Discrete Optimization 2024 (EPFL): Problem set of week 2

March 14, 2024

1. Show that the three medians in a triangle with vertices v_1, v_2, and v_3 meet at the point $\frac{1}{3}(v_1 + v_2 + v_3)$.

   Solution. The point in the middle of the edge v_1v_2 is $\frac{v_1+v_2}{2}$. Notice that $\frac{1}{3}(v_1+v_2+v_3) = \frac{1}{3}v_3+\frac{2}{3}(\frac{v_1+v_2}{2})$. Therefore, the point $\frac{1}{3}(v_1+v_2+v_3)$ lies on the median going from v_3 to the opposite edge. By symmetry it lies also on the other two medians.

2. Find the hyperplane passing through $(1, 1, 1)$ that is perpendicular to both hyperplanes $\{x + 2y + z = 2\}$ and $\{x - y - 3z = 8\}$ in $\mathbb{R}^3$.

   Solution. The hyperplane equation is $ax+by+cz = d$. We are looking for (a, b, c) that is perpendicular to both $(1, 2, 1)$ and $(1, -1, -3)$. We find it: $(5, -4, 3)$, or any multiple of it. Then we need $(1, 1, 1)$ to be on the hyperplane. This gives: $d = 5 - 4 + 3 = 4$. The answer is $\{5x - 4y + 3z = 4\}$.

3. Find the closest point to $(3, 5, 4)$ on the hyperplane $\{2x + 4y - z = 3\}$ in $\mathbb{R}^3$.

   Solution. We are looking for a point (a, b, c) such that $(a, b, c)-(3, 5, 4)$ is a multiple of $(2, 4, -1)$. In addition it must be on the hyperplane and so $2a + 4b - c = 3$.

   Alternatively, we are looking for t such that $(3, 5, 4)+t(2, 4, -1)$ (which is the point we are looking for) is on the hyperplane. This gives $2(3 + 2t) + 4(5 + 4t) - (4 - t) = 3$. We find $t = -\frac{19}{21}$.

4. Find the distance of the origin O to the line of intersection of the hyperplanes $\{x + y + z = 1\}$ and $\{2x - y + 3z = 1\}$ in $\mathbb{R}^3$.

   Solution. We first find the direction of the line of intersection of the two hyperplanes. It is in the direction perpendicular to both $(1, 1, 1)$ and $(2, -1, 3)$. We can take $(4, -1, -3)$ or any constant multiple of it. We now find a point on the line of intersection of the two hyperplanes: $(1, \frac{1}{4}, -\frac{1}{4})$, for example. Now we need to find t such that $(1, -\frac{1}{4}, \frac{1}{4}) + t(4, -1, -3)$ is perpendicular to $(4, -1, -3)$. Then find the distance from the origin to $(1, -\frac{1}{4}, \frac{1}{4}) + t(4, -1, -3)$. This just follows the approach of Exercise 3. We find $t = -\frac{9}{52}$.

5. Find a point that is inside the tetrahedron whose facets are: $\{x+y+z = 1\}, \{2x-3y-z = 2\}, \{x-3y+z = 4\}$, and $\{2x-y+3z = 1\}$.

   Solution. One way is to find the four vertices of the tetrahedron. We do this by solving four times a system of three equations in three variables. This equivalent to intersecting each three of the hyperplanes. So, $A = h_1 \cap h_2 \cap h_3 = (0.5, -0.75, 1.25)^\top$, $B = h_1 \cap h_2 \cap h_4 = (8/7, 3/14, -5/14)^\top$, $C = h_1 \cap h_3 \cap h_4 = (5, -0.75, -13/4)^\top$ and $D = h_4 \cap h_2 \cap h_3 = (-1, -1.5, 0.5)^\top$.

   The point $\frac{1}{4}(A + B + C + D) = (\frac{79}{64}, -\frac{39}{64}, -\frac{13}{28})$ must be inside the tetrahedron since it is a convex combination of the vertices and the tetrahedron, as it is a polytope, is defined as the convex hull of its vertices. It is a convex combination since the sum of the four coefficients is $4 \cdot \frac{1}{4} = 1$ and also $0 \leq \frac{1}{4} \leq 1$. Furthermore, $\frac{1}{2}A + \frac{1}{2}(\frac{A+B+C}{3})$ is inside the tetrahedron for the same reasoning.