Discrete Optimization 2024 (EPFL): Problem set of week 3

March 14, 2024

1. A set K in $\mathbb{R}^n$ is called *convex* if for every x and y in K and for every $0 \leq \lambda \leq 1$ also the point $\lambda x + (1-\lambda)y$ is in K. In other words, the entire line segment with endpoints x and y is in K.

   a) Prove that any intersection of convex sets is convex.

   b) For any given $\overrightarrow{a} \in \mathbb{R}^n$ and any $b \in \mathbb{R}$ prove algebraically (from the algebraic definitions) that the half-space $\{\overrightarrow{x} \mid < \overrightarrow{x}, \overrightarrow{a} > \leq b\}$ is convex.

   c) Conclude from a) and b) that every intersection of half-spaces is convex.

   Solution (only for part b. Parts a+c are not difficult). Assume that both $\overrightarrow{x}$ and $\overrightarrow{y}$ belong to the half-space. Let $0 \leq \lambda \leq 1$ and consider $\lambda\overrightarrow{x} + (1-\lambda)\overrightarrow{y}$. We need to show that it also belongs to the same half-space.

   Indeed,

   $$\begin{aligned} < \lambda\overrightarrow{x} + (1-\lambda)\overrightarrow{y}, a > &= \lambda < \overrightarrow{x}, \overrightarrow{a} > + (1-\lambda) < \overrightarrow{y}, \overrightarrow{a} > \\ &\leq \lambda b + (1-\lambda) b = b \end{aligned}$$

   Notice that it is crucial in the inequality that neither of λ, nor $(1-\lambda)$ is negative.

2. Let Q be the quadrangle in the plane whose vertices are $(4,3)$, $(3,4)$, $(2,3)$, and $(3,2)$. Find a matrix A and a vector $\overrightarrow{b}$ such that $Q = \{\overrightarrow{v} = (x,y) \mid A\overrightarrow{v} \leq \overrightarrow{b}\}$.

Solution. The idea of the exercise is to develop an intuition for hyperplanes. In this sense a strategy to solve the exercise is to draw the quadrangle and figure out the conditions on the hyperplanes (here lines) between the adjacent vertices of the quadrangle and the induced halfspace descriptions. For the adjacent vertices $(2,3)$ and $(3,2)$ the induced halfspace is given by $x+y \geq 5$ for $(2,3)$ and $(3,4)$ it is $y \leq x+1$ for $(3,4)$ and $(4,3)$ it is $x+y \leq 7$ and finally for $(4,3)$ and $(3,2)$ it is $y \geq x-1$.

Plugging this into the desired matrix form leads to:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix} \; b = \begin{pmatrix} 7 \\ 1 \\ -5 \\ 1 \end{pmatrix}$$

A way to prove that A and b are indeed a solution of the problem, one can check that each vertex is the intersection of each two of the edges induced by the inequalities.

3. Let B be the box in $\mathbb{R}^3$ defined by $B = \{\overrightarrow{v} = (x,y,z) \mid 1 \leq x \leq 5, \quad -2 \leq y \leq 6, \;\; 0 \leq z \leq 2\}$. Find a matrix A and a vector $\overrightarrow{b}$ such that $B = \{\overrightarrow{v} = (x,y,z) \mid A\overrightarrow{v} \leq \overrightarrow{b}\}$.

   Solution. The box is defined by the intersection of the following 6 hyperplanes:

   $$x \leq 5, \quad -x \leq -1, \quad y \leq 6, \quad -y \leq 2, \quad z \leq 2, \quad -z \leq 0.$$

   Put them into the matrix form we get:

   $$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ -1 \\ 6 \\ 2 \\ 2 \\ 0 \end{pmatrix}$$

4. Let P be the three dimensional pyramid with vertices $(1,1,-6)$, $(1,3,-4)$, $(-1,-2,5)$, and $(3,5,1)$. Find $\overrightarrow{c} \in \mathbb{R}^3$ such that the function $\langle \overrightarrow{c}, (x,y,z) \rangle$ attains its maximum on P precisely at the vertex $(1,3,-4)$.

   Solution. There are infinitely many solutions here. One simple solution (perhaps the simplest) is to take c that defines a hyperplane parallel to the hyperplane through the three vertices $(1, 1, -6)$, $(-1, -2, 5)$, and $(3, 5, 1)$.

The direction of c should point to the same halfspace to which $(1, 3, -4)$ belongs. Therefore, we first find a vector perpendicular to both $(1, 1, -6) - (-1, -2, 5)$ and $(3, 5, 1) - (-1, -2, 5)$.

One may want to take the vector $(65, -36, 2)$. Then the three vertices $(1, 1, -6)$, $(-1, -2, 5)$, and $(3, 5, 1)$ belong to the hyperplane $\{\langle (x, y, z), (65, -36, 2)\rangle = 17\}$.

We notice that $\{\langle (1, 3, -4), (65, -36, 2)\rangle = -51 < 17$. This means that $\overrightarrow{c} = -(65, -36, 2) = (-65, 36, -2)$ is a good choice.

5. Let P be the polyhedron defined by $P = \{v \mid A\overrightarrow{x} \le \overrightarrow{b}\}$. Assume that $v_1, \ldots, v_k$ are k points in P and $v_1 + \ldots + v_k = 0$. Show that $0 \in P$.

   Solution: We need to show that $A\overrightarrow{0} \le \overrightarrow{b}$. This is equivalent to showing that no coordinate of $\overrightarrow{b}$ is negative. Assume to the contrary that say $b_1 < 0$. Then it follows that the first coordinate of each of $Av_1, \ldots, Av_k$ is negative. We get a contradiction by noticing that $Av_1 + \ldots + Av_k = A(v_1 + \ldots + v_k) = A0 = 0$ and the first coordinate of $\overrightarrow{0}$ is (equal to 0 and is) not negative.