Discrete Optimization 2024 (EPFL): Problem set of week 1

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1. Let $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$ be such that the angles between every two are equal. Find this angle if it is known that it is different from 0.

   Solution: We may assume that the length of each v_i is equal to 1. Let A be the $(n+1) \times n$ matrix whose rows are $v_1, \ldots, v_{n+1}$. Then AA^t is an $(n+1) \times (n+1)$ matrix whose entries are all equal to some c except the main diagonal where it is all 1's. This matrix can be written as $(1-c)I_{n+1} + cJ_{n+1}$, where J_{n+1} is the matrix all of whose entries are equal to 1. Notice that J_{n+1} is a matrix of rank 1 with one eigen value that is equal to $n+1$ and all the rest are 0's. Then the eigen values of AA^t are all equal to $1-c$ except for one that is equal to $1-c+c(n+1)$. Because AA^t cannot be of rank $n+1$ (why?), then it must have an eigen value that is equal to 0. So either $c=1$ which corresponds to all v_i's being equal, or $c = \frac{-1}{n}$. Then the angle between any two vectors is $\arccos \frac{1}{n}$. (Check for example in the plane, where $n=2$.)

2. Let Δ be a triangle in $\mathbb{R}^n$ such that all the coordinates of its vertices are integers. Show that the area of Δ is at least $\frac{1}{2}$.

   Solution: We recall that the area of a triangle determined by two vectors is $\frac{1}{2}$ the area of the parallelogram determined by these vectors. This area is a square root of an expression that is an integer if all the coordinates are integers. Therefore, the area of such a triangle is at least $\frac{1}{2}$ times the square root of an integer (which is at least 1).

3. Find all the vectors in $\mathbb{R}^3$ that are perpendicular to $(1,1,1)$ and create an angle of 60 degrees with $(1,2,3)$

   Solution: Denoting such a vector by (x,y,z) we need $x+y+z=0$ and $\frac{x+2y+3z}{\sqrt{14}\sqrt{x^2+y^2+z^2}} = \frac{1}{2}$. Because we care only about the direction of the vectors we may assume $z = 1$ (and check the case $z = 0$ separately). Then we get a quadratic equation in one variable (after setting $y = -1 - x$). We should get two solutions unless we are unlucky.

4. $(1, 2, 3), (2, -4, 5)$, and $(-2, 0, 9)$ are three vertices of a parallelogram in $\mathbb{R}^3$. What are the possibilities for the forth vertex?

   Solution: Denoting the three points by A, B, C the center of the parallelogram is in one of the midpoints $(A+B)/2$, $(B+C)/2$, or $(A+C)/2$. Then for example if the center is $(A + B)/2$, then the fourth vertex is the reflection of C through $(A+B)/2$, This is $C+2((A+B)/2-C) = A + B - C$. The two other options are $A + C - B$ and $B + C - A$.

5. Show that it is impossible to find $n + 1$ nonzero vectors $\mathbb{R}^n$ every two of which are perpendicular.

   Solution: Not unlikely that you have seen it before in Linear algebra. There is more than one way to do this. Assume to the contrary that there are such vectors $v_1, \ldots, v_{n+1}$. Then let A be the $(n + 1) \times n$ matrix whose rows are $v_1, \ldots, v_{n+1}$. Then AA^t is a diagonal $(n + 1) \times (n+1)$ matrix with non-zeroes on the main diagonal. This is impossible because the rank of A is at most n. We have seen a similar trick already in Problem 1.