Discrete Optimization 2024 (EPFL): Problem set of week 8

May 2, 2024

Reminder: The min-max theorem for zero-sum games with mixed strategies says that for every $m \times n$ matrix A we have

$$\min_y \max_x yAx = \max_x \min_y yAx,$$

where the minimum is over all $y = (y_1, \ldots, y_m) \geq 0$ such that $\sum y_i = 1$. The maximum is over all $x = (x_1, \ldots, x_n) \geq 0$ such that $\sum x_i = 1$.

1. Let A be an $m \times n$ matrix. Assume that there is an entry in A that is the minimum in its column and the maximum in its row. Prove that this entry is the value of the zero-sum game with for two players with mixed strategies.

   Solution: Let the value of that entry be equal to M and assume it is in the i'th row and j'th columns.

   By having player R choose the i'th row with probability 1, then no matter what player C chooses, the value of the game will be at most M because M is the largest in its row. This shows that $\min_y \max_x yAx \leq M$.

   By having player C choose the j'th column with probability 1, then no matter what player R chooses, the value of the game will be at least M because M is the smallest in its columns. This shows that $\max_x \min_y yAx \geq M$.

   We are done because we know that

   $$\min_y \max_x yAx = \max_x \min_y yAx.$$

2. Find $\max_x \min_y yAx$ for the matrix

$$A = \begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}$$

What is the (mixed) strategy for the column player to guarantee the maximum possible result?

Solution: We follow the example we did in class. Denoting $x = (a, 1-a)$ and $y = (b, 1-b)$ we find that $yAx = (5b-1)(a-\frac{3}{5}) + 3\frac{2}{5}$. From here it is not hard to see that the value of the game is $3\frac{2}{5}$ and the best strategy for the column player is to use the vector $(\frac{3}{5}, \frac{2}{5})$ in order to guarantee this optimal value.

3. Find the min-max value for the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ on the main diagonal.

Solution: If $\lambda_i \geq 0$ and $\lambda_j \leq 0$ (possibly $i = j$), then the min-max value is equal to 0. This is because we can apply Problem 1 on the entry $a_{ij} = 0$ that is the largest in its column and smallest in its row.

Therefore, assume $\lambda_1, \ldots, \lambda_n > 0$.

Then $yAx = \sum x_i y_i \lambda_i$. Fixing y we have $\max_x \sum x_i y_i \lambda_i$ is when $x_i = 1$ for the i such that $y_i \lambda_i$ is maximum.

Then $\max_x yAx = \max_i y_i \lambda_i$. Therefore, if we want to find $\min_y \max_x yAx$ we better have y such that $\max_i y_i \lambda_i$ is minimum. Similar to what we did on Problem 2, this happens when all the $y_i \lambda_i$ are equal. Then $y_i = \frac{1}{\lambda_i}$ times a constant that does not depend on i. Because $\sum y_i = 1$, then we must have

$$y_i = \frac{1}{\sum \frac{1}{\lambda_j}} \frac{1}{\lambda_i}.$$

Then the value of the min-max is $\frac{1}{\sum \frac{1}{\lambda_j}}$.

What is the λ_i's are all negative?

In this case notice that we have (we use the fact that $\min f = -\max(-f)$)

$$
\begin{aligned}
\frac{1}{\sum \frac{1}{\lambda_j}} &= -\min_y \max_x y(-A)x \\
&= -\min_y \max_x -yAx \\
&= -\min_y(-\min_x yAx) \\
&= \max_y \min_x yAx \\
&= \max_y \min_x xAy \\
&= \min_x \max_y xAy \\
&= \min_y \max_x yAx
\end{aligned}
$$

4. Show that in a zero-sum game with a matrix A with mixed strategies the following is true: If one player knows the mixed strategy of the other player, then the best response (strategy) for him is a pure strategy. That is, the best response is choosing just one row or column.

   Solution: Assume for example that the Column player knows the strategy y_0 of the Row player. Then the Column player wants to maximize $\max_x y_0 Ax$ subject to $\sum x_i = 1$ and $x \geq 0$. However, this is a linear program. The simplex algorithm will find a vertex of the polyhedron $\sum x_i = 1$ and $x \geq 0$. The vertices of this polyhedron are the n vectors that have all their coordinates equal to 0 except one coordinate that is equal to 1. This gives a pure strategy.

5. Find a hyperplane separating the point $v = (4, 3, 6)$ from the ball $\{(x, y, z) \mid (x-1)^2 + (y-2)^2 + (z-3)^2 \leq 16\}$.

   Solution: The radius of the ball is $\sqrt{16} = 4$ The center of the ball is $u = (1, 2, 3)$. The closest point of the ball to v is $w = u + 4(v-u)/|v-u|$. Let z be the midpoint of the segment $[v, w]$, that is $(v + w)/2$. It is enough to take the hyperplane perpendicular to $v - u$ through z.