HW Puzzles: Week 2

February 26, 2026

Puzzle 1. *There are n kids and $2n-1$ dogs. Every kid likes some of the dogs. Show it is always possible to give every kid a different dog in such a way that it never happens that there is a kid that gets a dog that he or she does not like and at the same time one of the dogs that he or she does like is given to another kid.*

Puzzle 2. *Let P be the collection of all 2^n subsets of $\{1, 2 \ldots, n\}$. A* chain *in P is a collection of sets: $S_1 \subset S_2 \subset \ldots \subset S_k$, for some k, where all sets are from P. Using Hall's theorem, prove that P is a union of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ disjoint chains. Conclude that among any $\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1$ subsets of $\{1, \ldots, n\}$ there must be two such that one contains the other.*

Puzzle 3. *Let $S_1, \ldots, S_n$ be any n nonempty subsets of a set of $n+1$ different balls. Show that we can always color some (at least one but perhaps not all) of the balls by red or blue such that each S_i either contains balls of both colors red and blue, or no ball in S_i is colored.*

Puzzle 4. *Let let $G(A \cup B)$ be a bipartite graph where both $A = B = \mathbb{N}$. Assume that for every $a \in A$ $N(a)$ is a finite set and that Hall's condition is satisfied for finite sets. That is, for every finite set $S \subset A$ we have $|S| \leqslant |N(S)|$. Show there is a matching in G for all vertices in A.*

Puzzle 5. *For a $m \times n$ matrix a BIG horizontal move is to simultaneously and independently permute each of the rows of the matrix. Similarly, a BIG vertical move is to simultaneously and independently permute each of the columns of the matrix. What is the minimum number of BIG moves that is always enough to perform in order to move from a matrix to any possible given permutation of this matrix?*

Puzzle 6. *There are $100n$ people and $100n^2$ presents. These $100n^2$ presents are in fact of only n different kinds, each of which appears $100n$ times.*

We start when each of the $100n$ people gets n of the presents at random. However, the desired situation is that each of the $100n$ people has all the n different presents.

The rules are that pairs of people are allowed to swap presents but once you received a present from someone you are not allowed to pass it further to another person.

Show that it is always possible to do the swapping in a clever way such that at the end we reach the desired situation.