

March 5, 2026

Puzzle 1. *Show that if m and n can be written as a sum of two squares, then also mn can be written as a sum of two squares. Based on this and on what we did in class, which are all the numbers n that can be written as a sum of two squares.*

Puzzle 2. *Let $f(n) = \sum_{p<n} p$. Show that $f(n)$ behaves like $\frac{n^2}{\log n}$ (in the sense that there are constants c_1 and c_1 such that $c_1 \frac{n^2}{\log n} < f(n) < c_2 \frac{n^2}{\log n}$).*

Puzzle 3. *Let T be a triangle whose vertices are all integer grid points. Assume that T does not contain (on its boundary and interior) any other integer grid points except for its vertices. Show how Minkowski's Theorem can be used to prove that the area of T is smaller than or equal to $\frac{1}{2}$. Conclude from here that it must be equal to $\frac{1}{2}$.*

Puzzle 4. *Let p be an odd prime. what is the cardinality*

$$|\{x^2 \mid x \in \mathbb{Z}_p\} \cap \{x^2 + 1 \mid x \in \mathbb{Z}_p\}|?$$

Puzzle 5. *Prove that for every two integers $n \geqslant m \geqslant 1$ the expression*

$$\frac{gcd(m,n)}{n} \binom{n}{m}$$

is an integer.